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SOLUTION OF PROBLEMS IN PURE AND APPLIED MATHEMATICS,
PAPERS ON MATHEMATICAL SUBJECTS, BIOGRAPHIES
OF NOTED MATHEMATICIANS, ETC.

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BOLYAI FARKAS [WOLFGANG BOLYAI.]

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BIOGRAPHY.

BOLYAI FARKAS. [WOLFGANG BOLYAI.]

BY DR. GEORGE BRUCE HALSTED.

FOR the treatment of parallels, what Frischauf calls "das anschaulichste Axiom," is due to the researches of Bolyai Farkas. He gives it in his "Kurzer Grundriss eines Versuchs" etc., p. 46, as follows: "Koennten jede 3 Punkte, die nicht in einer Geraden sind, in eine Sphaere fallen; so waere das Eucl. Ax. XI. bewiesen." Thus the space whose every three points are costraight or concyclic is Euclidean.

But in his Autobiography written in Magyar, of which my forthcoming life of the Bolyais contains the first translation ever made, he says: "Yet I was not satisfied with my attempts to prove the Problem of Parallels, which was ascribable to the long discontinuance of my studies, or more probably it was due to myself that I drove this problem to the point which robbed my rest, deprived me of tranquility."

Hitherto what was known of the Bolyais came wholly from the published works of the father, Bolyai Farkas, and from a brief article by Architect Fr. Schmidt of Budapest, "Aus dem Leben zweier ungarischer Mathematiker, Johann und Wolfgang Bolyai von Bolya. Grunerts Archiv, Bd. 48, 1868, p. 217.

In two communications sent me in September and October, 1895, Herr Schmidt has very kindly and graciously put at my disposal the results of his subsequent researches which I will here reproduce. But meantime I have from entirely another source come most unexpectedly into possession of original documents so extensive, so precious that I have determined to issue them in a separate volume devoted wholly to the life of the Bolyais; but these are not used in the sketch here given.

Bolyai Farkas was born February 9th, 1775, at Bolya in that part of Transylvania (Erdély) called Székelyföld. He studied first at Enyed, afterward at Klausenburg (Kolozsvár), then went with Baron Simon Kemény to Jena and afterward to Goettingen. Here he met Gauss, then in his 19th year, and the two formed a friendship which lasted for life.

The letters of Gauss to his friend were sent by Bolyai in 1855 to Professor Sartorius von Walterhausen, then working on his biography of Gauss. Everyone who met Bolyai felt that he was a profound thinker and a beautiful character.

Benzenberg said in a letter written in 1801 that Bolyai was one of the most extraordinary men he had ever known.

He returned home in 1799, and in January, 1804, was made professor of mathematics in the Reformed College of Maros-Vásárhely. Here for 47 years of active teaching he had for scholars nearly all the professors and nobility of the next generation in Erdély.

Sylvester has said that mathematics is poesy.

Bolyai's first published works were dramas.

His first published book on mathematics was an arithmetic: Az arithmetica eleje. 8vo. I—XVI, 1—162 pp. The copy in the library of the Reformed College is enriched with notes by Bolyai János.

Next followed his chief work, to which he constantly refers in his later writings. It is in Latin, two volumes, 8vo. with title as follows: TENTAMEN | JUVENTUTEM STUDIOSAM | IN ELEMENTA MATHESEOS PURÆ, ELEMENTARIS AC | SUBLIMIORIS, METHEDO INTUITIVA, EVIDENTIA— | QUE HUIC PROPRIA, INTRODUCENDI. | CUM APPENDICE TRIPLICE. |

Auctore Professore Matheseos et Physices Chemiæque | Publ. Ordinario. | Tomus Primus. | *Maros Vásárhelyini*. 1832. | Typis Collegii Reformatorum per JOSEPHUM, et | SIMEONEM KALI de felsö Vist. | At the back of the title: Imprimatur. | M. Vásárhelyini Die | 12 Octobris 1829. |

The now world renowned Appendix by Bolyai János was an afterthought of the father, who prompted the son not 'to occupy himself with the theory of parallels,' as Staeckel says, but to translate from the German into Latin a condensation of his treatise, of which the principles were discovered and properly appreciated in 1823, and which was given in writing to J. W. von Eckwehr in 1825.

The father, without waiting for Vol. II., inserted this Latin translation, with separate paging (1—26), as an Appendix to his Vol. I., where, counting a page for the title and a page 'Explicatio signorum,' it has twenty-six numbered pages, followed by two unnumbered pages of Errata.

The treatise itself, therefore, contains only twenty-four pages—the most extraordinary two dozen pages in the whole history of thought!

Milton received but a paltry 5 pounds for his Paradise Lost; but it was at least plus 5. Bolyai Janos, as we learn from Vol. II., p. 384 of '*Tentamen*,' contributed for the printing of his eternal 26 pages, 104 florins 54 kreuzers.

That this Appendix was finished considerably before the Vol. I., which it ›ws, is seen from the references in the text, breathing a just admiration for Appendix and the genius of its author,

Thus Bolyai Farkaś says, p. 452: Elegans est conceptus *similium*, quem B. *Appendicis Auctor* dedit; again, p. 489: *Appendicis Auctor*, rem nine singulari aggressus, Geometriam pro omni casu absolute veram posuit; mvis e magna molé, tantum summe necessaria, in Appendice hujus tomi ex- ıerit, multis (ut tetraedri resolutione generali, pluribusque aliis disquisition- ı elegantibus) brevitatis studio omissis. And the volume ends as follows, p. : Nec operae pretium est plura referre; quum res tota ex altiori contempla- is puncto, in ima prenetranti oculo, tractetur in Appendice sequente, a quovis li veritatis purae alumno digna legi.

The father gives a brief resumé of the results of his own determined, life- ;, desperate efforts to do that at which Saccheri, J. H. Lambert, Gauss also failed, to establish Euclid's theory of parallels *a priori*.

He says, p. 490: "tentamina idcirco quae olim feceram, breviter expo- da veniunt; ne saltem alius quis operam eandem perdat." He anticipates J. boeuf's "Prolégomènes philosophiques de la géométrie et solution des postu- ," with the full consciousness in addition that it is *not* the solution,—that the l solution has crowned not his own intense efforts, but the genius of his son.

This son's Appendix which makes all preceding space only a special case, r a species under a genus, and so requiring a descriprive adjective, *Euclidean*, wonderful production of pure genius, this strange Hungarian flower ı saved for the world after more than thirty-five years of oblivion, by the rare dition of Professor Richard Baltzer of Dresden, afterward professor in the Uni- sity of Giessen. He it was who first did justice publicly to the works obachevski and Bolyai.

Incited by Baltzer, 1866, J. Hoüel issued a French translation of Loba- vski's Theory of Parallels and in a note to his Preface says: "M. Richard tzer, dans la seconde édition de ses excellents *Éléments de Géométrie*, a, le mier, introduit ces notions exactes à la place qu'elles doivent occuper." nor to Baltzer! But alas! father and son were already in their graves!

Fr. Schmidt in the article cited (1868) says: "It was nearly forty years ore these profound views were rescued from oblivion, and Dr. R. Baltzer, of sden, has acquired imperishable titles to the gratitude of all friends of science the first to draw attention to the works of Bolyai, in the second edition of his ellent Elemente der Mathematik (1866-67). Following the steps of Baltzer, fessor Hoüel, of Bordeaux, in a brochure entitled: Essai critique sur principes fondamentaux de la Géométrie élémentaire, has give extracts from lyai's book, which will help in securing for these new ideas the justice they rit."

The father refers to the son's Appendix again in a subsequent book, Ürtan mei Kezdöknek [Elements of the science of space for beginners] (1850-51), pp. . In the College are preserved three sets of figures for this book, two by the author, and one by his grandson, a son of János. The last work of Bolyai Farkas, the only one composed in German, is entitled: Kurzer Grundriss eines Versuchs

I. Die Arithmetik, durch zvekmässig Konstruirte Begriffe, von eingebildeten und unendlich-kleinen Grössen gereinigt, anschaulich und logisch-streng darzustellen.

II. In der Geometrie, die Begriffe der geraden Linie, der Ebene, des Winkels allgemein, der winkellosen Formen, und der Krummen, der verschiedenen Arten der Gleichheit u.d.gl. nicht nur scharf zu bestimmen; sondern auch ihr Seyn im Raume zu beweisen: und da die Frage, *ob zwey von der dritten geschnittene Geraden, wenn die summe der inneren Winkel nicht* $=2R$, *sich schneiden oder nicht?* Niemand auf der Erde ohne ein Axiom (wie Euklid das XI) aufzustellen, beantworten wird; die davon unabhängige Geometrie abzusondern; und eine auf die *Ja*-Antwort, andere auf das *Nein* so zu bauen, das die Formeln der letzten, auf ein Wink auch in der ersten gültig seyen.

Nach ein lateinischen Werke von 1829, M. Vásárhely, und eben daselbst gedruckten ungrischen.

Maros Vásárhely 1851. 8vo. pp. 88.

In this book he says, referring to his son's Appendix: "Some copies of the work published here were sent at that time to Vienna, to Berlin, to Goettingen. . . . . From Goettingen the giant of mathematics, who from his pinnacle embraces in the same view the stars and the abysses, wrote that he was surprised to see accomplished, what he had begun, only to leave it behind in his papers." This refers to 1832. The only other record that Gauss ever mentioned the book is a letter from Gerling written October 31st, 1851, to Wolfgang Bolyai on receipt of a copy of 'Kurzer Grundriss.' Gerling, a scholar of Gauss, had been from 1817 Professor of Astronomy at Marburg. He writes: "I do not mention my earlier occupation with the theory of parallels, for already in the year 1810—1812 with Gauss, as earlier as 1809 with J. F. Pfaff I had learned to perceive, how all previous attempts to prove the Euclidean axiom had miscarried. I had then also obtained preliminary knowledge of your works, and so, when I first [1820] had to print something of my view thereon, wrote it exactly so, as it yet stands to read on page 187 of the latest edition.

We had about this time [1819] here a law professor Schweikart, who was formerly in Charkow, and had attained to similar ideas, since without help of the Euclidean axiom he developed in its beginnings a geometry which he called Astralgeometry. What he communicated to me thereon, I sent to Gauss, who then informed me, how much farther already had been attained on this way and later also expressed himself about the great acquisition, which is offered to the few expert judges in the Appendix to your book."

The 'latest edition' mentioned appeared in 1851, and the passage refarred to is: "This proof [of the parallel-axiom] has been sought in manifold ways by acute mathematicians, but yet until now not found with complete sufficiency. So long as it fails, the theorem, as all founded on it, remains a hypothesis, whose validity for our life indeed is sufficiently proven by *experience*, whose *general, necessary exactness* however could be doubted without absurdity."

Alas! that this feeble utterance should have seemed sufficient for more than thirty years to the associate of Gauss and Schweikart, the latter certainly one of the independent discoverers of the non-Euclidean geometry. But then since neither of these sufficiently realized the transcendent importance of the matter to publish any of their thoughts on the subject, a more adequate conception of the issues at stake could scarcely be expected of the scholar and colleague. How different with Bolyai János and Lobachévski, who claimed at once, unflinchingly, that their discovery marked an epoch in human thought so momentous as to be unsurpassed by anything recorded in the history of philosophy or of science, demonstrating as had never been proven before the supremacy of pure reason at the very moment of overthrowing what had forever seemed its surest possession, the axioms of geometry.

Austin, Texas, December 16th, 1895.

THE DUPLICATION OF THE NOTATION FOR IRRATIONALS.

By JOS. V. COLLINS, Ph. D., State Normal School, Stevens Point, Wisconsin.

Authorities agree in crediting Rudolff (1525), the German cossist, with the introduction of the radical sign, $\surd$, not precisely as we use it, but one such mark for a square root, three for a cube, and two for a fourth root. Cantor thinks it probably originated from a West-Arabian custom of using dots, by makings *lines* of the dots, and connecting them in the making by lighter lines. These dots in turn originated, it is thought, in the use of the letter, dschim, the first in the Arabian word for *root*. Rudolff was followed by Stifel in the employment of this notation, and afterwards Girard (1633) changed it to the present form. By the middle of the 17th century the mark had come into general use. The exponent notation, though first used by Rudolff and Stifel in a crude form, was employed as we now have it for integral values of the exponents by Descartes. Soon after, Wallis, in his *arithmetica infinitorum* (1656), interpreted and used the simpler forms of fractional exponents, though Stevin (1585) had suggested the meaning to be assigned them. Then in 1676 Newton wrote to Oldenburg "since algebraists write a^2, a^3, a^4, etc., for *aa*, *aaa*, *aaaa*, etc., so I write $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{5}{3}}$, for $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{e.a^5}$." Newton went further in connection with his binomial theorem, and generalized this use of exponents into the exponential function. The question naturally arises why was it that the old notation for roots was not replaced by the new as had been done in numerous instances before? Doubtless the best reason for this is the fact that the radical signs were firmly intrenched by extended use before the fractional exponents as we have them were even thought of.

Now from one standpoint at least this duplication of marks for one of the commonest operations in mathematics is unfortunate. It certainly complicates unnecessarily a rather difficult part of elementary algebra. Doubtless all would agree that one or the other should be given up unless there is a good and sufficient reason for its retention. If either is to be discarded there is no question for a moment as to which should go. The use of fractional exponents is in perfect accord with that of integral ones, and introduces no new marks or conventions, while the radical sign notation is out of harmony with everything else in the algebraic notation. The radical sign and index are new marks, while the fractional exponent is an old quantity in a new place whose interpretation is quite natural. However, it should be said that the fractional exponent notation is ambiguous, since, in general,*

$\left(a^m\right)^{\frac{1}{n}}$ will not be the same as $\left(a^{\frac{1}{n}}\right)^m$, though each reduces to $a^{\frac{m}{n}}$. Nevertheless, even here the fractional exponent notation is to be preferred to the others, since the elementary treatment of irrationals virtually depends on the ignoring of this difference. (See, for example, Todhunter's Algebra, ed. 1877, p. 153; Chrystal's Treatise, Chapter X, Part II.) Not a few authors succeed by their manner of treatment in slurring this over. In this connection it ought to be said that some authors' books show distinct traces of their having been confused by the double surd notation. If authors themselves are not clear in their treatment of irrationals, it is likely that their students also will be more or less puzzled. This of itself would be a sufficient justification of an effort to remove the difficulty.

One obstacle in the way of dispensing entirely with the radical signs consists in the practical difficulty of writing and printing fractional exponents. But this, one is constrained to believe, can readily be overcome. And first it may be remarked that there is the same justification for omitting the numerator 1 in a fractional exponent that there is for never writing the integral exponent 1. When omitted it can be understood. Then again there is the same justification for dropping the denominator 2 in the exponent that there is for understanding the radical index 2 when no index is written. Thus all that is left of the fractional exponent $\frac{1}{2}$ is the horizontal line or the solidus oblique line. To make the changes suggested clear to the reader, some expressions are written below with their values in the three notations:

RADICAL NOTATION.		FRAC. EXPONENT NOT.		PROPOSED NOTATION.
$2\sqrt{a}$	=	$2a^{\frac{1}{2}}$	=	$2a^{/}$ *

*The marks for primes would differ from this sign in being shorter and vertical. However, it would be better to write subscripts in place of them.

$$3\sqrt[3]{26} = 3(26)^{\frac{1}{3}} = 3(26)^{/3}$$

$$\sqrt{a+m} = (a+m)^{\frac{1}{2}} = (a+m)^{/}$$

$$\sqrt[4]{\frac{a^2+b^2+c^2}{2abc}} = \left(\frac{a^2+b^2+c^2}{2abc}\right)^{\frac{1}{4}} = \left(\frac{a^2+b^2+c^2}{2abc}\right)^{/4}$$

$$\sqrt[5]{(x^3+3xy^2)^3} = (x^3+3xy^2)^{\frac{3}{5}}\text{, or } (x^3+3xy^2)^{3/5}$$

The proposed notation would do away with vinculums and would use preferably the solidus sign for division as is the tendency now in English mathematical and scientific books. In printing, $\sqrt{\ }$ would be replaced by $)^{/}$ on one type, and in script the latter would be made, without lifting the pen, in loop form. However, when the numerator of the fractional exponent is other than unity, the usual fractional exponent notation (which for this case is preferable to the radical sign notation) would be employed. Notice that by the simple changes proposed, which are perfectly natural ones, all the advantages of the duplicate notation would be preserved with none of its disadvantages, such as the use of the unsightly hieroglyphic-like radical sign (giving as it does a forbidding appearance to the printed page), and the confusion which arises from the simultaneous use of two distinct notations for the same operation.

In conclusion it should be emphasized that mathematicians themselves are not likely to feel the need or approve of any change in the algebraic notation. Like the reform in spelling, it is in the interest chiefly of the hundreds of thousands of students of elementary mathematics yet to come, and not in that of those who have already mastered the two notations, that this reform is urged. Surely it is not too much to ask that the fractional exponents as now written be employed exclusively (instead of largely as now) in all higher works involving the use of algebraical symbols. The abridgments would then be likely to come as a matter of course.

Stevens Point, Wisconsin, May 11, 1895.

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from December Number.]

THE CONSTRUCTION OF NON-PRIMITIVE GROUPS WITH THREE SYSTEMS OF NON-PRIMITIVITY.

Let the degree of the required group G be $3n$. G must be a subgroup (using subgroup in its broad sense in wnich it includes the group itself and identity) of

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all}(c_1 c_2 \ldots\ldots c_n)\text{all}.$$

If G_1 is not identity,* its constituents must be conjugate transitiv subgroups of these three systems.

If we designate the systems by A, B, and C, the permutations of the sys tems must correspond to a group of these three letters, for if these permutation would not form a group of operations G itself could not be a group. Hence every non-primitive group with three systems must correspond to one of the following groups :

$$(ABC)\text{cyc} \qquad (ABC)\text{all}$$

Since the former of these is a subgroup of the latter it follows that at least a part of every non-primitive group in three systems corresponds to

$$(ABC)\text{cyc}$$

we proceed to find this part. By a course of reasoning similar to that employed under two systems it follows that all the substitutions which transform any G_1 according to ABC must be contained in

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all}(c_1 c_2 \ldots\ldots c_n)\text{all}\ a_1 b_1 c_1 . a_2 b_2 c_2 \ldots\ldots a_n b_n c_n$$

and all those which transform G_1 according to ABC must be contained in

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all}(c_1 c_2 \ldots\ldots c_n)\text{all}\ a_1 c_1 b_1 . a_2 c_2 b_2 \ldots\ldots a_n c_n b_n$$

These sets are not independent, for if

$$s_\gamma \qquad \gamma = 1, 2, \ldots . (n!)^3$$

represents the substitution of one set then will the $(n!)^3$ different corresponding values of

$$s_\gamma^{-1}$$

represent the substitutions of the other set.

If in any non-primitive group G_2 stands for the substitution belonging to the first set and G_3 for those belonging to the second set, and if g_2 and g_3 represent the number of substitutions in G_2 and G_3 respectively we derive from the fact that if a group contains s_γ it must also contain s_γ^{-1} that

$$g_2 = g_3$$

If in any non-primitive group we multiply any substitution of G_2 by all

*This case was not considered under two systems of non-primitivity. It was unnecessary to consider it. For, since a transitive group contains substitutions which replace a given letter by all of the letters involved it follows that the order of a non-primitive group is always equal to its degree. It can easily be shown that the order of any transitive group is a multiple of its degree.

the substitutions of G_3 we obtain g_3 different substitutions of G_1, hence

$$g_1 \geqq g_3.$$

If we multiply a given substitution of G_3 into all the substitutions of G_1 we obtain g_1 different substitutions of G_3, hence

$$g_3 \geqq g_1.$$

Combining the last two relations with the preceding we obtain for any non-primitive group with three systems of non-primitivity

$$g_1 = g_2 = g_3.$$

Since the relation between G_2 and G_3 is such that we can derive one directly from the other we shall generally consider only G_2. But G_2 can be directly obtained from G_1 provided we have given one of the substitutions of G_2. Hence to construct the non-primitive group (or the part of a non-primitive group) corresponding to

$$(ABC)\text{cyc}$$

it is only necessary to find G_1 and one substitution (s_γ) corresponding to ABC.

s_γ must clearly satisfy the following conditions:

(1) Its cube is found in G_1.

(2) It transforms G_1 into itself.

(3) It permutes the systems according to ABC.

These three conditions are sufficient for if any substitution s_γ fulfills these conditions then is

$$G_1 + G_1 s_\gamma + G_1 s_\gamma^{-1}$$

a non-primitive group for

$$G_1 s_{\gamma_1} G_1 = G_1 s_{\gamma_1} G_1 s_\gamma^{-1} s_{\gamma_1} = G_1 s_{\gamma_1}$$

$$G_1 s_\gamma^{-1} G_1 = G_1 s_\gamma^{-1} G_1 s_{\gamma_1} s_\gamma^{-1} = G_1 s_\gamma^{-1}$$

etc., etc., etc.

It remains to prove that the three given conditions are necessary as well as sufficient, i. e., we have to show that none of the three pair of conditions is sufficient. The pair which excludes the last condition is evidently insufficient, and the following examples prove that the other two pair are also insufficient.

1	1	1	1
abc	*def*	*ghi*	*abc.def.ghi*
acb	*dfe*	*gih*	*acb.dfe.gih*
			ab.de.gh
			ac.df.gi
			bc.ef.hi

For *aehbdg.efi* satisfies the second and third but not the first of the thre conditions if we take the first of these groups for G_1, and *aehbficdg* satisfies th first and third but not the second if we take the second of these groups for G_1 Hence we see that the three given conditions are necessary as well as sufficient

If the transitive constituents of G_1 admit only a cyclical (not a symmetric permutation then it is impossible to construct a G corresponding to (ABC)al and involving the given G_1. If they admit a symmetric permutation we hav to add to the part of G corresponding to (ABC)cyc sufficient substitution to make it correspond to (ABC)all. By a course of reasoning similar to tha which we have just pursued we prove that it is only necessary to find one substi tution s_β corresponding to AB, and that s_β must satisfy the following conditions

(1) Interchange the first two systems.

(2) Have its square in G_1.

(3) Transform the group corresponding to ABC into itself.

To fix these ideas we proceed to the construction of the non-primitive groups of degree six which contain three systems of non-primitivity. We shall then have found all the non-primitive groups up to degree eight as no such groups can exist for degree seven, or any other prime degree.

Non-Primitive Groups of Degree Six with Three Systems of Non-Primitivity.

G_1 must be one of the following four groups: $(ab)(cd)(ef)$, $\{(ab)(cd)$ $(ef)\}$pos, $(ab.cd.ef)$, 1 G_2 must be contained in

$$(ab)(cd)(ef)\ ace.bdf$$

(*a*) If $G_1=(ab)(cd)(ef)$ then will *ace.bdf* evidently satisfy the three neces sary conditions, we thus obtain a non-primitive group corresponding to ABC whose order is 24, viz:

(1) $(ab)(cd)(ef)\ (ace.bdf)\text{cyc}=(abcdef)_{24}$*

For s_β we may take *ac.bd*. This leads to a group of order 48 which ha the preceding group as a self-conjugate sub-group. The group is

(2) $(ab)(cd)(ef)(ace.bdf)\text{cyc}(ac.bd)=(abcdef)_{48}$

(*b*) If $G_1=\{(ab)(cd)(ef)\}$pos we can again use *ace.bdf* for s_γ. We thu obtain a second non-primitive group of order 12, viz:

(3) $\{(ab)(cd)(ef)\}\text{pos}\ (ace.bdf)=(abcdef)_{12}$ †

This is the only group that corresponds to ABC since the negative substi tutions which correspond to the most general G_2 do not have their cubes in thi

*The foot note in regard to $(abcdef)_{8_2}$ applies also to this group.

†The foot note in regard to $(abcdef)_{8_3}$ applies also to this group.

G_1. For s_β we may take both *ac.bd* and *adbc*. We thus obtain two additional groups of order 24, viz:

(4) $\quad \{ (ab)(cd)(ef) \}$ pos $(ace.bdf)(ac.bd) = (+abcd)_{24}$

(5) $\quad \{ (ab)(cd)(ef) \}$ pos $(ace.bdf)(adbc) = (\pm abcd)_{24}$

(*c*) If $G_1 = (ab.cd.ef)$, s_γ may again equal *ace.bdf*. The two substitutions *ab.cd.ef* and *ace.bdf* generate the group. The first interchanges the two cycles of the second and the second interchanges the three cycles of the first. The resulting group must therefore have two as well as three systems of non-primitivity, and hence is found in the former list. All the other three possible groups corresponding to *ABC* are conjugate to this.

For s_β we may use *ac.bd*, but with *ace.bdf* this will generate (*ace.bdf*)all. Hence this group is also found in the list of non-primitive groups with two systems of non-primitivity. Hence there is no additional non-primitive group for $G_1 = (ab.cd.ef)$.

(*d*) If $G_1 = 1$ the second condition of s_γ is satisfied by every substitution. The substitutions that may correspond to *ABC* must be of the third order and are therefore all conjugate so that we need to consider only one of them. We thus obtain the intransitive group

(*ace.bdf*)cyc.

If we take *ac.bd* for s_β we obtain an intransitive group corresponding to (*ABC*)all. If we take *ab.dc.ef* for s_β we obtain a non-primitive group which is also non-primitive in two systems as is evident. Hence $G_1 = 1$ leads to no new non-primitive group.

We have now examined the entire region through degree six with a view to its non-primitive groups and have found the following

LIST OF NON-PRIMITIVE GROUPS THROUGH DEGREE SIX.

Degree	Order	No.	Group
4	4	1	$(abcd)_4$
		2	(*abcd*)cyc
	8	1	$(abcd)_8$
6	6	1	$(abcdef)_6$
		2	(*abcdef*)cyc
	12	1	$(abcdef)_{12}$
		2	$(abcdef)_{12,}$
	18	1	$(abcdef)_{18}$
	24	1	$(+abcdef)_{24}$
		2	$(\pm abcdef)_{24}$
		3	$(abcdef)_{24,}$
	36	1	$(abcdef)_{36}$

	2	$(abcdef)_{36}$
48	1	$(abcdef)_{48}$
72	1	$(abcdef)_{72}$

General Remarks on the Construction of Non-Primitive Groups.

Let it be required to find the non-primitive groups of degree n, n being a composite positive integer greater than three, and let

$$m_1, m_2, \quad . \quad . \quad . \quad m_e$$

be all the positive integral factors of n (excepting unity) which satisfy the relation

$$m\alpha \overline{\overline{<}} \sqrt{n} \qquad \begin{array}{l} \alpha = 1, 2, \quad . \quad . \quad . \quad e \\ \sqrt{\ } \text{ indicates only the arithmetic root.} \end{array}$$

Hence we may divide n as follows:

No. of Systems	No. of Letters in Each System
m_1	$\frac{n}{m_1}$
m_2	$\frac{n}{m_2}$
.	.
.	.
.	.
m_e	$\frac{n}{m_e}$
$\frac{n}{m_1}$	m_1
$\frac{n}{m_2}$	m_2
.	.
.	.
.	.
$\frac{n}{m_e}$	m_e

Two of these relations will become identical when $m_\alpha = \sqrt{n}$ for some value of α in the series

$$1, 2, \quad . \quad . \quad . \quad e.$$

Otherwise they will all be different. From these we see that the number of different ways of dividing n into systems is odd or even as n is or is not a perfect square.

The work of finding all the non-primitive groups for any one of these divisions into systems, *e. g.* the one which contains m_1 systems, may be resolved into the following steps:

(1) Contruct the groups (the G_1's) which have conjugate transitive constituent groups from each of these systems and are so constituted that their con-

ıents admit of the permutations of some transitive group of degree m_1. The stituent transitive groups are clearly of degree $\frac{n}{m_1}$ unless $G_1=1$. The last ı does not need consideration when the order of the transitive group of degree is not a multiple of n.

[To be Continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from December Number.]

SCHOLION IV: *In which is expounded on a figure a certain consideration which Euclid probably thought, in order to establish that Postulate of his as 'per evident.*

I premise first: within any acute angle BAX g. 12.) can be drawn from any point X of AX a cerı straight XB, which under designated even if obtuse ıle R, which only with this acute BAX falls short of ı right angles; a certain XB, say I, can be drawn, ich at a finite remove meets this AB in a certain nt B. For just that I have demonstrated in a Schol- after P. XIII. I premise secondly: these AB, AX g. 25) can be understood as produced into the infinite n to certain points Y, and Z; and likewise the aforesaid XB (into the infinite and itself produced even to a certain point Y) can be understood to be so moved above this AB toward the parts of the point Z, that the angle at the point X toward the parts of the point A is always equal to the certain given obtuse angle R.

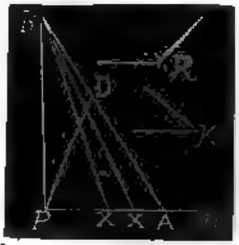


Fig. 12.

Fig. 25.

I premise thirdly: that Euclidean Postulate would be liable now to no doubt, if the aforesaid XY in this however great motion above straight AZ cuts always that AY in certain points B, D, H, P, and so successly in other points more remote from this point A.

The reason is evident; since thus any two straights AB, XH lying in the e plane, upon which any straight AX cutting makes two angles toward the same parts BAX, HXA, less than two right angles, must at length meet toward those parts in one and the same point H.

I premise fourthly: likewise will be no doubt over the truth of the preceding hypothetical assumption, if those later external angles YHD, YDP and so any other succeeding ones, either always are equal to the preceding external angle YBD, or at least always will be not so much less but that any one of them always will be greater than any little designated acute angle K. For, this holding, it is manifest that this XY in that however great motion of its toward the parts of the point Z, never will cease to cut the aforesaid AY; which assuredly (from the preceding note) is sufficient for establishing the controverted postulate.

Solely therefore remains, that a certain adversary may say those external angles at greater and greater distance from that point A may become always less without any determinate limit.

But thence would follow, that that XY in that motion of its above the straight AZ would at length meet AY in a certain point P without any angle with the segment PY, so that indeed a segment of the two straights APY, and XPY would be in this way common.

But this is evidently repugnant to the nature of the straight line. [The possibility that P may be a point at infinity is here overlooked.]

But if indeed to anyone may seem less opportune the obtuse angle at that point X toward the parts of the point A, it may easily be supposed right; so that indeed (in the motion of the aforesaid XY at angles always right above the straight AZ) more manifestly may appear that the single points of that XY are always moved equably relatively to the basal AZ; and therefore the aforesaid XY cannot go over from a secant into a non-secant of the other indefinite AY, unless either once in some point it precisely touches it, or meets it in some point P, where it has with this AY a common segment PY; each of which I will show contrary to the nature of the straight line in P. XXXIII.

Therefore in accordance with the true idea of the straight line, must that XY, in however great distance of the point X from the point A, always meet in some point this AY. And that this indeed (however small is supposed the acute angle at the point A) is sufficient for demonstrating, against the hypothesis of acute angle, the Euclidean Postulate, will follow from P. XXVII.

[To be Continued.]

THE BOND PROBLEM.

By J. K. ELLWOOD, A. M., Colfax School, Pittsburg, Pennsylvania.

What should an investor pay for one 7 per cent. $100. bond to run 20 years, interest payable semi-annually, in order to realize 8 per cent. per annum, payable semi-annually?

Let X=the price paid; R=4%, the semi-annual rate the investor realizes; t=the whole number of interest payments; r=3½%, the rate the bond draws semi-annually; v=\$100.

Besides the interest, the investor gains $v-x$, which will be due in ½t years. To liquidate both of these by equal payments requires each semi-annual payment to include the interest (rv) and such portion of the discount ($v-x$) as would, compounded semi-annually at R%, amount to $v-X$ in ½t years.

Let y be such a sum; then

$y(1+R)^{t-1}$= amount of 1st installment at end of ½t years.
$y(1+R)^{t-2}$= " " 2nd " " " " "
$y(1+R)$= " " $(t-1)^{th}$ " " " " "
$y(1+R)^0$ = " " t^{th} " " " " "

Hence, $y[(1+R)^{t-1}+(1+R)^{t-2}+\ldots\ldots(1+R)+1]=v-X.$

Summing the geometrical progression within the brackets, we have

$$y\left[\frac{(1+R)^t-1}{R}\right]=v-X,$$

$$\text{whence } y=\frac{R(v-X)}{(1+R)^t-1}.$$

Therefore each of the t equal payments is

$$vr+\frac{R(v-X)}{(1+R)^t-1},$$

which divided by X gives R.

$$\text{Hence, } vr+\frac{R(v-X)}{(1+R)^t-1}=RX.$$

Solve this equation for X and we have:

$$X=\frac{v(R-r)+vr(1+R)^t}{R(1+R)^t}\ldots\ldots(A).$$

In the above general equation substitute values from the problem and we have:

$$X=\frac{100(.04-.03\tfrac{1}{2})+3\tfrac{1}{2}(1.04)^{40}}{.04(1.04)^{40}}=\frac{\tfrac{1}{2}+3\tfrac{1}{2}\times 1.04^{40}}{.04\times 1.04^{40}}$$

The easy numerical computations are as follows:

40log 1.04=0.0170333×40=0.681332, which corresponds to 4.801.

$$\frac{\tfrac{1}{2}+3\tfrac{1}{2}\times 4.801}{.04\times 4.801}=\frac{17.3035}{.19204}=90.1036.$$

∴ \$90.1036 is the price to be paid for a 7% \$100 bond, interest payable semi-annually for 20 years, in order to realize 8% per annum, payable semi-annually.

The general equation (A) can be applied to the solution of the quarterly bond. In so applying it "we solve the government problem which confronted the Secretary of the Treasury when he placed the late \$50,000,000 loan on the market." This problem has been admirably solved by Theodore L. DeLand, the distinguished Examiner of the U. S. Civil Service Commission, first by algebraic analysis in THE AMERICAN MATHEMATICAL MONTHLY, and later by using the Calculus of Finite Differences. The latter solution was issued under cover of the *Mathematical Magazine*, January, 1895.

Secretary Carlisle desired to sell 10-year 5% \$100 bonds, interest payable quarterly, at a price that would enable the purchaser to realize 3%, interest payable quarterly.

Using these data, we have $R=\frac{3}{4}\%$, $r=1\frac{1}{4}\%$, $t=40$. Substituting values, equation (A) becomes:

$$x=\frac{100(.00\tfrac{3}{4}-.01\tfrac{1}{4})+1\tfrac{1}{4}(1.0075)^{40}}{.0075\times 1.0075^{40}}=\frac{1\tfrac{1}{4}\times 1.0075^{40}-\tfrac{1}{2}}{.0075\times 1.0075^{40}}.$$

40log 1.0075 = 0.0032451 × 40 = 0.129804, which corresponds to 1.34835.

$$\frac{1\tfrac{1}{4}\times 1.34835-\tfrac{1}{2}}{.0075\times 1.34835}=117.223.$$

∴ \$117.22$\frac{3}{10}$ is a just price for the bonds mentioned.

Problems of this kind may be solved very readily by arithmetic, as follows:

Take the first problem above. The bond yields \$7. per annum, which is 8% of \$87.50. This would be the price if only \$87.50 were to be paid the investor at maturity. But he will receive \$12.50 more, hence he must now give, in addition to the \$87.50, a sum that will in 20 years at 8% compound semi-annual interest amount to \$12.50.

$$\$12.50 \div \$4.80102 = 2.6036.$$

∴ \$87.50 + \$2.6036 = \$90.1036, the price.

When bonds are bought at a premium, the present value must be *deducted* from the sum that would be the price to be paid provided that sum were to be paid the investor at maturity.

Such problems are readily solved, but the arithmetician requires a very complete compound interest table to cover all cases.

The tables used by brokers give the same prices as those obtained by the methods herein set forth; but they extend only to 6% bonds to run 60 years.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

46. Proposed by I. J. SCHWATT, Ph. D., University of Pennsylvania, Philadelphia, Pennsylvania.

The Simson line belonging to one point of intersection of Brocard's Diameter of a triangle with the circumcircle of this triangle, is either parallel or perpendicular to the bessector of the angle formed by the side BC of the triangle ABC and the corresponding side $B'C'$ of Brocard's triangle.

Solution by the PROPOSER.

We shall first prove the following lemma:

1. If upon the sides of the triangle ABC are constructed similar isosceles triangles BA_2C, CB_2A, and AC_2B, and if the perpendicular A_2M_a is produced below BC, so that A'_2M_a is equal to A_2M_a then is $AC_2A'_2B_2$ a parallelogram.

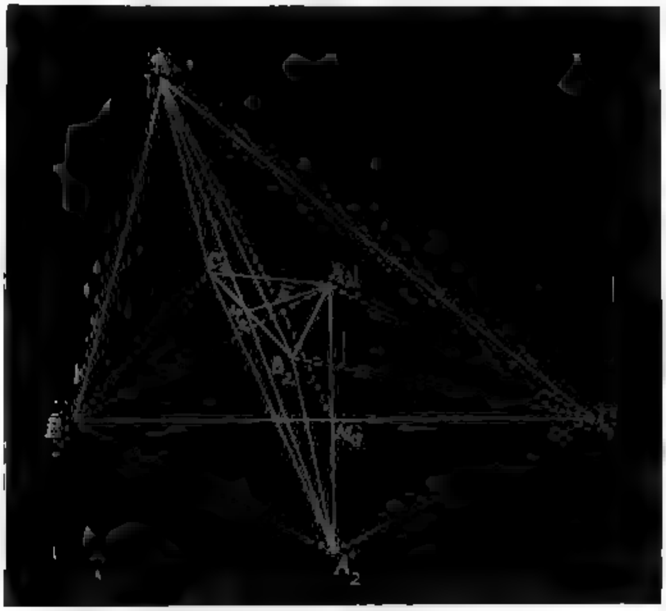


Fig. 1.

$\measuredangle C_2BA'_2 = \measuredangle C_2BC + \measuredangle CBA'_2$;

but $\measuredangle CBA'_2 = \measuredangle A_2BC$;

therefore $\measuredangle C_2BA'_2 = \measuredangle C_2BC + \measuredangle A_2BC$;

but $\measuredangle A_2BC = \measuredangle C_2BA$,

hence $\measuredangle C_2BA'_2 = \measuredangle C_2BC + \measuredangle C_2BA = \measuredangle ABC$.

The triangle A_2BM_a is similar to triangle C_2BM_c (since they are right triangles having $\measuredangle A_2BC = \measuredangle C_2BA$).

Therefore $A_2B : C_2B = BM_a : BM_c = \frac{a}{2} : \frac{c}{2} = a : c$;

but $A_2B = A'_2B$;

hence $A'_2B : C_2B = a : c$,

and since the $\measuredangle A'_2BC_2 = \measuredangle ABC$, therefore is triangle A'_2BC_2 similar to triangle ABC. In a similar manner can be proved that the triangle $B_2CA'_2$ is also similar to triangle ABC, and therefore A'_2BC_2 and $B_2CA'_2$ are similar to one another. But $A'_2B = A'_2C$ and consequently are the triangles A'_2BC_2 and $B_2CA'_2$ not only similar but also equal and therefore $B_2A'_2 = C_2A$. In a similar manner can be proved that $AB_2 = C_2A'_2$ or $AC_2A'_2B_2$ is a parallelogram.

2. The triangles ABC and $A_2B_2C_2$ have the same median point E.

Since $AC_2A'_2B_2$ is a parallelogram, the diagonals AA'_2 and A_2C_2 will bisect each other at the point M'_a. $A_2M'_a$ is a median line in the triangle $A_2B_2C_2$ as well as in the triangle $AA_2A'_2$. A second median line in the triangle $AA_2A'_2$ is AM_a (since $A_2M_a = A'_2M_a$); we have, therefore, that $A_2E = 2EM'_a$ and $AE = 2EM_a$. But AM_a is also a median line in the triangle ABC, therefore is E the median point in the triangle ABC as well as in the triangle $A_2B_2C_2$.

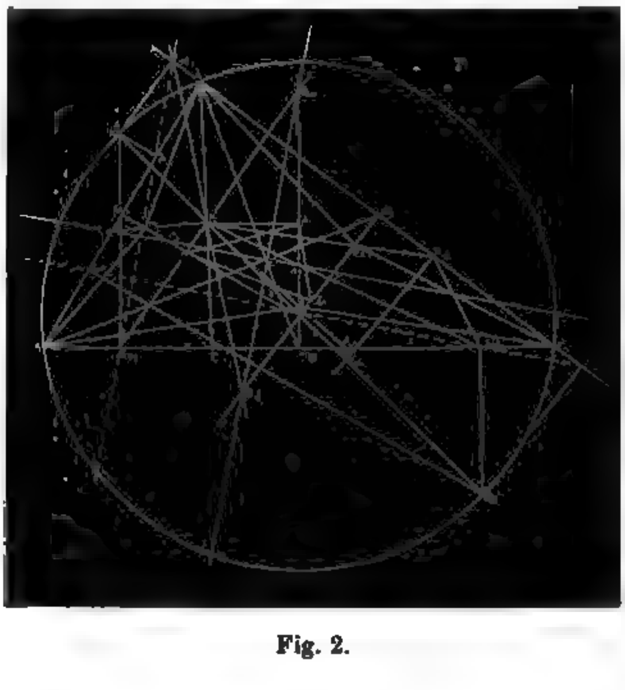


Fig. 2.

A_2, B_2, and C_2 were the vertices of similar isosceles triangles constructed upon the sides of the triangle ABC, and let KA_2, KB_2, and KC_2 meet the sides BC, AC, and AB respectively at $A_{2\alpha}$, $B_{2\beta}$, and $C_{2\gamma}$, then it can be proved that triangle $A_{2\alpha}\ B_{2\beta}\ C_{2\gamma}$ is similar to triangle $A_2\ B_2\ C_2$, the center of similitude being K. If we erect a perpendicular at $A_{2\alpha}$ to BC to meet Brocard's Diameter at Q_2, then, putting for A_1M_a, B_1M_b, their equals KK_a, KK_b respectively, ($A_1B_1C_1$ is Brocard's triangle), we have

$$\frac{A_2M_a}{KK_a}=\frac{A_{2\alpha}A_2}{A_{2\alpha}K}=\frac{Q_2M}{Q_2K}.$$

Since the triangles A_2BC and B_2AC are similar, we have

$$\frac{A_2M_a}{B_2M_b}=\frac{M_aC}{M_bC}=\frac{a}{b}=\frac{A_1M_a}{B_1M_b},$$

or
$$\frac{A_2M_a}{A_1M_a}=\frac{B_2M_b}{B_1M_b}=\frac{B_{2\beta}B_2}{B_{2\beta}K}=\frac{Q_2M}{Q_2K}.$$

Therefore
$$\frac{A_{2\alpha}A_2}{A_{2\alpha}K}=\frac{B_{2\beta}B_2}{B_{2\beta}K}.$$

Similarly we get
$$\frac{B_{2\beta}B_2}{B_{2\beta}K}=\frac{C_{2\gamma}C_2}{C_{2\gamma}K}=\frac{Q_2M}{Q_2K},$$

or, triangles $A_2B_2C_2$ and $A_{2\alpha}B_{2\beta}C_{2\gamma}$ are similar, and K is the center of similitude. From the equation

$$\frac{B_{2\beta}Q_2}{B_{2\beta}K}=\frac{Q_2M}{Q_2K},$$

it follows that $B_{2\beta}Q_2$ is parallel to B_2M, and since B_2M is perpendicular to AC, therefore $B_{2\beta}Q_2$ is also perpendicular to AC, or the perpendicular at $B_{2\beta}$ to AC passes through Q_2. Similarly, the perpendicular at $C_{2\gamma}$ to AB passes through Q_2. If, now, Q_2 is made to coincide with either Q_3 or Q_4, the points of intersection of Brocard's Diameter and the circumcircle of the triangle ABC, the triangle $A_{2\alpha}B_{2\beta}C_{2\gamma}$ will then degenerate into the straight lines $Q_{3a}\,Q_{3b}\,Q_{3c}$ and $Q_{4a}\,Q_{4b}\,Q_{4c}$ which are the Simson lines belonging to Q_3 and Q_4 with respect to the circumcircle of the triangle. The triangle $A_2B_2C_2$ will degenerate into the straight lines $A_3B_3C_3$ and $A_4B_4C_4$, which will be parallel to the Simson lines belonging to Q_3 and Q_4; and they will pass through the median point E, for the lines $A_3B_3C_3$ and $A_4B_4C_4$ still have the median point E in common with ABC.

Also, A_3, B_3, C_3 and A_4, B_4, C_4 are on the perpendiculars at the middle points of the respective sides of the triangle ABC. Since the Simson lines to Q_3 and Q_4 correspond to the extremities of a diameter, they are perpendicular to each other, and therefore their parallels $A_3B_3C_3$ and $A_4B_4C_4$ are also perpendicular to each other.

Furthermore, $Q_{3a}M_a=Q_{4a}M_a$,

$$Q_{3a}M_a : M_aK_a = Q_{4a}M_a : M_aK_a,$$

$$Q_{3a}M_a : M_aK_a = Q_{3a}A_3 : A_3K = A_3M_a : A_3A_1,$$

and
$$Q_{4a}M_a : M_aK_a = Q_{4a}A_4 : A_4K = A_4M_a : A_4A_1,$$

or
$$A_3M_a : A_3A_1 = A_4M_a : A_4A_1,$$

whence $\{M_aA_1,\ A_3A_4\}$ is an harmonic range, and $E\{M_aA_1,\ A_3A_4\}$ is an harmonic pencil. Since $\measuredangle A_4EA_3=90^\circ$, EA_3 will bisect the angle A_1EM_a.

Now, in two similar triangles the bisectors of the angles formed by any line in one triangle with the corresponding line in the other triangle are parallel to each other, hence the bisector of the angle formed by A_1E and EM_a, or the line AE, *i. e.*, the line $A_3B_3C_3$,—which is parallel to the Simson line belonging to one of the points of intersection of Brocard's Diameter, and the circumcircle about the triangle ABC,—is parallel to the bisector of the angle formed by B_1, C_1, and BC. (For particulars I can refer to my Geometrical Treatment of curves which are isogonal conjugate to a straight line with respect to a triangle, published by Leach, Shewell and Sanborn, New York.)

An excellent solution of this problem was also received from Professor G. B. M. Zerr.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

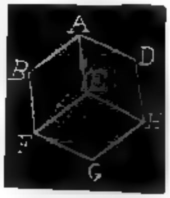
SOLUTIONS OF PROBLEMS.

36. Proposed by H. C. WHITAKER, B. Sc., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

A cube is revolved on its diagonal as an axis. Define the figure described and calculate its volume.

III. Solution by the PROPOSER.

The $\triangle$ BDE has each side$=\sqrt{2}a$, hence the radius of its circumscribed circle$=\frac{1}{3}\sqrt{6}a$. Hence the distance of A to the plane of $BDE=\frac{1}{3}\sqrt{3}a$. Take the origin at the center of the cube and the line AG as the axis of Z. The revolution will bring each line of the gauche hexagon $EHDCBF$ into either he position of DH or BF. The equations of DH are $x=\frac{1}{2}\sqrt{2}$, $y=-\sqrt{2}z$, and the equations of BF are $x=-\frac{1}{2}\sqrt{2}$, $y=-\sqrt{2}z$. In either case $x^2=\frac{1}{2}$ and $y^2=2z^2$ and $x^2+y^2=2z^2+\frac{a^2}{2}$ which is the equation of the surface generated by the gauche hexagon $EHDCBF$. This surface could also be generated by the hyperbola $x^2=2z^2+\frac{1}{2}$. Hence the volume of the hyperboloid of one nappe generated$=\int \pi x^2 dz$, the upper limit being $\frac{1}{6}\sqrt{3}a$ and the lower limit $-\frac{1}{6}\sqrt{3}a$. This integral is $\frac{5}{27}\pi\sqrt{3}a^3$.



The lines AB, AE, and AD generate a cone, radius$=\frac{1}{3}\sqrt{6}a$, altitude$=\frac{1}{3}\sqrt{3}a$, volume$=\frac{2}{27}\pi\sqrt{3}a^3$.

The lines GF, GH, and GC generate another cone of the same size.

The sum of the volumes of the three solids$=\frac{1}{3}\pi\sqrt{3}a^3=1.8138a^3$.

[NOTE.—This solution by the Proposer is fuller than that given in the November number, and is published because several of our contributors failed to comprehend the abbreviated solution previously published. Prof. Whitaker asserts that the solution by Dr. Zerr in the September-October number is incorrect, while the latter says he does not as yet see Prof. Whitaker's hyperboloid. The above seems to be correct, but we shall be glad to have the criticisms of other contributors.—EDITOR.]

43. Proposed by Professor C. B. WHITE, A. M., Trafalgar, Indiana.

Prove that $\int_0^1 \frac{x^{a-1}+x^{-a}}{1+x}\frac{dx}{x}=\log(\tan\frac{a\pi}{2})$, when $a>0$ and <1.

[*Williamson's Integral Calculus*, p. 154.]

Comment by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

There seems to be an error in No. 43, as I find the following in my copy of *Williamson* :

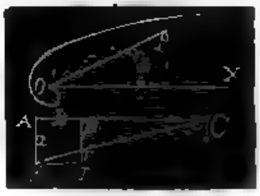
$$\int_0^1 \frac{x^{a-1}+x^{-a}}{1+x}\,\frac{dx}{\log x},$$

which gives the required result.

[In *Williamson's Integral Calculus*, edition of 1881, the problem is given as published, but the mistake has doubtless been corrected in the later edition.—EDITOR.]

44. Proposed by DE VOLSON WOOD, C. E., Professor of Mechanical and Electrical Engineering in Stevens Institute of Technology, Hoboken, New Jersey.

Find the equation of a curve in which $\rho=f(\theta)$, in which ρ is equal to BC, an intercept of any secant drawn from the corner E of the rectangle $AEDB$, and prolonged to cut AB prolonged in C. Let equal increments of θ be proportional to the equal increments of DB as divided by the secant EF, θ being zero when EC coincides with ED, and $\theta=2\pi$ when EF passes through B. Determine the asymptotes.



I. Solution by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas; and ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University P. O., Mississippi.

Referring to the diagram given by the Proposer of this problem, July-August (1895) MONTHLY, we have from the similar triangles FBC and FDE the following proportions: $BC : DE :: BF : DF$, or $\rho : b :: 2\pi-\theta : \theta$.

$\therefore$ $(\rho+b)\theta=2\pi b$......(1), which is the polar equation of *The Thistle of Scotland*, adopting the suggestion of Prof. MacCord.

Since $\rho^2(d\theta/d\rho)=-[(2\pi-\theta)^2/2\pi]b$, there is a *rectilinear asymptote* parallel to the initial line and at a distance $2\pi b$ *above* it. Making $\theta=\infty$, we have from (1) the equation $\rho=-b$; and this equation characterizes an *asymptotic circle* of radius b, or a *circular asymptote* of same radius, of the curve.

NOTE.—The derivation of (1) can be affected in, at least, *three* different ways; and, according to the conditions of the problem, (1) may also be written

$$(\rho+b)\theta=ab\text{......}(2).$$

II. Solution by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, P. O. Sebastopol, California.

From figure given $\dfrac{BC}{ED}=\dfrac{BF}{DF}=\dfrac{BD-DF}{DF}=\dfrac{BD}{DF}-1$,

or $\dfrac{\rho}{b}=\dfrac{2\pi}{\theta}-1$; $\rho=\dfrac{2b\pi}{\theta}-b$, the equation of the curve.

When $\theta=0$, $\rho=\infty$, and subtangent $=-2b\pi$.

The curve has, therefore, an asymptote parallel to OX at a distance above it, $2b\pi$, the circumference of a circle with radius AB.

The curve is concave toward the pole and intersects the axis perpendicularly and at a distance b to the left of the pole.

Elaborately solved by *O. W. ANTHONY*, and *C. W. M. BLACK*.

ERRATA.—On page 363, of last issue, line 4. omit $\sqrt{3}$ in the numerator of the second term; line 9, in the numerator, for "(a^2+x^2)" read (a^2-x^2); line 11, in the denominator of the second term, for "4" read 4^2; line 14, for "+" read =, before the last expression; page 364, line 15, for "of" read *to*; line 17, insert comma after "length"; line 17, for "$2n$" read 2π; line 18, for "π_2" read π^2; on same page, problem No. "43" should be No. 42; page 365, line 1, for "z^{n-2}" read z^{n-1}; and in line 2, of solution III., for "n^2+y^2" in the exponent, read x^2+y^2.

PROBLEMS.

51. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the maximum ellipsoid that can be cut out of a given right conic frustrum.

52. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

There are two lights of intensities m and n. Where must a target, whose surface is parallel to the line joining the two lights, be set up in order that it shall receive the maximum illumination per unit of area.

EDITORIALS.

In this issue will be found a bill with the amount due us to the end of 1896 marked thereon.

The great work of preparing the list of contributors and the index for Vol. II. is to be credited to Editor Colaw.

Prof. G. H. Harvill is now permanently located at Athens, Texas, from which place the *Mathematical Messenger* will be issued.

Persons wishing to discontinue their subscriptions to the MONTHLY, and who are not in arrears, should return this number with their names written upon the wrapper.

Mr. John McDowell of Philadelphia, writes as follows: "Find enclosed three dollars, being amount of subscription for your valuable journal, THE AMERICAN MATHEMATICAL MONTHLY, for '96."

This number of the MONTHLY has been cut short in order that we may catch up in its publication. We shall cut the February number some also. The March number will contain the regular departments again.

Prof. P. S. Berg, Larimore, North Dakota, writes, "Enclosed find three dollars as my subscription to THE AMERICAN MATHEMATICAL MONTHLY. . . . I should not be without it if the subscription price were five dollars."

Dr. G. A. Miller, Leipzig, Germany, writing in reference to the MONTHLY, says, "When I return I hope to be able to do much more towards aiding such efforts towards advancing the cause of mathematics in the United States. You are doing a great work. I hope you will not be discouraged in it."

Our valued contributor, Dr. Alexander Macfarlane, has an article on Quaternions in *Science* of January 17th. He has also prepared the article on Vector Analysis and Quaternions in *Higher Mathematics for Engineering Colleges*, a work edited by Drs. Mansfield Merriman and Robert S. Woodward, and which is expected to be ready in July.

Dr. G. B. M. Zerr, of Texarkana College, says, in a letter of January 7th, "I will remit subscription for '96 in a few weeks. I will remit $3.00 and am willing to pay $5.00 if necessary. I find myself very much benefited by the excellent solutions and excellent papers that appear in each number of the MONTHLY. Do not allow its publication to cease, rather raise the subscription price, I am satisfied the subscribers will stand by you."

NOTES.

Drs. Fisher and Schwatt's translation of Dr. H. Durège's Elements of the Theory of Functions is now ready.

Alexander Macmillan, the younger of the two brothers of the firm Macmillan & Co., died in England on January 25.

THE LOBACHEVSKI PRIZE.

On May 1, 1895, the Lobachévski Fund had reached, beyond all expenses, 8840 roubles, 95 kopeks.

This sum permits the accomplishment of the double aim of the committee: to found an international prize for research in geometry, especially non-Euclidean geometry, and to erect a bust of the celebrated scientist.

The prize, 500 roubles, will be adjudged every three years to the best works or memoirs on geometry, especially non-Euclidean geometry.

The prize will be given for works printed in the Russian, French, German, English, Italian, or Latin, sent to the Physico-Mathematical Society of Kazán by the authors, published during the six years which precede the adjudication of the prize. Works to compete must be sent to the Society at the latest one year before the day of award, October 22 old style (November 3).

The first prize will be adjudged October 22 (November 3), 1897.

To award the prize, the Society will form a commission to choose judges among Russian or foreign scientists.

The work of the judges (reporters) will be recompensed by medals of gold, bearing the name of Lobachévski.

As a fixed capital to found this prize, 6000 roubles were invested.

Of the sum collected, an additional 2000 roubles goes to share the expense of erecting a bust of Lobachévski in the park bearing his name in front of the University edifice in Kazán, the remainder of the cost to be borne by the Municipal Council.

A special committee, consisting of representatives of the Municipal Council and of the Physico-Mathematical Society, has made a contract with Mlle. Dillon, who engages for 3000 roubles to furnish a bronze bust of Lobachévski, to be placed on a granite pedestal, the height of the monument to exceed 3 mèters.

It is hoped to unveil the bust between the 15th and 25th of September 1896.

This 'fête mathématique' will follow the 'congrès des savants russes naturalistes et mathématiciens' at Kiev from 1st to 12th of September, 1896, and be during the grand Russian Exposition artistic and industrial at Nijny-Novgorod in the summer and autumn of 1896. Foreigners in any way identified with the name of Lobachévski are invited to the fête, and such as accept will be the guests of the city and University of Kazán.

For a second bust of Lobachévski to be placed in the Assembly Hall of th

University, 200 roubles have been given from the Lobachévski fund, the remainder of the cost to be borne by the professors of the University.

The residue of the sum already collected (640 r. 95 k.) will be added to the fixed capital. The augmentation of the capital will permit of a new edition of Lobachévski's works in a few years, the first volume of the Kazán edition having already become rare (out of print).

The Physico-Mathematical Society of Kazán has already received a large number of works and memoirs relating to Lobachévski and non-Euclidean geometry, and now having added its own collection of the printed and manuscript works of Lobachévski, the Society has inaugurated a separate library under the name *Bibliotheca Lobachévskiana*. It is hoped that in time this library will collect all the literature of non-Euclidean geometry and be an indispensable aid to those engaged in its development.

All writers on this fecund subject are begged to send to this library copies of their works.

Alas! That the Mathematico-physical Society of Hungary, a country having an equal claim to all the honors of the non-Euclidean geometry through the genius of Bolyai János, should have been content with placing in 1894 a monumental stone on his long neglected grave in Maros-Vásárhely!

GEORGE BRUCE HALSTED.

Austin, Texas.

THE UNIVERSITY OF CHICAGO: SUMMER, 1896.

The following mathematical courses will be offered: By Professor *Moore*, Theory of numbers, Differential equations (with introduction to Lie's continuous transformation groups); by Professor *Bolza*, Theory of substitutions, Theory of functions of a complex variable; by Professor *Miller*, of the University of Indiana, Analytical geometry of three dimensions; by Dr. *Young*, Conferences on mathematical pedagogy, Theory of equations, College algebra; by Mr. *Slaught*, Advanced integral calculus, Introductory course in differential and integral calculus; and by Mr. *Baker*, Analytical geometry of the plane. The pedagogical conferences are two hours weekly for six weeks and the other courses are four or five hours weekly for twelve weeks from July 1, 1896. Those who expect to work in mathematics in the University of Chicago during the coming summer as well as those who desire further information are requested to communicate with Professor Moore.

BOOKS AND PERIODICALS.

Elementary Mensuration. By F. H. Stevens, M. A., Formerly Scholar of Queen's College, Oxford; A Master of the Military Side, Clifton College. 12mo. cloth, 243 pp. Price, 90 cents, net. New York: Macmillan & Co.

This text-book of Elementary Mensuration is divided into two parts. The first part provides for those students whose knowledge of Geometry is confined to Euclid's First Book, and Algebra to the meaning of the simplest symbols. In the second part more difficult questions are offered to students who have mastered the Sixth Book of Euclid, have attained some facility in ordinary Algebraical methods as far as the Binomial Theorem and have made a beginning with Trigonometry.

Under each rule is given an illustrative solution neatly worked out, and proofs of formulæ have been given or indicated whenever they seemed likely to be intelligent to the learner. The book is in every way worthy of the consideration of teachers who are needing a good elementary text on Mensuration. B. F. F.

Problems in Differential Calculus Supplementary to a Treatise on Differential Calculus. By W. E. Byerly, Ph. D., Professor of Mathematics in Harvard University. 8vo. cloth. viii and 72 pp. Price, 80 cents. Boston and Chicago: Ginn & Co.

An excellent collection of about 350 problems to supplement the author's Treatise on the Differential Calculus. While these problems were especially prepared to use in connection with Dr. Byerly's Calculus they will be found useful wherever the subject is studied. B. F. F.

Computation Rules and Logarithms with Tables for other Useful Functions. By Silas W. Holman, Professor of Physics at the Massachusetts Institute of Technology. 8vo. cloth, 73 pp. Price, $1.00, net. New York: Macmillan & Co.

Besides a Table of Five Place Logarithms containing an abbreviated Table for One and Two Place Numbers, a table for five place numbers from 1.0 to 1.1, avoiding interpolation, a table for all four place numbers with interpolation tables for the fifth place; a table of logarithms of sines, cosines, tangents, and cotangents to four places; and a table of logarithms of sines, cosines, tangents, and cotangents to five places; there is also a four place logarithm table of numbers from 1 to 10; a table of square roots and squares of numbers from 1 to 100; a table of reciprocals of numbers from 1 to 1000; a table of slide wire ratios; a table of natural sines, cosines, tangents, and cotangents, and a number of tables of mathematical constants.

A very useful book for the practical computor. B. F. F.

Algebra for Schools and Colleges. By William Freeland, A. B., Head Master of the Harvard School, New York City. 8vo. cloth, 310 pp. Introduction Price, $1.12. New York: Longmans, Green & Co.

With the exception of two or three instances, the author sets no claim to originality. The book is designed to meet the requirements of those students who present themselves for the maximum courses in Freshman work for students who have advanced through the subject of Quadratics only.

Throughout the course tests for revision have been inserted, and a collection of 500 carefully graded Miscellaneous Examples has been given at the end of the book. The number of examples in the book is 5,200. It is very neatly printed on a good quality of paper. B. F. F.

A Primer of the History of Mathematics. By W. W. Rouse Ball, Fellow d Tutor of Trinity College, Cambridge, England. 12mo. cloth, 162 pp. ice, 65 cents. New York: Macmillan & Co.

This most charming little book ought to be used in all Algebra and Geometry sses in order to awaken early an interest in the History of Mathematics. A few years), I gave a short lecture to a class of about 60 students in Algebra, on the Arabic stem of Notation. After the lecture, a young man said to me, "Is it possible that ithmetic and Algebra have come down to us in their present form by a gradual develop-nt. I thought they were always as they are now." Were some such work as Mr. Ball's mer used in our classes in Algebra and Geometry, such dense ignorance concerning one the greatest departments of human knowledge would not exist. No one having then idied Arthemetic would suppose that the subject sprung from the human mind as rfect as Minerva from the head of Jupiter. B. F. F.

The Elements of Physics. A College Text Book. By Edward L. Nichols id William S. Franklin. In three volumes. Vol. I. Mechanics and Heat. ro. cloth, 228 pp. Price, $1.50. New York: Macmillan & Co.

In this valuable treatise on Physics, the authors have not attempted to lift the udent over difficulties and set him down in easy places. The work, it appears, is written ith a view of giving the student the best possible advantage of the subject. The authors ive squarely faced the difficulties of the subject and have, as occasion demanded, used e Calculus rather than encounter a subject by long, laborious and indirect methods oiding the use of the Calculus. However, the degree of mathematical experience of the idergraduate reader has been kept in view and the various proofs and demonstrations ve been given the simplest possible form. The concepts of directed and distributed antity are briefly treated in Chapter II of Vol. I.

From what we know of the first volume we believe that this Treatise will prove to be e best that has yet appeared in this country. B. F. F.

The Basis. A Monthly Magazine. Devoted to Good Citizenship. Edited · Judge Albion W. Tourgee, Mayville, New York. Price, $1.50 per year.

The Basis for January is a pleasant surprise in its new cover. The leading editorial nounces the retirement of the greenback as an "Epoch-Making Crime." In "A stander's Notes", Judge Tourgee treats especially the lack of earnest effort on the part the colored race for the betterment of their condition. The Mob-Record, the Depart-nt of Good Government Clubs and "Today's Thought" are well in evidence. There is ood short story and other characteristic matter. The number speaks well of the new inagement of *The Basis* and its new home on the Chautauqua Hills.

The Review of Reviews. An International Illustrated Monthly Magazine. lited by Albert Shaw. Price, $2.50 per year. Single number, 25 cents. The eview of Reviews Co., New York City.

The *Review of Reviews* for February contains an article which, in the compass of two iges, makes perhaps the most telling and effective exposure of the recent Turkish assacres that has yet been attempted in the English language. The article is based pon full accounts of the massacres, written on the ground by trustworty and intelligent ersons—French, English, American, Turk, Kurd, and Armenian—who were eye-witnesses f the terrible scenes. The article estimates the number of killed in the massacres thus ar at 50,000, the property destroyed at $40,000,000, and the number of starving survivors it 350,000.

Elements of the Theory of Functions of a Complex Variable with especial reference to the methods of Riemann. By Dr. H. Durege, late Professor in the University of Prague. Authorized translation from the fourth German Edition. By George Egbert Fisher, M. A., Ph. D., Assistant Professor of Mathematics in the University of Pennsylvania, and Isaac J. Schwatt, Ph. D., Instructor in Mathematics in the University of Pennsylvania. Large 8vo. cloth, 288 pp. Price, $2.50. Philadelphia: G. E. Fisher and I. J. Schwatt.

This valuable work comes to us just in time for notice in this issue of the MONTHLY. From only a cursory examination of it, we do not hesitate to emphasize what was said of it in the last issue. The work will afford a most excellent introduction to the study of the Theory of Functions and the intelligent reading of the larger Treatises—such as Forsyth's. The mechanical and typographical execution of the book is first class. B. F. F.

The Number Concept. Its Origin and Development. By Levi Leonard Connant, Ph. D., Associate Professor of Mathematics in the Worcester Polytechnic Institute. 8vo. cloth, 218 pp. Price, $2.00. New York: Macmillan & Co.

This work forms a most valuable addition to the literature of mathematics. The first chapter treats on Counting; the second, Number System Limits; the third and fourth, Origin of Number Words; the fifth, Miscellaneous Number Bases; the sixth, The Quinary System; the seventh, The Vigesimal System.

The treatment of these subjects is very interesting and evince careful study and research. B. F. F.

EMILE-MICHEL-HYACINTHE LEMOINE

THE AMERICAN MATHEMATICAL MONTHLY.

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BIOGRAPHY.

EMILE-MICHEL-HYACINTHE LEMOINE.

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SO EXTENSIVE has become the modern geometry of the triangle that one scarcely realizes that it has almost entirely developed within the last quarter of a century, and that most of its discoverers are still among the living. Lemoine, Brocard, Neuberg, Tucker, and W. J. C. Miller whose mathematical work in the *Educational Times* has done so much for the subject,—these and many others have lived to see their labors crowned with honor by lovers of geometry.

To none of these more than to Emile-Michel-Hyacinthe Lemoine is due the honor of having started this movement, and to him is the following brief sketch devoted.

M. Lemoine was born at Quimper, Finistère, in the west of France, Nov. 22, 1840. His father, a retired captain, who had been in all of the campaigns of the Empire after 1807, placed him as foundation scholar in the military Prytanée of La Flèche, whence he proceeded to the École Polytechnique. He entered this great breeding place of mathematicians at the age of twenty, the year of his father's death, and completed the course in due time. Instead of accepting any of the careers offered by the State to all graduates of the Polytechnic School, M. Lemoine determined to make his own way. Indeed, for the next few years, although engaged in science teaching in Paris, he seems to have run the round of pleasure of which that city is the home *par excellence.* Of great versatility and exceptional conversational powers, with an originality that fascinated and a personality that impressed his large circle of friends, he lived the life of a *dilettante* in the best sense of the term, and drank at the fountains of pleasure, of politics, of the arts, and of the sciences.

In these days Lemoine led as varied a life in education as in the less scholastic walks. We find him a student in the École des Mines, then *preparateur* of M. Janssen at the École d'Architecture,—supplying the place of his former professor, M. Kiœs, in the preparatory course of the École des Beaux Arts,—perfecting his knowledge of chemistry in the laboratory of Wurtz, for whom he always had a great admiration and between whom and himself there was much affection,—frequenting the courses of the École de Médecine, the hospitals and the clinics,—dabbling in philology,—and ending up by trying the law for a year. This last fancy he was forced to forego because he found himself in disgrace with the Empire through his republican principles and his liberal views on church matters. During these years, too, Lemoine traveled as his income would allow, and when his income failed him he not infrequently traveled as tutor in some wealthy family. Thus it was that he started out in his work as a teacher, full of life and health and hopes, although possibly scattering his attention too much for a career of highest success.

But however the result may have been, an unforeseen accident nipped the experiment in the bud. In 1870, when only a little more than twenty-nine years old, a laryngeal difficulty put an end to his teaching, and required him to leave Paris and seek rest at Grenoble. In the army for a time, he returned to Paris a couple of months after the Commune, and for a number of years filled divers positions in the engineering line. Finally, in 1886, he was appointed city engineer at the head of the gas department, a position which he still holds.

It is, however, with his mathematical work that we are concerned directly. In 1871 he, together with eight or ten other mathematicians, issued the circular which started the Société Mathématique of France. He was among the first to follow and to assist d'Almeida in founding the Journal de Physique and the Société de Physique. He joined with Wurtz, Friedel and others in the organization of the Association Française pour l'Avancement des Sciences. It was while yet a boy in his teens at La Flèche, that, in 1858, he published a short note in the Nouvelles Annales de Mathématiques, which discussed certain properties of the triangle. But it was at the Congrès de Lyon of the Association Française pour l'Avancement des Sciences, in 1873, that he presented his brief but noteworthy paper *Sur quelques propriétés d'un point remarquable de triangle*, and thus, as Casey says, made himself known as the founder of the modern geometry of the triangle. In the same year he published a short note in the Nouvelles Annales on the same subject. In 1874 he presented at the Congrès de Lille a second paper on the geometry of the triangle, entitled *Note sur les propriétés du center des médianes antiparalleles dans un triangle*, a point which has since been quite generally known as the *Lemoine point*, althoueh it is also called the *symmedian point* in England, and the *Grebe point* in Germany. The first paper (1873) contains among others the familiar theorem which may now be ated thus: "The three parallels to the sides of a triangle through its Lemoine oint meet the sides in six concyclic points (the first Lemoine circle)." By the emoine (symmedian) point is meant the point of concurrence of the symmedians f a triangle. Since the appearance of these two papers, Lemoine's name has een familiar to all readers of the mathematical journals in every country, and it for these contributions that he seems destined to be known, rather than for his *Géométrographie* which he considers his greatest work.

La Géométrographie, of which he had the first ideas in 1888, was suggestd by him in a memoir, on a more general theme, presented to the Congrès d' Oran of the Association Française pour l'Avancement des Sciences. The title of he paper is *De la mesure de la simplicité dans less Sciences mathématiques*, but or lack of time the study was limited to the simplicity of geometric construcions. On the same subject he published a short note in the *Comptes Rendus* of he Academy for that year,—more strictly *Sur la mesure de la simplicité dans les constructions géométriques*. Since then he has published numerous articles on the same or kindred subjects, in various journals, among them Mathesis (1888), Journal des mathématiques élémentaires (1889), Nouvelles Annales de Mathématiques (1892), in which last named article he considers especially the Problem of Apollonius. Finally, in 1892, at the Congrès de Pau and again at Besançon in 1893 and at Caen in 1894, a series of papers was presented on *La Géométrographie ou l'art des constructions Géométriques*, which may be considered as closing the subject of "geometrography" as applied either to the geometry of the rule and compasses alone, or to those constructions which admit the square, as in descriptive geometry.

Next in importance to the subject of "geometrography," M. Lemoine ranks his work on Continuous Transformation which permits of forming without effort, almost mechanically, a great number of formulæ and theorems relative to the triangle and to the tetrahedron. The principal memoirs which he has presented on this subject are the following: *Sur les transformations systématiques des formules relatives au triangle*, Congrès de Marseille 1891; *Étude sur une nouvelle transformation dite transformation continue*, in Mathesis for 1891; *Une règle d'analogies dans le triangle et la specification de certaines analogies à une transformation dite transformation continue*, in the Nouvelles Annales for 1893; and finally a memoir entitled *Applications au tétraèdre de la transformation continue.*

Three other geometric studies have been undertaken by M. Lemoine, which deserve especial mention. One is the study of *Triangles Orthologiques*. Steiner demonstrated that if two triangles ABC, $A'B'C'$ are such that the perpendiculars drawn from A, B, C, respectively, on $B'C'$, $C'A'$, $A'B'$ are concurrent, then, reciprocally, the perpendiculars drawn from A', B', C', on BC, CA, AB, respectively, are concurrent. Lemoine calls these triangles *orthologiques* and makes them the basis of a theory developed in several memoirs, notably in one presented at the Congrès de Limoges in 1890. He has also published three papers on the application of geometry to the calculus of probabilities, in the Bulletin de la Société Mathématique (1883), the Nouvelles Annales (1884), and the proceedings of the Congrès de Grenoble (1885). And finally, there should be mentioned a memoir presented at the Congrès de Nantes in 1875, entitled *Étude systématique du tétraèdre equifacial* (in which the four faces have equal area.)

But in some respects the crowning labor of M. Lemoine is the creation of *L'Intermédiaire des Mathématiciens*, the details of which should be told as a matter of historic interest, especially as they have not heretofore appeared. This publication, although still in its infancy, is known throughout the mathematical world. It consists simply of questions and answers, questions which one asks for information and not for the mere pleasure of displaying some puzzle, questions which bring one into a kind of personal relation to his co-workers whether they be in Russia or South Africa. The idea of the journal is purely M. Lemoine's, and for some time it had been in his mind, but unhappily with no thought of its realization, until the genial influence of a quiet dinner and some good cigars brought about its fruition. M. Laisant had long been a friend of Lemoine's, and it was no uncommon thing for the former to dine with the latter at his home in Rue Littré. On such an occasion, in March, 1893, as they were enjoying a quiet smoke after dinner, the talk ran as usual into mathematics, and Lemoine suggested the idea of the journal. Laisant at once saw the value of the scheme and urged his friend to join him in carrying it out. M. Lemoine replied that it seemed impossible both because he was much occupied with other matters, and because of ill health (from which, unhappily, he is still suffering). Nevertheless, M. Laisant was so persuasive and the influence of the dinner and the cigars so happy that before they separated the project had taken such form that the very next day it was laid before their friend Gauthier-Villars, the great mathematical publisher, and the journal was ushered into being. "Before dinner, nothing could have persuaded me," M. Lemoine writes, "that this idea which I had formed for others would ever be realized by me ; after dinner, the journal was a possibility ; the next day, it was an accomplished fact." Its publication began in January, 1894, and each editor serves during six months of the year.

As one surveys the labors of Lemoine it would seem, from present appearances, that his most valuable work is the foundation of *L'Intermédiaire*, a publication which bids fair to continue for generations because it is really needed. His most original mathematical work seems to be his "geometrography,"—purely a creation of his own, and a contribution which enters into the mathematical work of the military schools of Brussels and Turin, the polytechnic schools of Zurich and Milan, and more or less in many other places. The work which will bring his name to the most readers is his study of the modern geometry of the triangle. In general it may be said that his contribution to geometry has been the very valuable work of showing that the synthetic field is by no means exhausted ; that Euclid left something for this generation to accomplish ; and that an original mind can find abundant material in even so simple a figure as the simplest polygon. How suggestive is this of the vast field which awaits investigators of the more complex geometric figures !

This sketch should not close without a brief reference to the influence that M. Lemoine has exerted in the realm of music. The soirées of M. and Mme. Lemoine are justly celebrated, and each week of the winter sees an assemblage representing the *anciens élevées* of the École Polytechnique, the École Normale, the Marine, and in general a good part of the scientific, literary, and artistic circles of Paris, to listen to a musical programme as original as the mathematical labors of the host. These soirées have exerted a great influence in a musical way, the type which they have fixed being adopted by many societies in and about Paris. One amusing feature of these meetings is the name which designates them. If the writer may be pardoned a personal allusion, he once attended an examination in the École Polytechnique by M. Hermann Laurent. It was one of the most severe he had ever seen,—an exceptionally bright young man submitted to an oral examination that would certainly have floored most American professors,—the examiner, a dyspeptic looking man as cold and as keen as steel and apparently as unsympathetic as ice, though in reality oné of the most genial of men. To this justly celebrated mathematician, M. Laurent, is due the name of M. Lemoine's soirées, "La Trompette." Long ago he one day remarked to M. Lemoine in a jesting way, as the latter was excusing himself to attend one of his musical reunions, "Stay here with me, let the trumpet alone." Struck by the name, Lemoine adopted it, and *La Trompette* has ever since designated the delightful soirées with which the Paris cultured world is familiar.

A final word concerning the modesty of M. Lemoine. He estimates his position exactly. He says that he is not a mathematician. He has no claim to rank with Hermite, Poincaré, Picard, Painlevé, Appell, Jordan, Bertrand, Tannery, Darboux, or any of that famous circle which is making Paris such a center of study in the fields of higher modern mathematics. But all mathematicians feel that he has done a noteworthy work in other lines, and for this his name will be known and prominently known in the history of mathematics.

Ypsilanti, Michigan, March, 1896.

WHERE MATHEMATICIANS ARE NEEDED.

By ERIC DOOLITTLE, A. M., Chicago, Illinois.

There is no study of which the conceptions are more grand, nor of which the theorems are more comprehensive and profound than the study of Physical Astronomy. There is no study affording an application of Pure Mathematics in which the perfect harmony of its various parts is more evident; none in which reason plays a greater part nor approximation a less one. The beauty and simplicity of its first propositions richly reward the early attention of the student, and in the end he is led to the wonderful theorems of *La Place* on the stability of the solar system and the conditions of its formation; theorems which *Baron Fourier* has justly named the highest which the human intelligence can propose.

It is remarkable that more young mathematicians do not enter this absorbing field. The common impression that it requires an unusual mathematical training is largely erroneous. Such a thorough knowledge of Calculus and Mechanics as is shown by the many contributors of the MONTHLY is fully sufficient. Physical Astronomy demands patient and steadfast work; mere brilliance and versatility can accomplish no more of fundamental importance here than in any other true science.

I would urge upon those who are now fitted to enter this or other like work, the great necessity of concentrating their energies upon it. It should be the one object of every devoted student to perfect and advance his own science. It is to this that his whole work must be directed. To such an one years of fragmentary study, first on one subject and then on another, are utterly wasted.

It is the disastrous mistake of many students that they do not realize how soon study for mere amusement or culture should give place to something higher. They fear, often mistakenly, that they are incapable of beginning work of real importance: instead of arranging then a definite series of studies to prepare themselves, they continue to dissipate their strength and accomplish nothing.

Physical Astronomy is calling in many directions for original work. In this country it is comparatively neglected. There are many who are being attracted by the pleasures of Photography and Spectroscopy, but there are few who realize the field which the Fundamental Astronomy opens to them. It contains many problems of the deepest interest. It is filled with questions whose answer requires, not an expensive observatory, but rather mathematical patience or skill.

Readers of the MONTHLY who are determined to accomplish something may well devote themselves to this science. The certainty of their adding to the sum of human knowledge is here greater than in Pure Mathematics, the reward of faithful work unaccompanied by special genius far more certain. The explanation of the variable stars, of the cause and nature of the sun's peculiar rotation, more complete theories of the satellites and of the figures and attractions of the Heavenly Bodies, the determination of the perturbations of the asteroids and other planets and the causes of the anomalies which occur, and the able discussion of a multitude of observations relating to these and other problems are a very few of the many directions in which original work is needed.

As with any true science, Physical Astronomy requires from those who enter upon it long and patient devotion. Its rewards are not bestowed by chance, nor are they on that account of less value. It is of little popular interest. Its discoveries are seldom sensational. But its dignity and importance cannot be over-estimated. Of American Astronomers, the names of Hill and Newcomb will go down through the ages: their researches will never lose their importance. And whoever adds to this science is contributing to a knowledge which shall endure forever.

Baron Fourier said of La Place:

"Your successors, gentlemen, will witness the accomplishment of the great phenomena whose laws he discovered. They will observe in the motions of the Moon the changes which he predicted and of which he alone was able to assign the cause. The continued observation of Jupiter's satellites will perpetuate the memory of the inventor of the laws which govern them in their courses. The great inequality of Jupiter and Saturn, running through their long periods, and giving to these bodies new situations will recall without ceasing one of his most astonishing discoveries. These are the titles of a true glory which nothing can extinguish. The spectacle of the heavens will be changed, but at those remote epochs the glory of the inventor will continue forever; the traces of his genius bear the seal of immortality."

Chicago University, February 9th, 1896.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from January Number.]

PROPOSITION XXII. *If two straights AB, CD existing in the same plane stand perpendicular to a certain straight BD; but AC joining these perpendiculars makes with them internal acute angles (in hypothesis of acute angle): I say (Fig. 26) the terminated straights AC, BD have a common perpendicular, and indeed within the limits fixed by the designated points A and C.*

Fig. 26.

Proof. For if AB, CD are equal, it follows (from P. II) that the straight LK, by which these two AC and BD are bisected, will be to them a common perpendicular. But if either be the greater, as suppose AB; let fall to BD (according to Eu. I. 12) from any point L of AC the perpendicular LK, meeting the other BD in K. But it will meet it in some point K existing between the points B and D; otherwise (contrary to Eu. I. 17) the perpendicular LK would cut either AB, or CD, perpendicular to the same BD. So if the angles at the point L are not right, one of them will be acute and the other obtuse. Let the obtuse be toward the point C. But now LK is understood so to proceed toward AB, that it always stands at right angles to BD, and likewise opportunely increased, or diminished, in some point of it cuts the straight AC. It follows that the angles at the intersection points with AC cannot all be obtuse toward the parts of the point C, lest at length in that point A, where the straight LK is congruent with the straight AB, the angle at the point A toward the parts of the point C should be obtuse, when toward these parts it is by hypothesis acute. Since therefore the angle at the point L of this LK is by hypothesis obtuse toward the parts of the point C, the straight LK will not change over in this motion so as to make in some point of it with the straight AC an angle acute toward the parts of the aforesaid point C, unless, before, it changes over so as to make in some point of it with this AC an angle right towards the parts of this same point C. Therefore between the points A, and L will be some one intermediate point H, in which HK perpendicular to this BD is also perpendicular to to the other AC.

In a similar manner is shown to be present a certain XK between LK, CD, which is perpendicular both to the straight BD, and to the straight AC, if namely an angle at the point L is assumed to be obtuse toward the parts of the point A.

It follows therefore that the strights AC, BD will have a common perpendicular, and indeed within the limits fixed by the designated points A, and C, when the joins AB, CD exist in the same plane and are perpendicular to BD.

Quod erat, etc.

[To be Continued.]

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from January Number.]

(2) For each generating substitution s_a in a transitive group of degree m_1 find a substitution s_β which (a) interchanges the systems in the same way as s_a interchanges its elements, (b) has its k^{th} power in G_1 where k is the *order* of s_a, *i. e.* the lowest positive value of x which satisfies the equation $s_a^x=1$, and (c) if s_β is

e first substitution which corresponds to a generating substitution in the group degree m_1, s_β needs only to transform G_1 into itself; otherwise s_β must transform the group already found in the same way as s_α transforms the corresponding art of its group. Continue until all the generating substitutions s_α have een used. We will thus obtain a non-primitive group.

(3) Determine whether the non-primitive group just found is different om each one of those already in the list.

The relation which exists between the required non-primitive group G and e given group of degree m_1 G^1 is called a g_1, 1 *isomorphism*, or a g_1, 1 *correspondence*. The problem of constructing all the non-primitive groups of degree n as its more difficult elements in common with the problem of establishing an a, correspondence between two groups as may at once be inferred from the given elation. We shall not pursue this subject for the present since only its most evident principles need to be employed when n is small.

To this development of the elementary methods pursued in the construction of non-primitive groups we will add a proof of the general theorem to which we referred in a foot-note. For the sake of simplicity we shall not give the theorem in its most general form.

Theorem. Given that the number of the systems of non-primitivity is n and that the group which does not interchange the systems G_1 is the product of n conjugate transitive groups of which one is found in each system, then there is only one non-primitive group based upon the given G_1 and isomorphic to a transitive group of n elements which is generated by a single substitution.

There certainly is one such group for we may chose s_β so that it will simply permute the systems in the same way as s_α permutes its elements and will have the same order as s_α. Since s_β simply permutes the systems, *i. e.* it permutes the systems as units without permuting the elements of the systems, it must also transform G_1 into itself. · Hence G_1 and s_β generate a non-primitive group whenever G_1 differs from identity.

Let t_1^1, t_2^2, ... t_n^n(the upper index standing for the systems in the same order as they are represented by the letters of s_α and the lower index for the particular substitution in the system) be any substitution in the n systems which transform the n constituents of G_1 into themselves. Then will

$$t_1^1\, t_2^2 \,\ldots\ldots t_n^n s_\beta$$

be a symbol for all the substitutions whose degree $\overline{\gtrless}$ the degree of the required group which transform G_1 into itself and permute the systems in the same way as s_α permutes its elements. If this general substitution satisfies the other condition which must be satisfied if it, with the given G_1, generates a non-primitive group we have

$$(t_1^1\, t_2^2 \,\ldots\ldots t_n^n s_\beta\,)^K = \text{some substitution in } G_1,$$

where K is the smallest positive value of x in the equation

$$s^x_\alpha=1.$$

Since $(t^1_1\, t^2_2 \ldots\ldots t^n_n s_\beta\,)^K = t^1_1\, t^1_2 \ldots\ldots t^1_n\, t^2_2\, t^2_3 \ldots\ldots t^2_1 \ldots\ldots t^n_n t^n_1 \ldots\ldots t^n_{n-1}$

we know that $t^1_1\, t^2_2 \ldots\ldots t^n_n s_\beta$ may be multiplied by some substitution of G_1 so as to give for the new t's

$$t^1_1\, t^2_2 \ldots\ldots t^n_n=1 \ldots\ldots\ldots\ldots\ldots\ldots (A)$$

Consider now the equations

$$(K^1_1\, K^2_2 \ldots\ldots K^n_n)^{-1} s_\beta\, K^1_1\, K^2_2 \ldots\ldots K^{n*}_n=$$

$$\overset{-1}{K}{}^n_n K^n_1\, \overset{-1}{K}{}^{n-1}_{n-1} K^{n-1}_n \ldots\ldots \overset{-1}{K}{}^1_1\, K^1_2\, s_\beta = t^1_1\, t^2_2 \ldots\ldots t^n_n s_\beta\,.$$

We see directly that the following is a solution of the last equation if (A) is satisfied :

$$K^1_1=1,\ K^2_2=t^2_1\,,\ K^3_3=t^3_1\, t^3_2\,, \ldots\ldots K^n_n=t^n_1\, t^n_2 \ldots\ldots t^n_{n-1}.$$

Hence all the possible groups are conjugate to the one already given and our theorem is proved. This theorem may be employed with respect to the first subgroups as well as with respect to the entire groups.

In our next paper we shall consider the construction of the third and last class of groups, viz: the *primitive* groups.

[To be Continued.]

ON AN INTERESTING SYSTEM OF QUADRATIC EQUATIONS.

By DR. E. H. MOORE, University of Chicago, and EMMA C. ACKERMANN, Michigan State Normal School.

In C. Smith's Algebra, fourth edition, p. 134, are given for solution, examples 61, 62, 63, which are as follows (the third with a slight modification):

61. The roots of the equation $x^2+mx+m^2+a=0$ are $x_1,\ x_2$; show that $x_1{}^2+x_1x_2+x_2{}^2+a=0$.

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*This equation follows from the simpler one

$$(ts)^{-1}=s^{-1}t^{-1}$$

and this is true because if we multiply both members by ts we obtain an identity.

62. The roots of the equation $(x^2+1)(a^2+1)-max(ax-1)=0$ are x_1, x_2; show that $(x_1^2+1)(x_2^2+1)-mx_1x_2(x_1x_2-1)=0$.

63. The roots of the equation $a(x^2+mx+m^2)+bm^2x^2=0$ are x_1, x_2; show that $a(x_1^2+x_1x_2+x_2^2)+bx_1^2x_2^2=0$.

The equations possess the following properties: (1), the equation is of the second degree in the variable x and the constant a; (2), the roots x_1, x_2 of the equation are related to each other exactly as are the variable x and constant a.

We seek to generalize these theorems and formulate this problem:

To determine all quadratic equations of the form

$$f(\overset{2}{\breve{x}}, \overset{2}{\breve{m}})=0,$$

where the function $f(\overset{2}{\breve{x}}, \overset{2}{\breve{m}})$ is a symmetric function $f(\overset{2}{\breve{x}}, \overset{2}{\breve{m}})\equiv f(\overset{2}{\breve{m}}, \overset{2}{\breve{x}})$ of its two arguments x and m of the second degree in each of them, characterized by the property that between the two roots x_1, x_2 which are functions of m the relation

$$f(\overset{2}{\breve{x}}_1, \overset{2}{\breve{x}}_2)\equiv 0$$

holds as an identity in m.

I. Let $f(\overset{2}{\breve{x}}, \overset{2}{\breve{m}})\equiv a+h(m+x)+bmx+g(m^2+x^2)+f(m^2x+x^2m)+cm^2x^2=0$.

II. $\because f(\overset{2}{\breve{x}}_1, \overset{2}{\breve{x}}_2)\equiv 0$, and x_1 and x_2 take the places of x and m, $f(\overset{2}{\breve{x}}_1, \overset{2}{\breve{x}}_2)\equiv a+h(x_1+x_2)+bx_1x_2+g(x_1^2+x_2^2)+f(x_1^2x_2+x_2^2x_1)+cx_1^2x_2^2\equiv 0$.

We are to investigate now the conditions on the parameters a, b, c, f, g, h that must hold in order that $f(x_1, x_2)$ may as a function of m be identically 0. The problem then is not necessarily to prove $f(x_1, x_2)\equiv 0$ for all equations, but to find all equations for which it is true that $f(x_1, x_2)\equiv 0$.

III. Let $Kx^2+Lx+M=0$ be the original equation; x_1 and x_2 the roots; then $-K(x_1+x_2)=L$; $K(x_1x_2)=M$.

Comparing this equation with I:

$$K\equiv g+fm+cm^2.$$

$$L\equiv h+bm+fm^2.$$

$$M\equiv a+hm+gm^2.$$

IV. Transform equation in I to this form:

$$a+h(x+m)+(b-2g)xm+g(x+m)^2+f(xm)(x+m)+c(xm)^2=0.$$

V. Also equation in II to this form:

$$a+h(x_1+x_2)+(b-2g)x_1x_2+g(x_1+x_2)^2+f(x_1x_2)(x_1+x_2)+c(x_1x_2)^2\equiv 0.$$

VI. Multiply V by K^2:

$$aK^2+hK^2(x_1+x_2)+(b-2g)K^2x_1x_2+gK^2(x_1+x_2)^2$$
$$+fK^2(x_1x_2)(x_1+x_2)+cK^2(x_1x_2)^2=0.$$

VII. VI becomes, by substituting for x_1x_2 and x_1+x_2 their values as given in III:

$$aK^2-hKL+(b-2g)KM+gL^2-fLM+cM^2\equiv 0$$

where K, L, M are given in terms of m in III.

Since VII is an identity in m, the coefficients of the different powers of m are each zero; ∴ the condition in VII requires that five polynomials homogeneous in a, b, c, f, g, h of degree three shall be zero. Since there are six letters there are five ratios; ∴ there are five unknowns in five equations. This system of five cubic equations turns out to be extremely simple.

For, in VII, substituting for K, L, M their values involving m as given in III, collecting terms with reference to m, and using detached coefficients, we have

1	m	m^2	m^3	m^4	
ag^2	$2afg$	af^2+2acg	$2acf$	ac^2	$\equiv aK^2$
$-gh^2$	$-(fh^2+bgh)$	$-(ch^2+bfh+fgh)$	$-(bch+f^2h)$	$-cfh$	$\equiv -hKL.$
$ag(b-2g)$	$(af+gh)(b-2g)$	$(ac+fh+g^2)(b-2g)$	$(ch+fg)(b-2g)$	$cg(b-2g)$	$\equiv(b-2g)KM$
gh^2	$2bgh$	$b^2g+2fgh$	$2bfg$	f^2g	$\equiv gL^2$
$-afh$	$-(abf+fh^2)$	$-(af^2+bfh+fgh)$	$-(f^2h+bfg)$	$-f^2g$	$\equiv -fLM.$
a^2c	$2ach$	ch^2+2acg	$2cgh$	cg^2	$\equiv cM^2.$

Simplifying and letting c_0, c_1 ...... be coefficient of m^0, m^1 ......

$$c_0\equiv a\{(b-g)g+ac-fh\}=0.$$

$$c_1\equiv 2h\{(b-g)g+ac-fh\}=0.$$

$$c_2\equiv(b+2g)\{(b-g)g+ac-fh\}=0.$$

$$c_3\equiv 2f\{(b-g)g+ac-fh\}=0.$$

$$c_4\equiv c\{(b-g)g+ac-fh\}=0.$$

This means that given $f(x, m)=0$ as in I, then $f(x_1, x_2)\equiv 0$, if, and only if either $a=2h=b+2g=2f=c=0$, or $(b-g)g+ac-fh=0$; the second alternative is *one* condition, homogeneous, of degree two, between the six homogeneous parameters. Therefore, *All quadratic equations of the form*

$$a+h(m+x)+bmx+g(m^2+x^2)+f(m^2x+x^2m)+cm^2x^2=0$$

(in which the first member is a symmetric function $f(\overset{2}{\breve{x}}, \overset{2}{\breve{m}})\equiv f(\overset{2}{\breve{m}}, \overset{2}{\breve{x}})$ *of its two arguments* x *and* m *of the second degree in each of them), whose parameters are related by the equation*

$$(b-g)g+ac-fh=0,$$

—and, apart from the relatively trivial equation

$$g(x^2-2mx+m^2)=0,$$

only those equations whose parameters are so related—are characterized by the property that between the two roots x_1, x_2 *which are functions of* m *the relation*

$$f(\overset{2}{\breve{x}}_1, \overset{2}{\breve{x}}_2)\equiv 0$$

holds as an identity in m.

November 26, 1895.

QUADRATURE OF THE CIRCLE.

By WILLIAM E. HEAL, A. M., Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

The problem of the quadrature of the circle, or what amounts to the same thing, drawing a straight line equal in length to the circumference of a given circle, occupied the attention of mathematicians at a very early date. Long before the time of Archimedes, geometers had attacked the problem with but one result: failure. And for more than twenty centuries mathematicians have been struggling with the problem. Many claimed to have solved it, but their analysis has been, in every case, found to be fatally defective. After centuries of attempt and failure mathematicians began to suspect that the problem might not admit of solution. James Gregory was the first to attempt a proof of the impossibility of the quadrature of the circle. In the opinion of Montucla he succeeded; but later mathematicians have not so decided. Not a score of years have passed since a rigid proof was given that the solution of the problem is really impossible under the conditions usually understood: that is, by the use of the rule and compass only.

It is well known from the geometry that the ratio of the circumference to the diameter of a circle is constant. This constant ratio is usually denoted by the Greek letter π, and it follows at once that if π is a number commensurable with unity that it can be constructed geometrically, and the problem is solved. Lambert, in a memoir presented to the Berlin academy in 1761, was the first to prove that π is incommensurable. Other proofs of this result have been given, especially by Hermite in Crelle's Journal, Vol. 76, which demonstration is reproduced in the Traite de Geometrie of Rouche and Comberousse, 4th edition. But this result, however interesting in itself, does not prove the impossibility of a geometrical construction of π. For example, the square root of 2 (or any non-quadrate number) is incommensurable but is easily constructed geometrically. The first real advance towards the solution of the problem was made by Hermite in 1873. Hermite succeeded in proving that the number e, the base of the Naperian system of logarithms is not only incommensurable but that it can not be a root of a rational algebraic equation of any degree whatever. Such a number is called transcendent. If the number π could be proved to be transcendent the vexed question of the quadrature of the circle would be settled once for all. For this problem requires to derive the number π by a finite number of elementary geometrical constructions. As two straight lines, or a straight line and a circle, or two circles, have not more than two intersections, these processes, or any finite combination of them, can be expressed algebraically in a comparatively simple form ; so that the solution of the problem of the quadrature of the circle would mean that π can be expressed as the root of an algebraic equation solvable by square roots. Hermite did not succeed in proving that π is a transcendent number, but in 1882 Lindemann extended Hermite's proof to include the number π as well as e among the transcendent numbers. Hermite and Lindemann's methods are complicated and obscure and many mathematicians attempted to simplify them. But not until very recently were these attempts rewarded with any degree of success. In January, 1893, Hilbert published a proof of the transcendency of e and π that reduces the problem to such simple terms as to be understood by mathematicians having only a moderate understanding of the principles of the calculus. Hilbert's proof depends upon certain properties of the definite integral

$$\int_0^\infty z^\rho\Big[(z-1)(z-2)(z-3)\ldots\ldots(z-n)\Big]^{\rho+1}e^{-z}dz,$$

suggested by the investigations of Hermite.

Immediately after the publication of Hilbert's proof, Hurwitz published a proof for the transcendency of e based on still more elementary principles. And finally, in May, 1893, Gordan published a proof of the transcendency of e and π in which only the known development of e^x in powers of x is made use of. This last proof is so simple that it should be introduced into university teaching everywhere. The numbers e and π are very intimately related, and before proceeding to Gordan's proof of the transcendency of these two fundamental numbers I wish to give here a well known proof that e is incommensurable. We have

$$e=1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\ldots\ldots\frac{1}{\lfloor r}+\frac{1}{\lfloor r+1}\ldots\ldots$$

Assume, now, that e is a rational number $\frac{a}{r}$, where a and r are integers, and the fraction $\frac{a}{r}$ is in its lowest terms. Multiply this equation by $\lfloor r$ and we see that all the terms preceding the term $\frac{1}{\lfloor r+1}$ are integers. The series

$$\frac{1}{r+1}+\frac{1}{(r+1)(r+2)}+\frac{1}{(r+1)(r+2)(r+3)}\ldots\ldots \text{ is less than}$$

$$\frac{1}{r+1}+\frac{1}{(r+1)^2}+\frac{1}{(r+1)^3}\ldots\ldots$$

That is less than $\frac{1}{r}$. Thus we have an integer equal to a proper fraction which is impossible. I will now give Gordan's proof of the transcendency of e and π. The proof for e will be seen to be an extension of the above well-known proof of the irrationality of e and apparently should have been discovered long ago.

The function e^x is defined by the series

$$e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}\ldots\ldots$$

This, if we introduce the symbolic notation

$$\lfloor r=h^r$$

and multiply by this quantity and any whole number c_r, passes into the form

(1) $$c_rh^re^x=c_r(x+h)^r+c_rx^ru_r$$

in which $$u_r=\frac{x}{r+1}+\frac{x^2}{(r+1)(r+2)}\ldots\ldots$$

If $$\mu=\text{mod.}x$$

we have $$\text{mod.}u_r<e^{\mu};$$

and if we put $$u_r=q_re^{\mu},$$

$$\text{mod.}q_r<1.$$

From (1) it follows:

$$c_r h^r e^x = c_r (x+h)^r + c_r x^r q_r e^{\mu}.$$

And
$$e^x \sum_{r=0}^{r=s} c_r h^r = \sum_{r=0}^{r=s} c_r (x+h)^r + e^{\mu} \sum_{r=0}^{r=s} c_r q_r x^r.$$

And if we put
$$\sum_{r=0}^{r=s} c_r\, x^r = \phi(x),\ \sum_{r=0}^{r=s} c_r\, q_r\, x^r = \psi(x);$$

(2)
$$e^x \phi(h) = \phi(x+h) + e^{\mu} \psi(x).$$

If, now, there is an equation with integral coefficients, satisfied b number e:

$$\sum_{k=0}^{k=n} c_k e^k = 0,$$

then from (2) we have

(3)
$$0 = \sum_{k=0}^{k=n} c_k \phi(k+h) + \sum_{k=0}^{k=n} c_k \psi(k) e^k.$$

If we choose for ϕ the function

$$\phi(x) = \frac{x^{p-1}}{\underline{|p-1}} [(x-1)(x-2 \ldots\ldots (x-n)]^p$$

and for ρ a prime number greater than the numbers n and c_0, then $\phi(k+h)$ in formula (3) become whole numbers.

$$\phi(h+1),\ \phi(h+2) \ldots\ldots \phi(h+n)$$

have the factor p, but

$$c_0 \phi(h)$$

has not. If we let p increase, then ϕ and ψ become as small as we please formula (3) is impossible, and the number e transcendent.

If $i\pi$ is a root of an equation with integral coefficients:

(4)
$$c(x-w_1)(x-w_2) \ldots\ldots (x-w_\rho) = 0,$$

then we have the formula

(5)
$$(1+e^{w_1})(1+e^{w_2}) \ldots\ldots (1+e^{w_\rho}) = 0.$$

If $c-1$ vanishing quantities are found among the sums

$$w_i\ ;\ w_i+w_k\ ;\ w_i+w_k+w_\lambda \ldots\ldots$$

and we designate those remaining by

$$a_1,\ a_2,\ a_3 \ldots\ldots a_n$$

and their moduli by

$$a_1,\ a_2,\ a_3 \ldots\ldots a_n,$$

formula (5) becomes

$$(6) \qquad 0=c+\sum_{k=1}^{k=n} e^{a_k}.$$

The symmetric functions of cw_k, as well as those of ca_k, are whole numbers. By formula (2) we have

$$(7) \qquad 0=c\phi(h)+\sum_{k=1}^{k=n}\phi(a_k+h)+\sum_{k=1}^{k=n}e^{a_k}\psi(a_k).$$

Let $\phi(x)=\dfrac{(cx)^{p-1}}{\underline{|p-1}}c^{np}[(x-a_1)(x-a_2)\ldots\ldots(x-a_n)]^p,$

and let p be a prime number greater than the numbers

$$c\ ;\ n\ ;\ c\ ;\ c^n a_1 a_2 \ldots\ldots\ a_n.$$

The quantities $\phi(h)$ and $\sum_{k=1}^{k=n}\phi(a_k+h)$ are whole numbers :

$$\sum_{k=1}^{k=n}\phi(a_k+h)$$

contains the factor p, but $c\phi(h)$ does not.

If p increases the moduli of ϕ and ψ become as small as we please.

Formula (7) is impossible, and therefore π is a transcendent number.

THE CENTROID OF AREAS AND VOLUMES.

By G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

It is the object of this paper to put on record, once for all, general values for the centroid of areas, represented by the curve $\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}=1$, and the centroid of volumes represented by the surface

$$\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}+\left(\frac{z}{c}\right)^{\frac{2}{2p+1}}=1.$$

I. *Areas.* Let the density vary as $x^{k-1}y^{l-1}$, the thickness being constant.

$$\text{Then } \bar{x}=\frac{\int\int x^k y^{l-1}dxdy}{\int\int x^{k-1}y^{l-1}dxdy},\quad \bar{y}=\frac{\int\int x^{k-1}y^l dxdy}{\int\int x^{k-1}y^{l-1}dxdy}.$$

$$\therefore\ \bar{x}=\frac{\dfrac{a^{k+1}b^l}{\frac{4}{(2m+1)(2n+1)}}\ \dfrac{\Gamma\left\{\frac{k+1}{2}(2m+1)\right\}\ \Gamma\left\{\frac{l}{2}(2n+1)\right\}}{\Gamma\left\{\frac{k+1}{2}(2m+1)+\frac{l}{2}(2n+1)+1\right\}}}{\dfrac{a^k b^l}{\frac{4}{(2m+1)(2n+1)}}\ \dfrac{\Gamma\left\{\frac{k}{2}(2m+1)\right\}\ \Gamma\left\{\frac{l}{2}(2n+1)\right\}}{\Gamma\left\{\frac{k}{2}(2m+1)+\frac{l}{2}(2n+1)+1\right\}}}.$$

$$\therefore\ \bar{x}=\frac{\Gamma(km+m+\frac{k+1}{2})\ \Gamma(km+ln+\frac{k+l}{2}+1)}{\Gamma(km+\frac{k}{2})\Gamma(km+ln+m+\frac{k+l+1}{2}+1)}a\ldots\ldots\ldots\ldots(A).$$

$$\text{Similarly, } \bar{y}=\frac{\Gamma(ln+n+\frac{l+1}{2})\Gamma(km+ln+\frac{k+l}{2}+1)}{\Gamma(ln+\frac{l}{2})\Gamma(km+ln+n+\frac{k+l+1}{2}+1)}b\ldots\ldots\ldots\ldots\ldots\ldots(B).$$

This gives the centroid of a quadrant of the area whatever be the values of k, l, m, n. Let $k=l=1$, so that the density is the same throughout the whole area.

$$\therefore\ \bar{x}=\frac{\Gamma(2m+1)\Gamma(m+n+2)}{\Gamma(m+\frac{1}{2})\Gamma(2m+n+\frac{5}{2})}a,\quad \bar{y}=\frac{\Gamma(2n+1)\Gamma(m+n+2)}{\Gamma(n+\frac{1}{2})\Gamma(m+2n+\frac{5}{2})}b.$$

Let $m=n=0$, then $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(1)\Gamma(2)}{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}a=\frac{4a}{3\pi},\quad \bar{y}=\frac{\Gamma(1)\Gamma(2)}{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}b=\frac{4b}{3\pi}.$$

Let $m=n=1$, then $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(3)\Gamma(4)}{\Gamma(\frac{3}{2})\Gamma(\frac{11}{2})}a=\frac{256a}{315\pi},\quad \bar{y}=\frac{\Gamma(3)\Gamma(4)}{\Gamma(\frac{3}{2})\Gamma(\frac{11}{2})}b=\frac{256b}{315\pi}.$$

Let $m=n=2$, then $\left(\frac{x}{a}\right)^{\frac{2}{5}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(5)\Gamma(6)}{\Gamma(\frac{5}{2})\Gamma(\frac{17}{2})}a=\frac{2.4.8.12.16.20}{3.5.7.9.11.13.15}\cdot\frac{4a}{\pi},$$

$$\bar{y}=\frac{\Gamma(5)\Gamma(6)}{\Gamma(\frac{5}{2})\Gamma(\frac{17}{2})}b=\frac{2.4.8.12.16.20}{3.5.7.9.11.13.15}\cdot\frac{4b}{\pi}.$$

Let $m=0$, $n=1$, then $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(1)\Gamma(3)}{\Gamma(\frac{1}{2})\Gamma(\frac{7}{2})}a=\frac{16a}{15\pi},\quad \bar{y}=\frac{\Gamma(3)\Gamma(3)}{\Gamma(\frac{3}{2})\Gamma(\frac{9}{2})}b=\frac{128b}{105\pi}.$$

Let $m=n=\frac{3}{2}$, then $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(4)\Gamma(5)}{\Gamma(2)\Gamma(7)}a=\frac{a}{5},\quad \bar{y}=\frac{\Gamma(4)\Gamma(5)}{\Gamma(2)\Gamma(7)}b=\frac{b}{5},$$ the centroid of the area between the parabola and its tangents as axes.

Let the density vary as xy, so that $k=l=2$.

$$\therefore\ \bar{x}=\frac{\Gamma(3m+\frac{3}{2})\Gamma(2m+2n+3)}{\Gamma(2m+1)\Gamma(3m+2n+\frac{7}{2})}a,\quad \bar{y}=\frac{\Gamma(3n+\frac{3}{2})\Gamma(2m+2n+3)}{\Gamma(2n+1)\Gamma(2m+3n+\frac{7}{2})}b.$$

Let $m=n=0$, then $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1.$

$$\therefore \bar{x}=\frac{\Gamma(\frac{3}{2})\Gamma(3)}{\Gamma(1)\Gamma(\frac{7}{2})}a=\frac{8a}{15},\ \bar{y}=\frac{\Gamma(\frac{3}{2})\Gamma(3)}{\Gamma(1)\Gamma(\frac{7}{2})}b=\frac{8b}{15}.$$

Let $m=n=1$, then $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$

$$\therefore \bar{x}=\frac{\Gamma(\frac{9}{2})\Gamma(7)}{\Gamma(3)\Gamma(\frac{17}{2})}a=\frac{128a}{429},\ \bar{y}=\frac{\Gamma(\frac{9}{2})\Gamma(7)}{\Gamma(3)\Gamma(\frac{17}{2})}b=\frac{128b}{429}.$$

Let $m=n=\frac{3}{2}$, then $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1.$

$$\therefore \bar{x}=\frac{\Gamma(6)\Gamma(9)}{\Gamma(4)\Gamma(11)}a=\frac{2a}{9},\ \bar{y}=\frac{\Gamma(6)\Gamma(9)}{\Gamma(4)\Gamma(11)}b=\frac{2b}{9}.$$

Let the density vary as x the distance from the axis of ordinates so th $k=2$, $l=1$.

$$\therefore \bar{x}=\frac{\Gamma(3m+\frac{3}{2})\Gamma(2m+n+\frac{5}{2})}{\Gamma(2m+1)\Gamma(3m+n+3)}a,\ \bar{y}=\frac{\Gamma(2n+1)\Gamma(2m+n+\frac{5}{2})}{\Gamma(n+\frac{1}{2})\Gamma(2m+2n+3)}b.$$

Let $m=n=0$, then $\left(\frac{x}{a}\right)^{2}+\left(\frac{y}{b}\right)^{2}=1.$

$$\therefore \bar{x}=\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(1)\Gamma(3)}a=\frac{3\pi a}{16},\ \bar{y}=\frac{\Gamma(1)\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})\Gamma(3)}b=\frac{3b}{8}.$$

Let $m=n=1$, then $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$

$$\therefore \bar{x}=\frac{\Gamma(\frac{9}{2})\Gamma(\frac{11}{2})}{\Gamma(3)\Gamma(7)}a=\frac{49.45\pi a}{2^{14}},\ \bar{y}=\frac{\Gamma(3)\Gamma(\frac{11}{2})}{\Gamma(\frac{3}{2})\Gamma(7)}b=\frac{63b}{384}.$$

Let $m=n=\frac{3}{2}$, then $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1.$

$$\therefore \bar{x}=\frac{\Gamma(6)\Gamma(7)}{\Gamma(4)\Gamma(9)}a=\frac{5a}{14},\ \bar{y}=\frac{\Gamma(4)\Gamma(7)}{\Gamma(2)\Gamma(9)}b=\frac{3b}{28}.$$

Let the density vary as y the distance from the axis of abscissas so that $k=1,\ l=2$.

$$\therefore\ \bar{x}=\frac{\Gamma(2m+1)\Gamma(m+2n+\frac{5}{2})}{\Gamma(m+\frac{1}{2})\Gamma(2m+2n+3)}a,\quad \bar{y}=\frac{\Gamma(3n+\frac{3}{2})\Gamma(m+2n+\frac{5}{2})}{\Gamma(2n+1)\Gamma(m+3n+3)}b.$$

Let $m=n=0$, then $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(1)\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})\Gamma(3)}a=\frac{3a}{8},\quad \bar{y}=\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(1)\Gamma(3)}b=\frac{3\pi b}{16}.$$

Let $m=n=1$, then $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(3)\Gamma(\frac{11}{2})}{\Gamma(\frac{3}{2})\Gamma(7)}a=\frac{63a}{384},\quad \bar{y}=\frac{\Gamma(\frac{9}{2})\Gamma(\frac{11}{2})}{\Gamma(3)\Gamma(7)}b=\frac{49.45\pi b}{2^{14}}.$$

Let $m=n=\frac{1}{2}$, then $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(4)\Gamma(7)}{\Gamma(2)\Gamma(9)}a=\frac{3a}{28},\quad \bar{y}=\frac{\Gamma(6)\Gamma(7)}{\Gamma(4)\Gamma(9)}b=\frac{5b}{14}.$$

[To be Continued.]

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

NOTE on the Solution of Problem 53, by J. M. COLAW, A. M., Principal of High School, Monterey, Va.

As originally proposed the problem read "with 7% *annual interest* from date," while, it would seem by inadvertence, as reproduced in the November number, it reads "with *interest at 7 per cent.* from date."

I do not find the subject of "Partial Payments on Notes *with Annual Interest*" treated in any of our Arithmetics, except in Olney's *The Science of Arithmetic*, but there are doubtless other exceptions.

On page 191 of *Science of Arithmetic* it is stated that when partial payments are made on notes which bear *Annual Interest*, at other times than those at which the annual interest falls due, the method *usually adopted* is as follows:

Find the interest on the note for 1 year; and find also the amount of the payments made during the year from the times they were severally made to the end of the year.

If the payments amount to more than the interest due, take *their amount* from the amount of the note, and make the remainder a new principal.

But if the amount of the payments does not equal the interest due, the principal remains unchanged, and the amount of the payments is taken from the interest, the remainder being treated as deferred interest.

Proceed in this manner with each year till the time of settlement, the last period being that from the time the last annual interest fell due to the time of settlement.

Mr. Wilke's solution does not follow in all points the rule here laid down as *the usual one.* The question is, what is the rule in Ohio where the note was drawn?

55. Proposed by J. C. CORBIN, Pine Bluff, Arkansas.

How long will it take to count a million, in the following manner: the counter is to pronounce each syllable in the names of the successive numbers at the rate of one per second?

Solution by B. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio.

One, *two*, ..., *nine*—10 syllables—of the first order, are each pronounced 9 times in every hundred.

$\therefore$ The total for these is $9\times10\times10000=$ 900000.

The same, of the fourth order, are each pronounced 9000 times in every hundred thousand.

∴. The total for these is $9000 \times 10 \times 10 =$ 900000.

Ten, *eleven*, ..., *nineteen*—20 syllables—of the first and second orders, are each pronounced once in every hundred.

∴. The total for these is $20 \times 10000 =$ 200000.

The same, of the fourth and fifth orders, are each pronounced 1000 times in every hundred thousand.

∴. The total for these is $10 \times 20 \times 10000 =$ 200000.

Twenty, *thirty*, ..., *ninety*—17 syllables—of the second order, are each pronounced 10 times in every hundred.

∴. The total for these is $10 \times 17 \times 10000 =$ 1700000.

The same, of the fifth order, are each pronounced 10000 times in every hundred thousand.

∴. The total for these is $10 \times 17 \times 10000 =$ 1700000.

One hundred, *two hundred*, ..., *nine hundred*—28 syllables—of the third order, are each pronounced 100 times in every thousand.

∴. The total for these is $28 \times 100 \times 1000 =$ 2800000.

The same, for the sixth order, are each pronounced 100000 times.

∴. The total for these is $28 \times 100000 =$ 2800000.

Thousand is pronounced 999000 times.

∴. The total for this word is 1998000.

The number of syllables in *one million* is 3.

The grand total is 13198003.

∴. 13198003 seconds=152 days, 18 hours, 6 minutes, 43 seconds, the time required.

[Chas. C. Crose, New Windsor, Maryland, sent in a solution of problem 49. The solution is by Algebra and is very good, but as the space in the MONTHLY is very limited even for unsolved problems, we reluctantly omit his solution. The published solution of problem 49 is not valuable because of its brevity, but because each step is the statement of a very elementary mathematical proposition, and hence can be comprehended by any one who has mastered these simple propositions. It is no discredit to a solution to be long if at the same time it is clear in its statements. EDITOR.]

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

54. Proposed by Professor E. W. MORRELL, Department of Mathematics, Montpelier Seminary, Montpelier, Vermont.

Transform $x^4+y^4+z^4-2y^2z^2-2z^2x^2-2x^2y^2$ into a product.

I. Solution by ROBERT E. MORITZ, B. Sc., Professor of Mathematics in Hastings Colleg Nebraska; and EDGAR KESNER, Boulder, Colorado.

Adding and subtracting $4y^2z^2$, we have

$$(x^4+y^4+z^4-2y^2z^2-2x^2z^2+2y^2z^2)-4y^2z^2,=(x^2-y^2-z^2)^2-(2yz)^2.$$

$$(x^2-y^2-z^2)^2-(2yz)^2=(x^2-y^2-z^2-2yz)(x^2-y^2-z^2+2yz)$$

$$=[x^2-(y+z)^2][x^2-(y-z)^2]$$

$$=(x-y-z)(x+y+z)(x-y+z)(x+y-z),$$

$$\text{or } -(y+z-x)(x+y+z)(x-y+z)(x+y-z).$$

Similarly solved by *O. W. ANTHONY, J. SCHEFFER, C. D. SCHMITT, H. C. WIL YANNEY*, and *G. B. M. ZERR*.

II. Solution by A. P. READ, A. M., Clarence, Missouri.

By the method of the last solution, we get

$$[x^2-(y+z)^2][x^2-(y-z)^2]=(x-y-z)(x+y+z)(x-y+z)(x+y-z)$$

In a similar way by adding and subtracting first $4x^2z^2$ and then 4

obtain $\qquad (y-x-z)(y+x+z)(y-x+z)(y+x-z),$

and $\qquad (z-x-y)(z+x+y)(z-x+y)(z+x-y).$

Also solved in this way by *M. A. GRUBER*.

III. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of University, Lafayette County, Mississippi.

$x^4+y^4+z^4-2y^2z^2-2z^2x^2-2x^2y^2$ may be expressed as the deter

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix}$$

which, as in Burnside and Panton's *Theory of Equations*, second edition, or, as in Weld's *Theory of Determinants*, pages 41 and 42, may be reso the factors

$$-(x+y+z)(y+z-x)(z+x-y)(x+y-z).$$

55. Proposed by MARCUS BAKER, M. A., U. S. Geological Survey, Washington, D. C.

Two right triangles ABC and ABD are so placed as to h side $x(=AB)$ in common. From P the intersection of their hypot drawn c perpendicular to x. Knowing the hypotenuses $a=39$ feet and b and the perpendicular $c=12\frac{6}{7}$ feet, find x. Note this theorem

$$\frac{1}{m}+\frac{1}{n}+\frac{1}{c} \text{ or } \frac{1}{\sqrt{a^2+x^2}}+\frac{1}{\sqrt{b^2-x^2}}=\frac{1}{c},$$

where m and n are the altitudes of the two triangles, respectively. locus of P. Discuss the case when the triangles are general (not right a

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Let $AB=x$, $PG=c$, $AC=a$, $BD=b$, $CB=m$, $AD=n$.

From the triangles ABC and AGP, we get

$$m:c=x:x-GB\ldots\ldots\ldots\ldots(1).$$

From the triangles ABD and BGP, we get

$$n:c=x:GB\ldots\ldots\ldots\ldots\ldots(2).$$

Eliminating GB between (1) and (2), we get

$$\frac{1}{m}+\frac{1}{n}=\frac{1}{c} \text{ or } \frac{1}{\sqrt{a^2-x^2}}+\frac{1}{\sqrt{b^2-x^2}}=\frac{1}{c}.$$

But $a=39$, $b=25$, $c=12\frac{6}{7}$.

$$\therefore \frac{1}{\sqrt{1521-x^2}}+\frac{1}{\sqrt{625-x^2}}=\frac{7}{90}. \quad \text{Let } 1521-x^2=y^2.$$

$$\therefore \frac{1}{y}+\frac{1}{\sqrt{y^2-896}}=\frac{7}{90}.$$

$$\therefore 7y^4-180y^3-6272y^2+161280y-1036800=0. \quad \therefore y=36,\ x=15.$$

For locus of P, let A be the origin. Using polar co-ordinates, we get

$$\tan\theta=\frac{\sqrt{a^2-x^2}}{x}, \text{ and } \frac{r\sin\theta}{x-r\cos\theta}=\frac{\sqrt{b^2-x^2}}{x}, \text{ for the equations to } AC \text{ and } BD.$$

The value of x from the first in the second gives

$$(a^2-b^2)(r\pm a)^2=(a^4\mp 2a^3r)\sin^2\theta. \quad \text{If } a=b,\ r=\pm\tfrac{1}{2}a.$$

For the general triangle, let $EF=x$, $P'G'=c$, $EC=a$, $DF=b$, $BC=m$, $AD=n$. Then from similar right triangles, we deduce the relation:

$$\frac{c}{n}(x-\sqrt{b^2-n^2})+\frac{c}{m}(x-\sqrt{a^2-m^2})=x.$$

II. Solution by Professor J. SCHEFFER, A. M., Hagerstown, Maryland, and COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

Let $BC=m$, $AD=n$, $PG=c$.

Then $$m:c::AB:AG. \quad \therefore AG=\frac{c.AB}{m};$$

$$n:c::AB:BG. \quad \therefore BG=\frac{c.AB}{n}.$$

Adding, $AG+BG=AB=\dfrac{c.AB}{m}+\dfrac{c.AB}{n}$,

$$\text{or } 1=\frac{c}{m}+\frac{c}{n}, \text{ whence } \frac{1}{m}+\frac{1}{n}=\frac{1}{c}.$$

$$m=\sqrt{a^2-x^2}=\sqrt{1521-x^2}\ ;\ n=\sqrt{b^2-x^2}=\sqrt{625-x^2}.$$

Whence $\dfrac{1}{\sqrt{1521-x^2}}+\dfrac{1}{\sqrt{625-x^2}}=\dfrac{7}{90}$. Solving, $x=15$.

[SCHEFFER, SCHMITT.]

Putting $AG=x$, $PG=y$, we find from $\sqrt{b^2-\overline{AB}^2}:y=AB:x$,

$\overline{AB}^2=\dfrac{b^2x^2}{x^2+y^2}$; and substituting this in $\dfrac{a^2-\overline{AB}^2}{y^2}=\dfrac{\overline{AB}^2}{(AB-x)^2}$, we get for the Cartesian equation of the locus $\dfrac{(a^2-b^2)x^2+a^2y^2}{y^2(x^2+y^2)}=\dfrac{b^2}{(b-\sqrt{x^2+y^2})^2}$.

Changing this into the polar equation by putting $x=r\cos\theta$, $y=r\sin\theta$, $x^2+y^2=r^2$, we obtain $r=\dfrac{b\sqrt{a^2-b^2\cos^2\theta}}{b\sin\theta+\sqrt{a^2-b^2\cos^2\theta}}$.

Also solved by *A. H. BELL* and *H. C. WILKES*. [See No. 24, Geometry, Vol. I, page 353, for another solution of a similar problem. Mr. Bell sends a trigonometrical solution, and says that his view of the problem in general is to have given a, b, c, and angles $ABC=BAD=\theta$, to find the base. EDITOR.]

ERRATA.—On page 359, line 16 of December issue, for "t_1+t_1" read t_1+t_2; page 360, line 7, in the denominator for "m_6" read m^6; page 360, under Case III., for "$-4m^6<n^2$" read $-4m^6>n^2$; in the last line on same page insert $\frac{1}{2}$ before the second radical; and on page 361, line 3, of problem 52, for "(2)" read (3).

PROBLEMS.

62. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

A man raises 1 chicken the first year; 6, the second; 35, the third; 180, the fourth; 921, the fifth; 4626, the sixth; 23215, the seventh; 116160, the eighth; and so on. How many does he raise the 20th year, and how many in the twenty years?

63. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A, B, and C bought unequal shares in 200 acres of land at same price per acre, which they sold for \$286.90. A gained as much per cent. on his part as he had acres, B gained $\frac{1}{2}$ as much per cent. on his part as A did, and C lost \$9.10 on the cost of his part; the total net gain was $43\frac{9}{20}$ per cent. How much land did each buy, and what did each receive per acre at the sale?

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

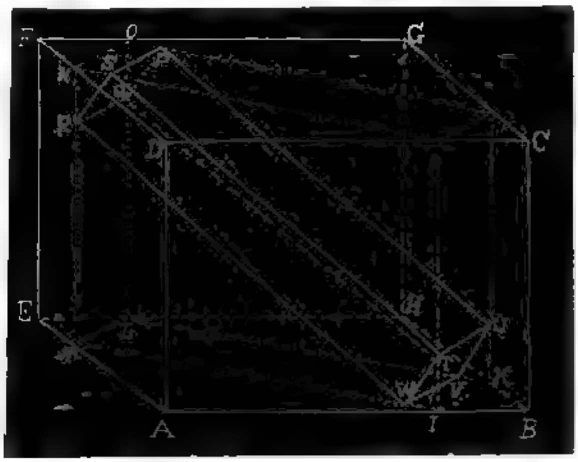
41. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pennsylvania.

Find the length (x) of a rectangular parallelopiped $b=5$ feet, and $h=3$ feet, which can be *diagonally inscribed* in a rectangular parallelopiped $L=83$ feet, $B=64$ feet, and $H=50$ feet.

II. Solution by A. H. BELL, Hillsboro, Illinois, and B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Let $AB=L=83$ feet, $AE=B=64$ feet, $AD=H=50$ feet, $TUVW-P=$the required inscribed parallelopiped, $VW=b=5$ feet, $WT=h=3$ feet, $BK=z$, $IM=y$, and $WR=TS=UP=VQ=x$.

Then $KH=AM=B-z$, $BI=LE=(b^2-z^2)^{\frac{1}{2}}$, and $AI=HL=L-(b^2-z^2)^{\frac{1}{2}}$.



In the right triangle IAM, $IA^2+AM^2=IM^2$, or

$$[L-(b^2-z^2)^{\frac{1}{2}}]^2+(B-z)^2=y^2 \qquad (1).$$

In the similar triangles IAM and IBK, we have

$$AI:BK=IM:IK, \text{ or } L-(b^2-z^2)^{\frac{1}{2}}:z=y:b;$$

whence

$$yz=b[L-(b^2-z^2)^{\frac{1}{2}}] \qquad (2).$$

Solving (2) for y and substituting its value in (1) we have, after reducing and freeing of radicals,

$$4z^4-4Bz^3+[L^2+B^2-4b^2]z^2+2Bb^2z-(L^2-b^2)b^2=0 \qquad (3).$$

Restoring numbers, we have

$$4z^4-2562z^3+10885z^2+3200z-171600=0 \ldots\ldots\ldots\ldots (4).$$

Solving this equation by *Horner's Method*, we find $z=4$. $\therefore \sqrt{b^2-z^2}=3$.

$$\therefore \text{From (1) or (2), } y=100.$$

If in (3) we let $L=100$, $B=50$, and $b=3$ and solve the equation again for z, we find $z=2.750413+=TI$. $\therefore IW=1.5248$. $\therefore WR=109.4494693746751+$ feet, the required length of the parallelopiped.

Had we solved equation (2) for z and substituted its value in (1), we would have obtained an equation which would give the length of the rectangle *IMLK*, but it would require a great deal of work to free the equation of radicals. We shall now obtain such an equation, or formula.

Let $AB=L$, $BH=B$, $\theta=\angle AIM$, and $x=IM$. Then $AI=x\cos\theta$, $AM=x\sin\theta$, $IB=b\sin\theta$, $BK=b\cos\theta$.

$$\therefore \text{Area of } ABHE=2[\tfrac{1}{2}AI\times AM+\tfrac{1}{2}BI\times BK]+IM\times IK$$

$$=(x^2+b^2)\cos\theta\sin\theta=ab \ldots\ldots\ldots\ldots (1).$$

Also

$$x\cos\theta+b\sin\theta=L \ldots\ldots\ldots\ldots (2),$$

$$x\sin\theta+b\cos\theta=B \ldots\ldots\ldots\ldots (3).$$

Squaring (2) and (3) and adding the results, we have

$$x^2+c^2+4cx\sin\theta\cos\theta=L^2+B^2 \ldots\ldots\ldots\ldots (4).$$

Equating $\sin\theta\cos\theta$ in (1) and (4), we have, after an easy reduction,

$$x^4-(L^2+B^2+2b^2)x^2+4LBbx-b^2(L^2+B^2-b^2)=0 \ldots\ldots\ldots (5),$$

an equation which gives the length of the longest rectangle of given width which can be diagonally inscribed in a given rectangle.

[NOTE.—It is but justice to Mr. Bell to say that he was obliged to protest long and vigorously before he received a proper hearing to his claim that the published solution of Dr. Mats and Mr. Burleson is wrong. It was simply a case of that injustice commonly done to men when we believe them to be wrong and refuse to examine their claims. This problem was proposed a few years ago in the *School Visitor*, and at that time we solved the problem though we did not try to obtain the numerical result. When Dr. Mats and Mr. Burleson sent in their solution, it seemed to us on cursory examination to be obtained on the same plan pursued by us a few years ago. But after Mr. Bell had written to us on several different occasions, we offered to publish his solution that our readers might compare the results. But before doing so, we examined the published solution in the May No. Vol. II and found that it was wrong. The numerical calculation of $z=WR$ is due to Mr. Bell, as is also the last equation and the method of obtaining it. EDITOR.]

49. Proposed by J. C. WILLIAMS, Rome, New York.

Of all triangles inscribed in a given segment of a circle, with the chord as base, the isosceles is the maximum.

I. Solution by M. A. GRUBER, War Department, Washington, D. C., and A. P. REED, Superintendent of Schools, Clarence, Missouri.

The bases being equal, the maximum triangle is the one having the greatest altitude.

In any segment of a circle, the greatest perpendicular that can be drawn to the chord, is the perpendicular to the middle of the chord. This perpendicular is the altitude of the isosceles triangle.

∴ The isosceles triangle is the maximum.

II. Solution by J. M. COLAW, A. M., Superintendent of Schools, Monterey, Virginia, and E. KESNER, Boulder, Colorado.

As the segment may be greater or less than a semi-circle, the general proof is for the circle. In the figure it is obvious that the isosceles triangle $P'BC$ is greater than any other triangle ABC, *as its altitude is greater*. Having the given chord as the common base, the *area* depends entirely on the altitude. But the isosceles triangle is a maximum both in *perimeter* and *area*.

Draw PM perpendicular to AB. Then the triangles APM, $P'PB$ are similar, and the diameter $P'P$ is $>AP$; ∴ $P'B$ is $>AM$.

But $2AM=AB+CA$ (*Richardson and Ramsey's Modern Plane Geometry, pp.* 24, 131).

∴ $P'BC$ has the maximum perimeter.

Also solved by *E. L. SHERWOOD* and *G. B. M. ZERR*.

[NOTE.—This problem, with the addition that the isosceles triangle has the maximum perimeter, is Theorem 11, page 131, *Richardson and Ramsey's Modern Plane Geometry*. EDITOR.]

PROBLEMS.

54. Proposed by I. J. SCHWATT, Ph. D., University of Pennsylvania, Philadelphia, Pennsylvania.

Prove geometrically:

If through the center of perspective D of a given triangle ABC and its Brocard triangle $A'B'C'$ be drawn straight lines so as to pass through S_a, S_b and S_c (S_a, S_b, and S_c are the middle points of the sides BC, AC, and AB of the triangle ABC) and if S_aD_1 is made equal to DS_a, S_bD_2 equal to DS_b, and S_cD_3 equal to DS_c then are (1) the figures $D_1O'AO$, $D_2O'BO$ and $D_3O'CO$ parallelograms (O and O' are Brocard's points), (2) the triangles $D_1D_2D_3$ and ABC are equal, and (3) D_1A, D_2B, and D_3C intersect in S, (S is the middle point of OO').

55. Proposed by FREDERICK R. HONEY, Ph. B., New Haven, Connecticut.

Let ab and cd be respectively the major and minor axes of an ellipse, and let α be the angle which a diameter th forms with the major axis; it is required to find the length of this diameter.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

NOTE ON PROBLEM 26, AVERAGE AND PROBABILITY.

BY G. B. M. ZERR.

In reply to Dr. Martin, for whom I have the utmost respect, I have the following remarks to make. The problem that gives the result $\frac{1}{6}a^2$ is different from the problem that gives the result $\frac{a^2}{2\pi}$. In the former the right angle remains fixed and does not lie on a circle as Dr. Martin states. The problem is as follows: Find the average area of all triangles formed by a straight line of constant length a sliding so that its extremities constantly touch two fixed straight lines at right angles to one another. In the problem under consideration the hypotenuse a is fixed and the right angle moves on the semi-circumference. In the first case the average length of one leg is $\int_0^a x dx \Big/ \int_0^a dx = \frac{1}{2}a$. In the second case the average length is $\int_0^a x ds \Big/ \int_0^a ds = a/\pi$. In the first case the average area of all the triangles is $\int_0^a \frac{1}{2}x\sqrt{a^2-x^2}\,dx \Big/ \int_0^a dx = \frac{1}{6}a^2$. In the second case the average area is $\int_0^a \frac{1}{2}x\sqrt{a^2-x^2}\,ds \Big/ \int_0^a ds = \frac{a^2}{2\pi}$, where ds represents an element of arc. It is plainly evident that in the result $\frac{1}{6}a^2$ the leg does not and cannot change its direction or its average length would not be $\frac{1}{2}a$. In the second case it is constantly changing its direction and the right angle is moving on a semicircumference. The problem calls for a given hypotenuse and not one that is constantly changing its direction; hence the result $\frac{a^2}{2\pi}$ is the correct result.

DR. MARTIN'S RESULT IS NOT CORRECT.

F. P. MATZ.

Cause the problem to read: "Find the average area of all right-angled triangles having a given hypotenuse, *if an arm of the triangle vary uniformly;*" then Dr. Martin's result, $\frac{1}{6}h^2$, is perfectly correct.

Strip the problem of this italicised condition; that is, make the problem read as originally proposed; then the number of possible right-angled triangles is proportional to the *length* of the semicircumference of which the given hypotenuse is the diameter. This is *the correct* plan of solution. By *adhering* to *this* plan of solution, the correct result, $h^2/2\pi$, is obtained, regardless as to choice of independent variable.

Dr. Martin's result, $\frac{1}{6}h^2$, is *too great;* for he, by making the number of possible right-angled triangles "proportional to the given hypotenuse," *ignores* the consideration of the areas of practically an infinitude of right-angled triangles of which the major portion have one *rather small* acute angle—thus giving them areas *smaller* than $\frac{1}{4}h^2$.

Since not only all of Dr. Martin's *ignored* right-angled triangles, but *all possible* right-angled triangles, have been properly averaged in my solutions leading to (the result) $h^2/2\pi$, I repeat that this result is the correct one.

Mechanicsburg, Pa.

A REPLY TO DR. MARTIN'S NOTE.

BY THE EDITOR OF THIS DEPARTMENT.

I will say at first, that I too, have profound regard for Dr. Martin, and his opinion on a subject in which he was the pioneer writer in America should not be assailed simply for the sake of controversy.

His argument is entirely sound as to fact but not as to interpretation. It is true the triangles are *not* uniformly distributed on the semicircumference IF the number of triangles is to be obtained by varying one of the legs of the triangle. That this is true may be easily shown from the figure. Let $AC=a$, the hypotenuse, and $BC=x$, angle $BDC=\theta$. Then $x=a\sin\angle BDC=a\sin\frac{1}{2}\theta$. Differentiating, we have $\frac{dx}{d\theta}=\frac{1}{2}a\cos\frac{1}{2}\theta=\frac{1}{2}a\sqrt{\frac{1}{2}(\cos\theta+1)}$. $\therefore dx$ increases $\sqrt{\frac{1}{2}(\cos\theta+1)}$ times as fast as $d\theta$. When $\theta=0$, dx and $d\theta$ are increasing equally, and when $\theta=\frac{1}{2}\pi$, dx is increasing $\frac{1}{2}\sqrt{2}$ times as fast as $d\theta$. Hence it is evident that a greater number of triangles exist for a certain length of arc in the vicinity of the vertex of the semicircumference whose diameter is the hypotenuse, than for the same length of arc near the origin C when the number of triangles is made a function of one of the legs of the triangle, and therefore Dr. Martin's conclusion is sound if we grant his assumption, namely, that the number of triangles is a function of one of the legs of the triangle.

But this assumption is what we refuse to grant. We believe that there are other triangles that are to be interpolated in the series in order that the totality of the triangles may be obtained and that these interpolated triangles are found by making the totality a function of the semicircumference.

From this consideration, it is evident that Dr. Martin's result, $\frac{1}{4}a^2$, is greater than the result, $\frac{a^2}{2\pi}$, which we are defending. The reason is, that according to his interpretation, the triangles are most numerous when $x=\frac{1}{2}\sqrt{2}a$, that is to say, when the vertex of the triangle coincides with the vertex of the semicircumference. Hence the sum of the areas of the triangles ought to be greater than when only as many triangles are taken in one portion of the arc of the semicircumference as in any other.

If the radius DB is made to revolve with uniform velocity about the point

D and its extremity *B* be joined with the points *A* and *C* then the totality of triangles will be formed and they will be uniformly distributed on the semicircumference whose diameter is the hypotenuse.

The question is not whether the triangles are uniformly distributed or not but what method gives the *totality* of the series.

Drury College, January 27, 1896.

NOTES.

ERRATA. Professor Beman calls my attention to a manifest error in Professor Klein's paper which I translated for the December number. Vol. II, page 350, should give the series $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots.$ The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots.$ $=\log_e 2$ instead of $\frac{\pi}{4}$. D. E. SMITH.

Dr. E. A. Bowser writes: Should not problem 43 [Calculus] read $\int_0^1 \frac{x^{a-1}-x^{-a}}{1+x}\,\frac{dx}{\log x}$, as in Price's Calculus, Vol. II, page 120?

NOTE ON THE SOLUTIONS OF PROBLEM 45, PAGES 274-75.

BY ARTEMAS MARTIN, LL. D.

There is but *one* case in Problem 45, Geometry, *as proposed*. Only the circumscribing circle is required.

The final result may be expressed in the more simple form

$$R=\frac{abc}{2\sqrt{[abc(a+b+c)]}-(ab+ac+bc)}.$$

In the second solution, page 275, the equation

"$\cos BCA=\cos(BCA+BCO)$"

should be $\cos BCA=\cos(ACO+BCO)$.

EDITORIALS.

We shall be pleased to receive a catalogue from each of the schools ‡d colleges where the MONTHLY is taken.

Charles De Medici, of New York City, writes, "Your magazine has cer-‡ıly more merit than any other of the kind and ought to be well supported."

George W. Howe, Professor of Mathematics, State Normal School, ‡ırrensburg, Mo., says, "The MONTHLY is a welcome visitor and I trust ‡t you will continue it."

This number was delayed more than two weeks because of the failure of ‡ıe proof reaching its destination. We feel confident that the March number ‡l be mailed by the last of the month, and that thereafter the MONTHLY will ‡ear regularly.

Cooper D. Schmitt, Professor of Mathematics, University of Tennessee, ‡es, "I enclose my subscription for the current year. I wish I had time to ‡ you how much I enjoy the MONTHLY and the good I get from it. It ‡caused me to study along certain lines that I had never before entered upon, ‡ I feel that it does me an immense amount of good."

We are very thankful for the kind words that come from many of our con-‡ıtors respecting the MONTHLY. That the work of getting out such a periodi-‡every month is very arduous is not realized by all. That some errors creep ‡ its pages is not surprising when every one thoroughly realizes the great ‡k connected with the enterprise.

Professor E. P. Thompson, of Miama University, Oxford, Ohio, writes, ‡end you with pleasure $2.00 for THE AMERICAN MATHEMATICAL MONTHLY for ‡3. I get many a useful item, or point in discussion from it, and I hope you ‡ prosper in the good work of putting into print the thoughts of the present ‡kers in our beloved science."

Professor Thompson promises to contribute a paper on the "Mechanics of ‡ Bicycle."

In *The Advance in Education* is an article, "A Class in Geometry under ‡ Laboratory Plan," by Adelia R. Hornbrook, High School, Evansville, Ind. ‡m this article we see that good practical use is made of the MONTHLY in the ‡sroom. She says, "A group of boys, most of whom hope to go to the Poly-‡nic school, are working on a problem given them yesterday. They are much ‡ressed with it, because it came out of the [*American*] *Mathematical Monthly*. ‡y had never seen any mathematical publications except the text books, and ‡ *Monthly*, with its intricate diagrams, mysterious figures, and unfamiliar ‡ıs was a revelation to them." There are thousands of teachers in our High

Schools that could most profitably follow the writer of the above article's plan. Not necessarily that the MONTHLY be used but that the teachers of mathematics carry the spirit of the great living subject into their classrooms. There is no better way to do this than for every teacher to take some good magazine especially devoted to his favorite study.

BOOKS AND PERIODICALS.

Elements of the Differential and Integral Calculus with Examples and Practical Applications. By J. W. Nicholson, A. M., LL. D., President and Professor of Mathematics, Louisiana State University and Agricultural and Mechanical College. 8vo. Cloth, 256 pp. New York and New Orleans: University Publishing Co.

We have long been expecting this unique work on the Calculus as Col. Nicholson apprised us more than a year ago that he was preparing a work on the subject which he expected would create a stir among mathematicians. In this, I think he will realize his expectation, as his work is a great departure from the long beaten path of the traditional Calculus. None of the metaphysical speculations of Newton, Leibinz, Carnot, D'Alembert, Berkeley, Duhamel, Cavalieri, Marquis de L'Hopital, etc., are met with, in reading this book. The idea is simply an extention of mathematical principles without assuming vague metaphysical propositions.

The chief distinctions of this treatise is that, (1) it is based on the conception of *Proportional Variations*, (2) the treatment of dx as a variable, (3) a rigorous deduction of simple tests of absolute convergency, without recourse to the remainder in Taylor's formula, (4) an extension of the ordinary rules for finding maxima and minima, (5) a chapter on Independent Integration, (6) integration by independent coefficients, (7) the introduction of *turns* in curve tracing, and (8) a new proof of Taylor's formula.

The treatment of dx as a variable is the only rational way of viewing dx as a quantity at all. We do not think that Col. Nicholson has wandered too far from the usual method of treating the subject and we are sure the beginner in Calculus will hail the work with joy.

It is time for the Calculus to be treated on sound mathematical principles and not those of metaphysics. We very gladly recommend this new work to the favorable consideration of teachers and students desiring a good text book on the Calculus. B. F. F.

The Science Absolute of Space. Independent of the truth or falsity of Euclid's Axiom XI (which can never be proved *a priori*). By John Bolyai. Translated from the Latin by Dr. George Bruce Halsted, President of the Texas Academy of Science. Fourth edition. Vol. three of the Neomonic Series. Published at the Neomon, 2407 Guadalupe Street, Austin, Texas. Cloth, 71 pp. Price, $1.00.

Dr. Halsted has just got out the fourth edition of his translation of Bolyai's "Science Absolute of Space." The work is enriched by many interesting particulars concerning the lives of the celebrated author of the Non-Euclidean Geometry, Bolyai Janos, and his father, Bolyai Farkas. This little work is worth a careful reading at least once a year.

B. F. F.

Concrete Geometry for Beginners. By A. R. Hornbrook, A. M., Teacher of athematics in High School, Evansville, Indiana. 12mo. Cloth, 201 pp. rice, 75 cents. New York and Chicago: American Book Co.

This book is designed as an Introductory Course to the Study of Demonstrative Geometry. The author is a very thorough and efficient teacher of mathematics, and intensely interested in the subject. The book is carefully and skillfully written, and can ot be too highly recommended for the place it is designed to occupy. Were all students refully instructed by the Laboratory Method, in Concrete Geometry, better results would obtained while studying Demonstrative Geometry. B. F. F.

The Review of Reviews. An International Illustrated Monthly Magazine. dited by Dr. Albert Shaw. Price, $2.50 per year. Single number, 25 cents. he Review of Reviews Co., New York City.

During these months of extraordinary unrest in foreign politics, the *Review of Reviews* evotes its attention in large measure to international affairs. Its editorial department iscusses matters in South Africa, the attitude of the great European powers, and the most cent phases of the movement among the nations for the arbitration of disbutes; the arch number also contains a most timely article on "The Government of France and Its ecent Changes," by Baron Pierre de Coubertin; "A Review of Canadian Affairs," by J. . Russell, and a character sketch of "Cecil Rhodes, of Africa," by W. T. Stead. It can ardly be said that the *Review of Reviews* is narrowly provincial in its outlook on men and ents!

The Cosmopolitan. An International Illustrated Monthly Magazine. Ed-ed by John Brisben Walker. Price, $1.00 per year. Single number, 10 cents. rvington-on-the-Hudson, New York.

The price of this magnificent magazine has been so reduced as to make it possible for ur-fifths of the homes of America to enjoy its choice reading. The March number is re-lete with the best literature of the present time.

See our club rate in December number and then give us your order for the *Review of eviews* and the *Cosmopolitan*.

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oL. III. MARCH, 1896. No. 3.

EW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

y BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Superintendent of Schools, Limaville, Ohio.

It is proposed in these papers to give a more or less complete list of proofs, h new and old, of this celebrated and practical theorem. An attempt is made classification based upon immediate principles used in the proofs. Due credit l be given in all known cases. A historical note will be appended to the com- ted list.

THEOREM.

The square described upon the hypotenuse of a right triangle is equivalent to sum of the squares described upon the other two sides.

PROOFS.

I. RESULTING FROM LINEAR RELATIONS OF SIMILAR RIGHT TRIANGLES.

Let ABC be a $\triangle$ right-angled at C. Draw perpendicular to AB. There are thus three sim- r right triangles.

Fig. 1.

Letting $AC=b$, $BC=a$, $AB=c$, $CD=x$, $AD=y$, $=c-y$, we obtain the following proportions, with ir resulting equations:

(1). $y:b::b:c$. $\therefore yc=b^2$ ....1.

(2). $y:b::x:a$. $\therefore bx=ay$ ....2.

(3). $b:c::x:a$. $\therefore cx=ab$ ....3.

(4). $y:x::x:c-y$. $\therefore x^2=cy-y^2$ 4

(5). $y : x :: b : a.$ $\therefore bx=ay$..............................

(6). $x : c-y :: b : a.$ $\therefore ax=b(c-y)$........................

(7). $c-y : a :: a : c.$ $\therefore c(c-y)=a^2$.... ....

(8). $c-y : a :: x : b.$ $\therefore ax=b(c-y)$........................

(9). $a : c :: x : b.$ $\therefore cx=ab$...............................

From the nine different proportions, there are derived but six diff equations, equation 2 being derived from proportion (2) or (5), 3 from (3) o and 5 from (6) or (8).

It is evident that from no single equation can we determine the rel between a, b, and c, the sides of the given right △.

It is also evident that there is but one set of twos which will give the tion desired, viz., equations 1 and 6. If we add these, member by membe get directly $c^2=a^2+b^2$. Giving to this form the usual geometrical interp tion, we thus have one proof of the theorem. This, though in a different 1 is one of the methods usually found in the books. It is credited to Legendr

We now proceed to find combinations of threes, which will give the re ed relation. There are $\frac{6.5.4}{|\underline{3}}=20$ sets of three equations out of the six. B these, four must be rejected, since they contain 1 and 6, which two alone p the theorem, as already shown; also the following three sets, since in eacl the equations are dependent: 1, 2, 3; 2, 4, 5; 3, 5, 6. There are, then, lef following thirteen sets, from each of which, if we eliminate x and y get $c^2=a^2+b^2$: 1, 2, 4; 1, 2, 5; 1, 3, 4; 1, 3, 5; 1, 4, 5; 2, 3, 4; 2, 3, 5; 2, 2, 4, 6; 2, 5, 6; 3, 4, 5; 3, 4, 6; 4, 5, 6.

Of these thirteen sets, there are six that contain one equation derived from either of two proportions; six sets containing two each such tions; and one containing three. Therefore, including the proof already g there are $1+6\times2+6\times2^2+2^3=45$ proofs, by this method.

II. Let ABC be a △ right-angled at C. Draw a line perpendicul AB from A, meeting BC produced, as at D.

Fig. 2.

Letting $AC=b$, $BC=a$, $AB=c$, $AD=x$, $DC=y$, $BD=y+a$, and proceeding as in the preceding case, we find that this method also yields 45 different proofs. The details are left to be carried out by the reader.

III. Let ABC be a △ right-angled at C. Draw DE perpendi to AB so that $DE=DC$. Then will $BE=BC$. △ ADE is similar to △ ABC.

Fig. 3.

Letting $AC=b$, $BC=a$, $AB=c$, $AE=c-a$, $DE=DC=x$, $AD=b-x$, we have :

(1). $c-a : b :: x : a.$ $\therefore x=\dfrac{ac-a^2}{b}$ ....1.

(2). $c-a : b :: b-x : c.$ $\therefore x=\dfrac{b^2-c^2+ac}{b}$ ....2.

(3). $x : a :: b-x : c.$ $\therefore x=\dfrac{ab}{a+c}$ ....3.

From the three equations, it is evident that we may obtain three proofs by this method.

IV. Let ABC be a $\triangle$ right-angled at C. Extend AB to D making $BD=BC$. Draw a line perpendicular to AD at D, meeting AC produced as at E. Then will $CE=DE$, and $\triangle\ AED$ will be similar to $\triangle\ ABC$.

Fig. 4.

Letting $AC=b$, $BC=a$, $AB=c$, $AD=c+a$, $DE=x$, $AE=x+b$, and proceeding as in the last case, we obtain three more proofs, making in all, thus far, 96 proofs.

In our next paper, we shall give a method whose results reach into the thousands.

[To be Continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from February Number.]

PROPOSITION XXIII. *If any two straights AX, BX (Fig. 27.) exist in the same plane, either they have (even in the hypothesis of acute angle) a common perpendicular, or prolonged toward either the same part, unless sometime at a finite distance one strikes upon the other, they mutually approach ever more toward each other.*

Proof. From any point A of AX is let fall to the straight BX the perpendicular AB. If BA makes with AX a right angle, we have the asserted case of the common perpendicular. But otherwise this straight makes toward one or the other part, as suppose toward the parts of the point X, an acute angle.

So in the aforesaid straight AX between the points A and X any points D, H, l are designated, from which are let fall to the straight BX the perpendiculars DK, HK, LK. If any one angle at the points D, H, L be acute toward the parts of the point A, it follows (from the preceding) that AX, BX will have a common perpendicular.

Fig. 27.

But if every angle of this sort be greater than acute; either some one will be right, and thus again we will have the asserted case of the common perpendicular, since all angles at the points K are supposed right; or all those angles toward the parts of the point A are obtuse, and therefore all therewith acute toward the parts of the point X, and so again I argue: Since i the quadrilateral $KDHK$ the angles at the points K are right, but the angle a the point D is acute, the side DK will be (from Cor. II. after P. III.) greate than the side HK.

In a similar way the side HK is shown to be greater than the side LK and so always, comparing to each other perpendiculars from any ever highe points of AX let fall upon the other BX. Wherefore AX, BX mutually approac each other ever more toward the parts of the point X: Which is the second par of the disjunct proposition.

From all which follows that any two straights AX, BX, which exist in th same plane, either have (even in the hypothesis of acute angle) a common per pendicular, or produced toward either the same part, unless sometime at a finit distance one strikes upon the other, mutually approach each other ever more.

Quod erat etc.

Corollary I. Hence the angles toward the base AB will be alway obtuse at each point of AX, from which is let fall a perpendicular to the straigh BX: will be, I say, always obtuse, as often as those two AX and BX mutually approach each other ever more toward the parts of the points X; which indee ought to be understood in a sane way, of course, of perpendiculars let fall befor the aforesaid meeting, if perchance one should strike upon the other at a finit distance.

Scholion. I see indeed that it may be here inquired, in what way can b shown the existence of that common perpendicular, as often as any straigh $PFHD$ (Fig. 28.) meeting two AX, BX in points F, and H, makes towar the same parts two internal angles AHF, BFH, not themselves indeed right, but nevertheless together equal to two rights. But behold that common perpendicular geometrically demonstrated.

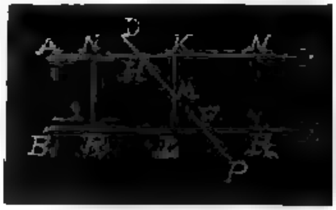

Fig. 28.

FH being bisected in M, perpendiculars MK, ML are let fall to AX and BX. The angle MFL will be equal (Eu. I. 13) to the angle MHK, which indeed is supposed to make up two right angles with the angle BFH. Moreover the angles at the points K and L ar t ; and again MF, MH are equal. Therefore (Eu. I. 26) so are the angles L, HMK equal. Wherefore the angle HMK makes two right angles with the HML, since with this the angle FML (Eu. I. 13) makes two right angles. re (Eu. I. 14) KML will be one continuous straight line, consequently the perpendicular of the aforesaid straights AX, BX. Quod erat etc.

[To be Continued.]

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from February Number.]

The Construction of the Primitive Groups.

We have shown that all the intransitive and the non-primitive groups of a n degree, may be made to depend upon groups of a lower degree. We shall prove a similar property of the primitive groups.

It must however not be inferred that this will solve, in a satisfactory man- the problem of constructing all the groups of a given degree. The elemen- methods to which we have confined ourselves require a large number rials if the degree is large. Some briefer methods will be given later even these will only tend to make the construction of all the groups given degree practical for somewhat larger degrees.

It is not difficult to give general theorems which include all the groups of ven type, as, for instance, the theorem at the end of our discussion of the truction of the non-primitive groups ; but new types arise continually and no -tentative method by means of which all the groups of any given degree may ound has yet been published.

We proceed to prove some theorems which apply to all transitive groups are especially useful in the construction of primitive groups. Unless the rary is stated the symbols G, g, and n will respresent respectively the group consideration, its order and its degree.

Let us consider the transitive group G which contains the letters a_1, ..., a_n. The substitutions of G which do not contain a_1 (*i. e.*, those replace a_1 by itself) may be represented by

$$s_1,\ s_2,\ \ldots\ldots,\ s_r \equiv G_1.$$

As every group must contain the identical substitution if the number of its letters is finite and this is the only kind we are considering now, the minimum value of r is unity.

Since G is transitive it must contain some substitution s_{r+1} which replaces a_1 by a_2. We desire to find all the substitutions of G which have this property. If s_k is such a substitution then will

$$s_k s_{r+1}^{-1}$$

belong to the first line, since s_k replaces a_1 by a_2 and s_{r+1}^{-1} replaces a_2 by a_1 $s_k s_{r+1}^{-1}$ must leave a_1 unchanged. Hence we have the equations

$$s_k s_{r+1}^{-1} = s_\alpha \qquad (\alpha = 1, 2, \ldots\ldots r)$$

$$s_k = s_\alpha\, s_{r+1}.$$

Since the condition expressed by the last equation is sufficient as well as necessary it follows that there are just r different substitutions in G, which transform a_1 into a_2. Similarly there are exactly r substitutions in G which replace a_1 by a_3, etc. From this we see that the number of substitutions which replace a_1 by itself is equal to the number of those which replace a_1 by any other letter of G. We have imposed no condition upon a_1 which is not satisfied by each of the other letters so that the property which we have proved in regard to a_1 belongs to all the letters. That r has the same value for each of the letters follows from the following considerations :

If the substitutions of G which do not contain a_2 are

$$s_1', s_2' \ldots\ldots s_{r'}' \equiv G_2,$$

then will

$$s_{r+1} G_2 s_{r+1}^{-1} = r^1 \text{ substitutions of } G \text{ which do not contain } a_1 \text{ and}$$

$$s_{r+1}^{-1} G_1 s_{r+1} = r \text{ substitutions of } G \text{ which do not contain } a_2.$$

From the first of these two equations we have $r' \overline{\gtrless} r$ and from the second $r \overline{\gtrless} r'$, hence $r' = r$. Similar remarks clearly apply to all the letters of G. We may embody the results at which we have arrived in the following

THEOREM: *The number of substitutions* (r) *of any transitive group* (G), *which do not contain any given letter, is equal to the number of substitutions which replace a letter by any required other letter of the group.*

Corollary I. $g = nr$, *i. e. the order of any transitive group is a multiple of its degree.*

*Corollary II. The average number of letters in all the substitutions of a transitive group of degree n is $n-1$.**

*Since every intransitive group may be resolved into transitive constituent groups whose separate elements enter an equal number of the substitutions of the intransitive group, the general statement of this corollary is as follows: *The average number of letters in all the substitutions of any group is $n-\alpha$, n being the degree of the group and α the number of its transitive constituents.*

The last corollary may be proved as follows: Since G contains only $g \div n$ substitutions that do not involve a_α it must contain $g-g/n=\frac{n-1}{n}g$ that involve a_α. Hence all the g substitutions of G contain $n\times\frac{n-1}{n}g=(n-1)g$ letters.

From this corollary we may directly derive the following:

Corollary III. Every transitive group contains at least $n-1$ substitutions of the n^{th} degree.

Corollary IV. If the order of a transitive group exceeds its degree it must contain substitutions of a lower than the n^{th} degree and hence n conjugate subgroups $G_1, G_2, \ldots\ldots, G_n$ whose degree is at most $n-1$. These n subgroups need not all be distinct.

We may divide the primitive groups into two classes. (1) Those whose order is equal to their degree—the *regular* primitive groups—and (2) those whose order is b times their degree, where b is a positive integer larger than 1.

We proceed to consider the first one of these classes. Since the average number of letters in its substitutions is $n-1$ it must contain $n-1$ substitutions of the n^{th} degree, *i. e.* all its substitutions except unity are of the n^{th} degree.

If any one of these $n-1$ substitutions consists of two or more cycles all of these cycles will be of the same order, *i. e.* they will all contain the same number of letters, otherwise some power of this substitution would at the same time differ from identity and not contain all the letters of the group.

We proceed to prove the following

THEOREM: *Whenever a regular group contains a substitution (s) which contains more than one cycle it is non-primitive.*

Let $s=a_1a_2 \ldots\ldots a_r, b_1 \ldots\ldots \ldots\ldots$ Some substitution of G (s_1) replaces a_1 by b_1. If we transform s with respect to s_1 we have

$$s_1^{-1}s\, s_1=b_1b_2\ldots\ldots b_r\ldots\ldots$$

If we assume that

$$b_\alpha=a_\beta \qquad (\alpha,\ \beta \overline{\gtrless} r)$$

we have as a consequence that s_1 replaces a_α by a_β. This is also done by $s^{\beta-\alpha}$. Since only one substitution of G can perform this operation we have as a second consequence of the given assumption

$$s_1=s^{\beta-\alpha}.$$

The latter of these transforms the cycle $a_1a_2\ldots\ldots a_r$ into itself and the former does not, the given assumption is therefore untenable and the cycle of b's must be distinct from the cycle of a's.

If these a's and b's do not include all the letters of G there must be some substitution of G (s_2) which replaces a_1 by some new letter c_1. We now derive the substitution

$$s_2^{-1}s\ s_2 = c_1 c_2 \ldots\ldots c_2 \ldots\ldots$$

We have already proved that these c's are all different from the a's. It remains to show that they do not include any b.

From $$c_\alpha = b_\beta$$

it would follow that s_2 replaced a_α by b_β and therefore that

$$s_2 = s^{\beta-\alpha} s_1 .$$

This is impossible since the second member replaces the a's by the b's and the first replaces a_1 by c_1.

Continuing in this manner we must finally exhaust the letters of G and obtain the l distinct cycles

$$a_1 a_2 \ldots\ldots a_r,\ b_1 b_2 \ldots\ldots b_r,\ \ldots\ldots,\ l_1 l_2 \ldots\ldots l_r$$

where $lr = n$, the degree of G.

We proceed to prove that these cycles may be used as systems of non-primitivity. This is, of course, included in the proof that the substitutions composed of these cycles

$$a_1 a_2 \ldots\ldots a_r.\ b_1 b_2 \ldots\ldots b_r \ \ldots\ldots\ l_1 l_2 \ldots\ldots l_r \equiv t$$

is transformed into itself by all the substitutions of G.

Let s_α represent any substitution of G ; we desire to prove that

$$s_\alpha^{-1} t s_\alpha = t.$$

If s_α replaces c_γ by b_β we have

$$s_a = s_2^{-1} s^{\beta-\gamma} s_1 .$$

The second member replaces $c_{\gamma+\rho}$ by $b_{\beta+\rho}$ where ρ satisfies the congruence

$$\gamma+\rho,\ \beta+\rho \equiv \delta \pmod r,\ (\delta = 1, 2, \ldots\ldots r).$$

Hence s_α must replace the c's in order by the b's in order. Since similar remarks apply to all the cycles it follows that s_α which is any substitution of G transforms t into itself and our theorem is proved.

By starting with the different cycles of G which contain the same letter we obtain different systems of non-primitivity for the same group.*

*Cf. Jordan, Traite des Substitutions, §75; and Netto, Theory of Substitutions (American Edition), §98.

From the last theorem we see that a regular group cannot be primitive unless it is generated by a single cycle involving a prime number of letters. Since such a group must be primitive we have the following

THEOREM : *The regular primitive groups and the prime numbers have a* 1,1 *correspondence ; i. e. for each prime number there is one regular primitive group and for each regular primitive group there is one prime number.*

[To be Continued.]

THE CENTROID OF AREAS AND VOLUMES.

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[Continued from February Number.]

II. VOLUMES. Let the density vary as $x^{h-1}y^{k-1}z^{l-1}$. Then

$$\bar{x}=\frac{\iiint x^hy^{k-1}z^{l-1}dxdydz}{\iiint x^{h-1}y^{k-1}z^{l-1}dxdydz},\quad \bar{y}=\frac{\iiint x^{h-1}y^kz^{l-1}dxdydz}{\iiint x^{h-1}y^{k-1}z^{l-1}dxdydz},$$

$$\bar{z}=\frac{\iiint x^{h-1}y^{k-1}z^ldxdydz}{\iiint x^{h-1}y^{k-1}z^{l-1}dxdydz}.$$

$$\therefore \bar{x}=\frac{\dfrac{a^{h+1}b^kc^l}{\frac{8}{(2m+1)(2n+1)(2p+1)}}\;\dfrac{\Gamma\left\{\frac{h+1}{2}(2m+1)\right\}\Gamma\left\{\frac{k}{2}(2n+1)\right\}\Gamma\left\{\frac{l}{2})2p+1)\right\}}{\Gamma\left\{\frac{h+1}{2}(2m+1)+\frac{k}{2}(2n+1)+\frac{l}{2}(2p+1)+1\right\}}}{\dfrac{a^hb^kc^l}{\frac{8}{(2m+1)(2n+1)(2p+1)}}\;\dfrac{\Gamma\left\{\frac{h}{2}(2m+1)\right\}\Gamma\left\{\frac{k}{2}(2n-1)\right\}\Gamma\left\{\frac{l}{2}(2p+1)\right\}}{\Gamma\left\{\frac{h}{3}(2m+1)+\frac{k}{2}(2n+1)+\frac{l}{2}(2p+1)+1\right\}}}$$

$$\therefore \bar{x} = \frac{\Gamma(hm+m+\frac{h+1}{2})\Gamma(hm+kn+lp+\frac{h+k+l}{2}+1)}{\Gamma(hm+\frac{h}{2})\Gamma(hm+kn+lp+m+\frac{h+k+l+1}{2}+1)} a \ldots\ldots\ldots (C).$$

$$\bar{y} = \frac{\Gamma(kn+n+\frac{k+1}{2})\Gamma(hm+kn+lp+\frac{h+k+l}{2}+1)}{\Gamma(kn+\frac{k}{2})\Gamma(hm+kn+lp+n+\frac{h+k+l+1}{2}+1)} b \ldots\ldots\ldots\ldots (D).$$

$$\bar{z} = \frac{\Gamma(lp+p+\frac{l+1}{2})\Gamma(hm+kn+lp+\frac{h+k+l}{2}+1)}{\Gamma(lp+\frac{l}{2})\Gamma(hm+kn+lp+p+\frac{h+k+l+1}{2}+1)} c \ldots\ldots\ldots\ldots (E).$$

This gives the centroid of the eighth part of the volume whatever may be the values of h, k, l, m, n, p.

Let $m=n=p$, and also let the density vary as xyz so that $h=k=l=2$.

$$\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(3m+\frac{3}{2})\Gamma(6m+4)}{\Gamma(2m+1)\Gamma(7m+\frac{9}{2})}.$$

Let $m=0$, $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(\frac{3}{2})\Gamma(4)}{\Gamma(1)\Gamma(\frac{9}{2})} = \frac{16}{35}$, for $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$

Let $m=1$, $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(\frac{9}{2})\Gamma(10)}{\Gamma(3)\Gamma(\frac{23}{2})} = \frac{2^{13}}{11.13.17.19}$,

for $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{z}{c}\right)^{\frac{2}{3}} = 1.$

Let $m=\frac{3}{2}$, $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(6)\Gamma(13)}{\Gamma(4)\Gamma(15)} = \frac{10}{91}$,

for $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} + \left(\frac{z}{c}\right)^{\frac{1}{2}} = 1,$

the centroid of the volume bounded by the positive portion of the co-ordinate planes.

Let $m=n=p$, and let the density be the same throughout the solid so that, $h=k=l=1$

$$\therefore\ \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(2m+1)\Gamma(3m+\frac{5}{2})}{\Gamma(m+\frac{1}{2})\Gamma(4m+3)}.$$

$$n=0,\ \therefore\ \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(1)\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})\Gamma(3)}=\frac{3}{8},\ \text{for}\ \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1.$$

$$n=1,\ \therefore\ \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(3)\Gamma(\frac{11}{2})}{\Gamma(\frac{3}{2})\Gamma(7)}=\frac{21}{128},\ \text{for}\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1.$$

$$n=\tfrac{3}{2},\ \therefore\ \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(4)\Gamma(7)}{\Gamma(2)\Gamma(9)}=\frac{3}{28},\ \text{for}\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}+\left(\frac{z}{c}\right)^{\frac{1}{2}}=1.$$

Let $m=n=p$, and let the density vary as xy, so that $h=k=2$, $l=1$,

$$\therefore\ \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\Gamma(3m+\frac{3}{2})\Gamma(5m+\frac{7}{2})}{\Gamma(2m+1)\Gamma(6m+4)},\quad \frac{\bar{z}}{c}=\frac{\Gamma(2m+1)\Gamma(5m+\frac{7}{2})}{\Gamma(m+\frac{1}{2})\Gamma(6m+4)}.$$

$$\text{Let } m=0,\ \therefore\ \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\Gamma(\frac{3}{2})\Gamma(\frac{7}{2})}{\Gamma(1)\Gamma(4)}=\frac{5\pi}{32},$$

$$\frac{\bar{z}}{c}=\frac{\Gamma(1)\Gamma(\frac{7}{2})}{\Gamma(\frac{1}{2})\Gamma(4)}=\frac{5}{16},\ \text{for}\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1.$$

$$\text{Let } m=1,\ \therefore\ \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\Gamma(\frac{9}{2})\Gamma(\frac{17}{2})}{\Gamma(3)\Gamma(10)}=\frac{5.7.11.13.15\pi}{2^{20}},$$

$$\frac{\bar{z}}{c}=\frac{\Gamma(3)\Gamma(\frac{17}{2})}{\Gamma(\frac{3}{2})\Gamma(10)}=\frac{5.11.13}{2^{13}},\ \text{for}\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1.$$

$$\text{Let } m=\tfrac{3}{2},-\ \therefore\ \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\Gamma(6)\Gamma(11)}{\Gamma(4)\Gamma(13)}=\frac{5}{33},$$

$$\frac{\bar{z}}{c}=\frac{\Gamma(4)\Gamma(11)}{\Gamma(2)\Gamma(13)}=\frac{1}{22},\ \text{for}\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}+\left(\frac{z}{c}\right)^{\frac{1}{2}}=1.$$

Let $m=n=p$, and let the density vary as x so that $h=2$, $k=l=1$.

$$\therefore \frac{\bar{x}}{a}=\frac{\Gamma(3m+\frac{3}{2})\Gamma(4m+3)}{\Gamma(2m+1)\Gamma(5m+\frac{7}{2})},\quad \frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(2m+1)\Gamma(4m+3)}{\Gamma(m+\frac{1}{2})\Gamma(5m+\frac{7}{2})}.$$

Let $m=0$, then for $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$

$$\frac{\bar{x}}{a}=\frac{\Gamma(\frac{3}{2})\Gamma(3)}{\Gamma(1)\Gamma(\frac{7}{2})}=\frac{8}{15},\ \frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(1)\Gamma(3)}{\Gamma(\frac{1}{2})\Gamma(\frac{7}{2})}=\frac{16}{15\pi}.$$

Let $m=1$, then for $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1$

$$\frac{\bar{x}}{a}=\frac{\Gamma(\frac{9}{2})\Gamma(7)}{\Gamma(3)\Gamma(\frac{17}{2})}=\frac{2^7}{3.11.13},\ \frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(3)\Gamma(7)}{\Gamma(\frac{3}{2})\Gamma(\frac{17}{2})}=\frac{2^{14}}{5.7.9.11.13\pi}.$$

Let $m=\frac{3}{2}$, then for $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}+\left(\frac{z}{c}\right)^{\frac{1}{2}}=1$

$$\frac{\bar{x}}{a}=\frac{\Gamma(6)\Gamma(9)}{\Gamma(4)\Gamma(11)}=\frac{2}{9},\ \frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(4)\Gamma(9)}{\Gamma(2)\Gamma(11)}=\frac{1}{15}.$$

Thus we could multiply examples almost without number.
If $a=b$ we get another series of areas.
If $a=b=c$ we get another series of solids.
If $b=c$ or $a=c$ we get still another series of solids.
But formulæ (A), (B), (C), (D), (E) apply to them all.

One more example and we will proceed to the discussion of surfaces. the density vary as x^3y^2z, and let the equation to the surface be

$$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{7}}=1,$$

so that $h=4$, $k=3$, $l=2$, $m=1$, $n=2$, $p=3$

$$\therefore \bar{x}=\frac{\Gamma(\frac{15}{2})\Gamma(\frac{43}{2})}{\Gamma(6)\Gamma(23)}a=\frac{5.7.9.11.13.23.29.31.37.39.41\pi a}{2^{50}}.$$

$$\bar{y}=\frac{\Gamma(10)\Gamma(\frac{43}{2})}{\Gamma(\frac{15}{2})\Gamma(24)}b=\frac{5.9.29.31.37.41b}{11.2^{26}},$$

$$\bar{z}=\frac{\Gamma(\frac{21}{2})\Gamma(\frac{43}{2})}{\Gamma(7)\Gamma(25)}c=\frac{5.7.9.11.13.13.17.19.29.31.37.41\pi c}{2^{51}}.$$

The prodigious amount of work to accomplish this by the ordinary m od would be impossible.

[To be Continued.]

THE ANGLE-SUM ACCORDING TO PLAYFAIR.

By Professor JOHN N. LYLE, Ph. D., Westminster College, Fulton, Mo.

In Playfair's Euclid, pages 295 and 296, there is given a short method for ng the angle-sum of a rectilineal triangle.

As the soundness of this method has been called in question by the Hyper-ıe theorists, it is incumbent upon teachers of geometry to examine both the ıod itself and the criticisms to which it has been subjected.

John Playfair in treating of the angle-sum says—"It is of importance in aining the Elements of Science, to connect truths by the shortest chain pos-ı; and till that is done, we can never consider them as being placed in their ıral order.

The reasoning in the first of the following propositions is so simple, that ems hardly susceptible of abbreviation, and it has the advantage of connect-ımmediately two truths so much alike, that one might conclude, even from bare enunciations, that they are but different cases of the same general theo-, viz., That all the angles about a point, and all the exterior angles of any lineal figure, are constantly of the same magnitude, and equal to four right es.

DEFINITION.

If, while one extremity of a straight line remains fixed at A, the line itself s about that point from the position AB to the tion AC, it is said to describe the angle BAC con-ed by the lines AB and AC.

Corollary. If a line turn about a point from position AC till it come into the position AC again, it describes angles which together equal to four right angles. This is evident from the second corollary he fifteenth, 1.

PROPOSITION I.

All the exterior angles of any rectilineal figure are together equal to four t angles.

1. Let the rectilineal figure be the triangle ABC, of which the exterior es are DCA, FAB, GBC; these angles are together ıl to four right angles.

Let the line CD, placed in the direction BC produced, about the point C till it coincide with CE, a part of the CA, and have described the exterior angle DCE or ı.

Let it then be carried along the line CA, till it be in position AF, that is, in the direction of CA produced, the point A remaining fixed, let it turn about A till it describe the angle ?, and coincide with a part of the line AB.

Let it next be carried along *AB* until it come into the position *BC*
by turning about *B*, let it describe the angle *GBC* so as to coincide with a p
BC.

Lastly, let it be carried along *BC* till it coincide with *CD* its first pos

Then, because the line *CD* has turned about one of its extremities
has come into the position *CD* again, it has by the corollary to the above d
tion described angles which are together equal to four right angles ; but the
les which it has described are the three exterior angles of the tri
ABC, therefore the exterior angles of the triangle *ABC* are equal to four
angles.

2. If the rectilineal figure have any number of sides, the propositi
demonstrated just as in the case of a triangle. Therefore all the exterior a
of any rectilineal figure are together equal to four right angles.

Corollary 1. Hence, all the interior angles of any triangle are equal t
right angles. For all the angles of the triangle, both exterior and interio
equal to six right angles, and the exterior being equal to four right angles
interior are equal to two right angles."

In this demonstration of the angle sum Playfair evidently regard
method employed by him as legitimate, simple, direct and brief.

The Riemannian division of the Hyper-Space theorists assumes
a plane is the surface of an immense sphere, and that straight lines are c
that come back to their starting points, and, hence, raises the objection tha
lines can not be slid along and then rotated as Playfair's demonstration req

This objection of the Riemannian School obviously rests on the false b
that a plane is a spherical surface and that straight lines are curves. The
dation being insecure, that which is built thereon can not stand. The obj
obliterates the distinction between spherical geometry and plane geometry.

If it be true that a plane is perfectly flat and that straight lines are d
of curvature, the objection that we are considering is seen to have no force.

The Riemannian theorists tell us that for ought they know straight
may be curves. They begin by doubting the truth of Euclid's second postu
"That a terminated straight line may be produced to any length in a straight
—and his Proposition XXXII, Book I. They are believers, also, as
as doubters. They believe that the angle-sum of a rectilineal triangle is g
than two right angles. They believe that if a straight line be extended i
ultimately return to the starting point.

The Euclidean geometers doubt these articles of Riemannian faith, an
lieve that the angle-sum of a rectilineal triangle is equal to two right angles
that the longer a straight line is the further apart are its ends.

The Riemannians doubt what the Euclideans believe and believe wh
Euclideans doubt. Those who undertake to teach both of these doctrines
contradict each other have failed to reckon with the logical laws of non-cont
tion and excluded middle. We notice further that the Riemannian objecti
Playfair's demonstration is in conflict with Proposition I of Lobatschew

Theory of Parallels. Says the Russian Pangeometer—"A straight line fits upon itself in all its positions. By this I mean that during the revolution of the surface containing it the straight line does not change its place, if it goes through two unmoving points in the surface: (*i. e.*, if we turn the surface containing it about two points of the line, the line does not move)." These statements can not be made of any arc of any circle, and, hence, can not be made of Riemannian straight lines that are assumed to have constant positive curvature. What Lobatschewsky says respecting the straight line in his theorem I is inconsistent, of course, with his doctrine that the angle-sum is less than two right angles. But we are not quoting Lobatschewsky now to show that his theory is inconsistent with itself, but with that of Riemann.

Another objection to Playfair's demonstration is that a triangle drawn on a blackboard is not bounded by lines perfectly straight, since the surface of the board is uneven.

This objection does not hold against the triangle whose vertices are the three points A, B, and C in space and whose sides are destitute of curvature.

The geometer, whether he proceeds analytically or synthetically, naturally regards space as extending beyond himself on all sides without bounds, and between any two points A and B located therein he can draw an absolutely straight line with his mind, although he may be unable to do so with his hand. Some metaphysicians doubt these facts. What is it that they have not doubted? The function of a metaphysician, however, is to explain facts, not to doubt or discredit them.

Three points A, B, and C not in the same straight line may be located in trinally extended objective space and connected by the straight lines AB, AC, and BC. Hence, a rectilineal triangle in objective space is possible. When we say rectilineal triangle we do not mean a bogus triangle with wrinkled sides, but a genuine triangle with straight sides. When we say straight sides we do not mean wrinkled sides. The rectilineal triangle ABC of the geometer is perfect. His ability to cognize such a triangle is shown in the fact that he does cognize it. This fact, too, has been doubted. What a wonderful endowment that must be that enables man to people space with faultlessly perfect forms! This lofty power of intelligence in man, nay even his own doubt respecting it, differentiates him from the lower animals.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

34. Proposed by R. H. YOUNG, West Sunbury, Pennsylvania.

Prove (1) that $\frac{n(n+1)(2n+1)}{6}$ is a whole number for all values of n; an

(2) prove that $\frac{n(n-1)(n+1}{24}$ is a whole number when n is odd.

I. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Kn ville, Tennessee.

(1). $1^2+2^2+3^2+4^2+\dots\dots\dots+n^2=\frac{n(n+1)(2n+1)}{6}$ = a whole numb

for all integral values of n.

(2). Let $n=2m+1$ = an odd number for all integral values of m.

$\therefore \frac{(n-1)n(n+1)}{24}=\frac{m(m+1)(2m+1)}{6}$ = same as (1).

II. Solution by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

As n and $n+1$ are consecutive numbers, one of them must be even and divisible by two. But n must be of the form of $3p$, $3p+1$, or $3p+2$. If of t form of $3p$, it is divisible by three; if of the form $3p+2$, then $n+1$ or $3p+3$ divisible by three; if of the form $3p+1$, then $(2n+1)$ becomes $6p+3$, and is visible by three. Hence $n(n+1)(2n+1)$ is divisible by twice three, or six, wh ever the value of n is.

2. $(n-1)n(n+1)$ of which the middle one is odd. One of every th consecutive numbers is always divisible by three: one of two consecutive e numbers is always divisible by four and the other by two. Hence $(n-1)n(n+$ when n is odd, is always divisible by $2\times3\times4$ or 24.

Also solved by *O. W. ANTHONY, M. A. GRUBER, EDGAR KESNER, E. W. MORRELL, SCHEFFER, E. L. SHERWOOD, B. F. YANNEY,* and *G. B. M. ZERR.*

35. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Te kana College, Texarkana, Arkansas-Texas.

Decompose into the sum of two squares the number $13^2.61^3$.

I. Solution by E. L. SHERWOOD, A. M., Professor of Mathematics in Mississippi Normal College, H ton, Miss., and E. W. MORRELL, Department of Mathematics in Montpelier Seminary, Montpelier, Vermont.

$13^2.61^3=13^2.61^2.61=13^2.61^2(5^2+6^2)=13^2.61^2.5^2+13^2.61^2.6^2$.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Put $13^2.61^3=(p^2+q^2)^2(m^2+n^2)^3$, in which $p=3$, $q=2$, $m=6$, $n=5$. decomposing into the sum of two squares, we find

$$(p^2+q^2)^2(m^2+n^2)^3=[m(p^2+q^2)(m^2-3n^2)]^2+[n(p^2+q^2)(3m^2-n^2)]^2=$$
$$[m(m^2+n^2)(p^2+q^2)]^2+[n(m^2+n^2)(p^2+q^2)]^2=$$
$$[m(p^2-q^2)(m^2-3n^2)\pm 2npq(3m^2-n^2)]^2+$$
$$[n(p^2-q^2)(3m^2-n^2)\mp 2mpq(m^2-3n^2)]^2=$$
$$[m(m^2+n^2)(p^2-q^2)\pm 2npq(m^2+n^2)]^2+[n(m^2+n^2)(p^2-q^2)\mp 2mpq(m^2+n^2)]^2,$$
making *six* sets of the sum of two squares.

Substituting the respective values of p, q, m, and n, we have

$$13^2.61^2=3042^2+5395^2=4758^2+3965^2=3810^2+4883^2$$
$$=6150^2+733^2=5490^2+2867^2=1830^2+5917^2.$$

Solved with these six sets of values by *A. H. BELL*, and with the five sets last in order by the *PROPOSER*. Also solved by *J. H. DRUMMOND*, *C. D. SCHMITT*, and *B. F. YANNEY*.

36. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the first six integral values of n in $\frac{n(n+1)}{2}=\square$.

I. Solution by Professor J. SCHEFFER, A. M., Hagerstown, Maryland, and O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

We have $n^2+n=2y^2$. Putting $n=\frac{t-1}{2}$, we obtain $t^2-8y^2=1$. Since $t=3, y=1$, satisfy this equation, we have $t=\frac{1}{2}[(3+2\sqrt{2})^m+(3-2\sqrt{2})^m]$, where for m successive integral numbers must be chosen. The required values of n we then obtain from the relation $n=\frac{t-1}{2}$.

For $m=1, 2, 3, 4, 5, 6$, in succession, we find in order the corresponding values of $t=3, 17, 99, 577, 3363, 19601$, and $n=1, 8, 49, 288, 1681, 9800$.

II. Solution by A. H. BELL, Hillsboro, Illinois.

Let $\frac{n(n+1)}{2}=\square=y^2$ say; then clearing of fractions, multiplying by 4, and adding 1 to both members, etc., $(2n+1)^2=8y^2+1=\square=x^2$ say.

$\therefore n=\frac{x-1}{2}$. Again $x^2-8y^2=1$. The 1st convergent for the $\sqrt{8}=\frac{3}{1}=\frac{x}{y}$ or the solution of this celebrated equation and the value of x and y can be found, on page 53, Vol. I. of MONTHLY.

The general value for x is $x_{n+1}=2x_1\times x_n-x_{n-1}$, hence $x_0=1$, $x_1=3$, $x_2=6\times3-1=17$, $x_3=6\times17-3=99$, $x_4=6\times99-17=577$, $x_5=6\times577-99=3363$, and $x_6=19601$, etc.

$\therefore$ The required values of $n=1, 8, 49, 288, 1681, 9800$, etc.

III. Solution by the PROPOSER.

When $\frac{n(n+1)}{2}=\square$, one of the factors, n and $n-1$, is a square and the other two times a square. Being known *one* of the values of n in $\frac{n(n+1)}{2}=\square$, the value next succeeding as well as the value just preceding can be found by t following formula which I deduced by inspection :

$$\frac{n(n+1)}{2}=\left(2n_1+1\pm3\sqrt{\frac{n_1(n_1+1)}{2}}\right)^2$$

in which n_1 is a known value of n. By inspection we find that when $n=$ $\frac{n(n+1)}{2}=\square=1^2$. Now put $n_1=1$, and substituting in the formula, we obt $\frac{n(n+1)}{2}=6^2$ or 0^2, 6^2 being the $\square$ next succeeding and 0^2 the square just p ceding 1^2. From $\frac{n(n+1)}{2}=6^2$, we obtain $n=8(=2\times2^2)$, or $-9(=-3^2)$, ɪ $n+1=9(=3^2)$ or $-8(=-2\times2^2)$. Now put $n_1=8$, and substituting in the foı ula, we get $\frac{n(n+1)}{2}=(35)^2$ or $(-1)^2$, the positive value being the next succe ing square and the negative value the one just preceding, the latter being square with which we started. From $\frac{n(n+1)}{2}=35^2$, we find $n=49$ or -50, ɪ $n+1=50$ or -49. By continuing this process, we find the first six positive tegral values of n in $\frac{n(n+1)}{2}=\square$, to be 1, 8, 49, 288, 1681, and 9800.

IV. Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Allianc

Let $n=p^2$ or p^2-1, since it must be a perfect power, or a perfect po less 1. Then $\frac{n(n+1)}{2}=\frac{p^2(p^2\pm1)}{2}=a^2$; whence, $p^2\pm1=\frac{2a^2}{p^2}=2q^2$ ......(

Adding $2q^2+4pq+p^2$ to each member of equation (1), we have,

$2q^2+4pq+2p^2\pm1=4q^2+4pq+p^2$; or $(2q+p)^2\pm1=2(q+p)^2$ ....(

Since equations (1) and (2) are the same in form, if we find one set of tegral values for p and q in (1), we can then readily find succeeding values (2). Now, for $p=1$, $q=1$. ∴ Other values are : 3 and 2 ; 7 and 5 ; 17 and 41 and 29 ; 99 and 70 ; and so on. Then by formula $\frac{n(n+1)}{2}=\frac{p^2(p^2\pm1)}{2}$, first positive integral values of n are found to be 1, 8, 49, 288, 1691, 9800.

Also solved by *J. H. DRUMMOND, C. D. SCHMITT, H. C. WILKES, G. B. M. ZERR*, and *PROPOSER*.

ERRATA. On page 368 of December issue, line 4, for "(10+2)" r (10^m+2) ; line 9, at end, for "B^2" read $B_1{}^2$; line 12, for "B^2" read $B_1{}^2$, and

$B+1+A_1$)" read $(B+1-A_1)$; line 19, for "hypothenuse" read hypotenuse; ne 22, leave out comma after 6; line 26, for "p, b, d," read p, d, b,; line 30, for 13, 14, 15," read 13, 15, 14; page 369, line 8, for "from" read for; line 25, for the" read their; line 35, for "$a^{mn}+1$" read a^m+1; page 370, line 2, insert a mma before the sign of equality; and credit J. H. Drummond with a solution No. 32.

NOTES, CRITICISMS, ETC., BY ARTEMUS MARTIN, LL. D.

On page 285 Mr. Adcock gives "An Equation for the sum of Squares equal Square" which he says he has not seen published. I used the same method n the *Mathematical Magazine*, Vol. II., page 71, to find *three* square numbers hose sum is a square; and in a paper I had read at the last meeting of the American Association for the Advancement of Science I found in the same way *four* quares whose sum is a square. It is easily seen that the formula may e extended so as to find any number of squares whose sum is a square.

Note on Solutions of Problem 27, pp. 329-331.—In the *Mathematical Magazine*, Vol. II., No. 9, page 157, I have given a general method of finding any number (greater than two) of positive cube numbers whose sum is a cube, and on age 158 applied it to the case of five cubes, obtaining the set

$$6^3+11^3+13^3+18^3+20^3=26^3.$$

In Problem 42, p. 332, "$2a^2+2b^2-c^2+d^2$" should be $2a^2+2b^2=c^2+d^2$.

PROBLEMS.

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45. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Penn.

Solve the equation $x^3+y^3=a^3$.

46. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

Give a general solution, finding such values of a and b in $x^2+x\sqrt{xy}=a$ and $y^2+y\sqrt{xy}=b$ as will make x and y integral.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

27. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Penn.

Find the mean area of the *dodecagonal surface* formed by joining in order the points taken at random, one in each *sectoral triangle* of a regular inscribed dodecagon.

Solution by O. W. ANTHONY, Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Let AOB and BOC be two adjacent sectors of the regular dodecagon. Let the dodecagon be determined by its apothem$=a$.

Let $\angle P_1OB=\theta_1$, $\angle P_2OB=\theta_2$, $OP_1=\rho_1$, $OP_2=\rho_2$.

Then area of triangle $P_1OP_2=$

$$\tfrac{1}{2}\rho_1\rho_2\sin(\theta_1+\theta_2).$$

And average area of triangle=

$$\Delta=\tfrac{1}{2}\frac{\int_0^{\frac{1}{6}\pi}\int_0^{\frac{1}{6}\pi}\int_0^{OM}\int_0^{ON}\rho_1\rho_2\sin(\theta_1+\theta_2)d\rho_1d\rho_2d\theta_1d\theta_2}{\int_0^{\frac{1}{6}\pi}\int_0^{\frac{1}{6}\pi}\int_0^{OM}\int_0^{ON}d\rho_1d\rho_2d\theta_1d\theta_2}.$$

$$OM=a\sec(\theta_1-\frac{\pi}{12}).$$

$$ON=a\sec(\theta_2-\frac{\pi}{12}).$$

Then $$\Delta=\tfrac{1}{8}a^2\frac{\int_0^{\frac{1}{6}\pi}\int_0^{\frac{1}{6}\pi}\sec^2(\theta_1-\frac{\pi}{12})\sec^2(\theta_2-\frac{\pi}{12})\sin(\theta_1+\theta_2)d\theta_1d\theta_2}{\int_0^{\frac{1}{6}\pi}\int_0^{\frac{1}{6}\pi}\sec(\theta_1-\frac{\pi}{12})\sec(\theta_2-\frac{\pi}{12})d\theta_1d\theta_2}.$$

The numerator may be written

$$\int_0^{\frac{1}{6}\pi}\Big[\sec^2(\theta_2-\frac{\pi}{12})\int_0^{\frac{1}{6}\pi}\sec^2(\theta_1-\frac{\pi}{12})\sin[(\theta_1-\frac{\pi}{12})+(\theta_2+\frac{\pi}{12})]d\theta_1\Big]d\theta_2$$

The part under the last integral sign may be written, after expansion and some minor reductions,

$$\cos(\theta_2+\frac{\pi}{12})\int_0^{\frac{1}{6}\pi}\sec(\theta_1-\frac{\pi}{12})\tan(\theta_1-\frac{\pi}{12})d\theta_1+\sin(\theta_2+\frac{\pi}{12})\int_0^{\frac{1}{6}\pi}\frac{d\theta_1}{\cos(\theta_1-\frac{\pi}{12})}$$

$$=\cos(\theta_2+\frac{\pi}{12})\int_0^{\frac{1}{6}\pi}\sec(\theta_1-\frac{\pi}{12})+\sin(\theta_2+\frac{\pi}{12})\int_0^{\frac{1}{6}\pi}\log_e\frac{1+\tan\frac{1}{2}(\theta_1-\frac{\pi}{12})}{1-\tan\frac{1}{2}(\theta_1-\frac{\pi}{12})},$$

$$=0+2\sin(\theta_2+\frac{\pi}{12})\log_e\frac{1+\tan\frac{\pi}{24}}{1-\tan\frac{\pi}{24}}.$$

∴ The numerator may be written:

$$2\log_e\left(\frac{1+\tan\frac{\pi}{24}}{1-\tan\frac{\pi}{24}}\right)\int_0^{\frac{1}{6}\pi}\sec^2(\theta_2-\frac{\pi}{12})\sin(\theta_2+\frac{\pi}{12})d\theta_2.$$

The integral may be written

$$\int_0^{\frac{1}{6}\pi}\sec^2(\theta_2-\frac{\pi}{12})\left[\sin(\theta_2-\frac{\pi}{12})\cos(\theta_2-\frac{\pi}{12})\sin\frac{\pi}{6}\right]d\theta_2$$

=(after reductions similar to those above)

$$\frac{1}{2}\int_0^{\frac{1}{6}\pi}\frac{d\theta_2}{\cos(\theta_2-\frac{\pi}{12})}=\frac{1}{2}\log_e\left(\frac{1+\tan\frac{\pi}{24}}{1-\tan\frac{\pi}{24}}\right)$$

∴ The numerator reduces to

$$\left[\log_e\left(\frac{1+\tan\frac{\pi}{24}}{1-\tan\frac{\pi}{24}}\right)\right]^2.$$

It may also be shown that the denominator reduces to

$$\left[\log_e\left(\frac{1+\tan\frac{\pi}{24}}{1-\tan\frac{\pi}{24}}\right)\right]^2.$$

∴ $\Delta=\frac{1}{8}a^2$.

And the mean area of dodecagon = area of 12 such triangles $=\frac{3}{2}a^2$.

Solutions of this problem were received from G. B. M. Zerr and the Proposer, the latter furnishing [illegible] solutions.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him

SOLUTIONS OF PROBLEMS.

30. Proposed by R. J. ADCOCK, Larchland, Warren County, Illinois.

When the sum of the distances of a point of a plane surface, from all oth points, is a minimum, that point is the center of gravity of the plane surface.

IV. Discussion by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, N Windsor, Maryland.

I. Consider the following problem: Find a point within a plane surfa such that the sum of the n^{th} power of the distances to all other points of the su face shall be a minimum.

$$S=\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^n\ dxdy.$$

For minimum—

$$\frac{dS}{dx_1}=2n\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1}(x_1-x)dxdy=0\ \ldots\ldots\ldots\ldots(1$$

$$\frac{dS}{dy_1}=2n\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1}(y_1-y)dxdy=0\ \ldots\ldots\ldots\ldots(2$$

(1) and (2) may be satisfied in several ways.

(A). The curve may be such that the integration in question perform over the surface reduce to zero.

(B). $\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1}dxdy=0$, or,

$$\int\int\Big[(x_1-x)^2+(y_1-y)^2\Big]^{n-1}dxdy=$$

$$(C).\ \begin{cases}(x_1-x)dxdy=0,\ \text{or}\int\int(x_1-x)dxdy=C_1\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\\ (y_1-y)dxdy=0,\ \text{and}\int\int(y_1-y)dxdy=C_2\ldots\ldots\ldots\ldots\ldots\ldots(\end{cases}$$

We shall only consider (C), as it is the only one which leads to the cons eration of the center of gravity.

From (3) and (4), $x_1 = \dfrac{C_1 + \int\int x dx dy}{\int dx dy}$, and $y_1 = \dfrac{C_2 + \int\int x dx dy}{\int dx dy}$.

Therefore (x_1, y_1) is the center of gravity only when C_1 and C_2 are zero. For this condition to be fulfilled the first member of (3) and (4) must be evidently zero. From (1) and (2) we see that this will be true generally only when $n-1=0$, *i. e.*, $n=1$. Hence there can be no *general* proposition except for the sum of the squares of the distances.

II. $u = \int\int \left[(x_1-x)^2 + (y_1-y)^2\right]^n dx dy.$

$$\frac{du}{dx_1} = 2n\int\int \left[(x_1-x)^2 + (y_1-y)^2\right]^{n-1} (x_1-x) dx dy = 0,$$

$$\frac{du}{dy_1} = 2n\int\int \left[(x_1-x)^2 + (y_1-y)^2\right]^{n-1} (y_1-y) dx dy = 0.$$

Whence $x_1 = \dfrac{\int\int \left[(x_1-x)^2 + (y_1-y)^2\right]^{n-1} x dx dy}{\int\int \left[(x_1-x)^2 + (y_1-y)^2\right]^{n-1} dx dy}$,

and $y_1 = \dfrac{\int\int \left[(x_1-x)^2 + (y_1-y)^2\right]^{n-1} y dx dy}{\int\int \left[(x_1-x)^2 + (y_1-y)^2\right]^{n-1} dx dy}$.

For (x_1, y_1') to be identical with center of gravity $n-1$ must be zero.

III. Prof. Zerr in his proof has $S = \int D dA$. He then writes $\dfrac{dS}{dx_1} = \dfrac{(x-x_1)dA}{D}$. He should have written $\dfrac{dS}{dx_1} = \int \dfrac{x-x_1}{D}.\ dA$; the integration was with respect to A and the differentiation with respect to x_1, and the two do not destroy each other.

IV. The proposition fails to hold for the simplest case imaginable, an indefinitely narrow rectangle, or straight line. Thus let AB be a straight line, P any point on that line. $AP=a$, $AB=l$, $PQ=x$. Then the sum of distances from

A ——— P ——— Q ——— B

$P=S=\int_{-a}^{l-a} x\,dx=\frac{1}{2}[l^2-2al]$. $\frac{dS}{da}=-2l=0$ for minimum, *i. e.*, $l=0$ which is an absurdity. The sum of squares a minimum *will* hold in this case.

V. The same proof that Prof. Zerr gives will hold for *any* power of the distance, which proposition is highly improbable.

31. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Penn.

In order that a vertical cylindric stalk may be severed by a blow of minimum force, the stalk must be struck at what inclination by a sharp wedge-shaped blade?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Ark.-Tex.

Let $f\phi(\theta)$ = the force necessary to sever a unit of area, where θ is the inclination to the horizon. Let r = radius of stalk.

$\therefore$ The area of section made in cutting is $\pi r^2\sec\theta$, the area of an ellipse with semi-axes r and $r\sec\theta$. $\therefore \pi r^2 f\sec\theta\phi(\theta)$ = a minimum. This can be made a minimum when $\phi(\theta)$ is known. If $f\phi(\theta)=a+b\cos^2\theta$, then $\theta=\sin^{-1}\sqrt{1-\frac{a}{b}}$.

32. Proposed by S. H. WRIGHT, M. D., A. M., Ph. D., Penn Yan, New York.

Intermittent reflections of flashes of light on a clear sky after dark, indicated a storm was progressing *below* the horizon. Refraction of 34′ on the horizon, brought the upper edge of the storm-cloud up to the horizon, and was just visible. How far off was the storm if the cloud was one mile above the earth?

I. Solution by the PROPOSER.

In the plane triangle ABC, let C be the center of the Earth, A the place of the observer, and B that of the cloud. Then AC = Earth's mean radius = 3959 miles, $=b$, BC = 3960 miles, $=a$, $AB=c$, the required distance. The angle BAC = the nadir distance of the cloud, being $90^\circ-34'=89^\circ26'=A$. Then

$\sin B=\frac{b\sin A}{a}$. $\therefore B=88^\circ35'36''$, and $180^\circ-(A+B)=1^\circ58'24''=C$, and

$c=\frac{b\sin C}{\sin B}=\frac{a\sin C}{\sin A}=136.367$ miles.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Ark.-Tex.

Let A be the position of the observer, B the cloud, O the center of the earth, R = mean radius of the earth = 3958 miles.

$\therefore AC=2R\sin\frac{1}{2}AOC$. $\angle ACB=\frac{\pi}{2}+\frac{1}{2}AOC$, $\angle BAC=\frac{1}{2}AOC-34'$.

$$\therefore AC = \frac{BC\cos(AOC-34')}{\sin(\frac{1}{2}AOC-34')} \ldots\ldots\ \ldots\ldots\ldots\ldots (1).$$

$$AB = \frac{BC\cos\frac{1}{2}AOC}{\sin(\frac{1}{2}AOC-34')} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$BC=1$ from (1). $\quad 2R\sin\frac{1}{2}AOC = \dfrac{\cos(AOC-34')}{\sin(\frac{1}{2}AOC-34')}$.

$\therefore \cos(AOC-34') = \dfrac{(1+R)\cos 34'-1}{R+1}$. $\therefore$ $AOC = 1°58'16''$. From (2) $AB = 136.778$ miles.

ERRATA. On page 56, second line from top, for "$-2562Z^2$" read $-256Z^2$; sixth line from top for "$Z=2.750413$" read $Z=2.750458368$, and for "$WR=1.5248$" read $WR=1.2963390864$; and in Note, second line from bottom, for "$z=WR$" read $x=WR$.

PROBLEMS.

29. Proposed by S. H. WRIGHT, M. D., A. M., Ph. D., Penn Yan, New York.

In latitude $42°30'$ north$=\lambda$, at what angle with the horizon, will the sun rise, its declination$=22°$ north$=\delta$?

30. Proposed by SETH PRATT, C. E., Assyria, Michigan.

The pendulum of a clock which gains 6 seconds in 1 hour and 13 minutes, makes 6000 vibrations in 1 hour and $9\frac{1}{2}$ minutes. What is the length of the pendulum? And what length should it have to keep true time?

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

The Origin of π.—At least from the time of Archimedes π has stood for the number expressing how many times the diameter the circumference is. It is the initial letter of the Greek word περιφέρεια, meaning periphery. If the diameter is taken as a unit, then π stands for the periphery, or circumference. This is in reply to query of Lottie Smith in December Number.

BENJ. F. YANNEY,
Mount Union College, Alliance, Ohio.

An Expression for π.—Though the result is not new, I have not seen it developed as follows:

Since $e^{i\pi}=-1$, $\therefore\ i\pi\log e=\log(-1)$.

$$\therefore\ \pi=\frac{\log(-1)}{\sqrt{-1}}.$$

BENJ. F. YANNEY.

Referring to the Note of R. Greenwood in the December Number, I would state that (1) probably the other root was infinite. Thus the equations $x^2-y^2=5$ and $x+y=5$ have roots $x=3$ or ∞, and $y=2$ or $-\infty$. (2) The proof that imaginary roots enter in pairs assumes that all the coefficients are real. The equation $x^2-bix=a^2-abi$ has roots: a and $-a+bi$ but its coefficients are not all real. (3) The equation $\sqrt{2x_2-2}-(3x-5)=0$ or A must be multiplied by $\sqrt{2x^2-2}+(4x-5)=B$ or 0 to give a quadratic equation. The given equation is not of the second degree as Mr. Greenwood seems to imply but of the $\frac{1}{2}$ degree. An infinite number of equations can be written that have no roots at all: for instance, $2x-5+\sqrt{x^2-7}=0$ (or ?). This when combined with its congeneric $2x-5-\sqrt{x^2-7}=?$ or 0 gives a quadratic; the last expression takes both roots leaving no root for the first form. The demonstration that every equation has a root referred to equations free from surds.

H. C. WHITAKER,
Manual Training School, Philadelphia.

Another Reply. By squaring the equation, we get $2x^2-2=(3x-5)^2\ldots(1)$ $=2x^2-2-(3x-5)^2=0$, which is equivalent to transposing the member $3x-5$, and then multiplying the equation by $\sqrt{2x^2-2}+(3x-5)=0$. By doing this we have really introduced a new equation, which is satisfied for $x=\frac{7}{7}$.

Observe that (1) is satisfied for both $x=3$, and $x=\frac{7}{7}$, for it contains both the original equation and the one introduced by the questionable operation of squaring. Therefore, if the given equation means the positive root of $(2x^2-2)=3x-5$, then 3 is the only value of x that will satisfy it.

If $\pm\sqrt{2x^2-2}=3x-5$, both 3 and $\frac{7}{7}$ will.

BENJ. F. YANNEY,
Mount Union College, Alliance, Ohio.

Note on Solution IV., Page 190. It does not follow that triangles AEL and ADK are equal because the triangles AEL and ADK are similar respectively to AFN and AGM, and the solution fails.

I would like to see a direct proof of this problem. It is said that the mathematician Todhunter failed to produce a direct proof of it.

GEORGE LILLEY,
394 Hall Street, Portland, Oregon.

Problem by Euler. One answer is given, but he adds there are many more. Legendre asks for a general solution as Euler's solution is lost: and he says such a solution would be very much prized by mathematicians, if it could be given.

1st. The sum of the squares of each, horizontal, vertical, or diagonal rows shall be equal,—10 conditions.

A	B	C	D
E	F	G	H
I	K	L	M
N	O	P	Q

68	-29	41	-37
-17	31	79	-32
59	28	23	61
-11	-77	8	49

=Euler's Numbers.

2nd. The sum of their products taken two and two=0, taking any two rows, horizontal, vertical, and diagonals,—12 conditions.

$$AE+BF+CG+DH=0=AD+FG+KL+NQ, \text{ etc.}$$

HILLSBORO, ILL., MATHEMATICAL CLUB.

Note on No. 4—Miscellaneous. In regard to No. 4 Miscellaneous, I had worked the problem with the assumption made by Prof. Hume, but rejected my solution, as on further thought I did not consider the assumption warranted. The constantly changing curvature carries with it a change in actual contact as well as in the amount ground off, which I have not been able to analyze. The assumption made would seem to apply if the stones were kept pressed together with such a force as would not yield, and would cause the particles to overlap for a constant distance. This also would require a constantly changing pressure or adjustment.

I should like to ask whether any one knows of a principle which will apply to the effect of friction in a case of this kind.

C. W. M. BLACK,
Wesleyan Academy, Wilbraham, Mass.

Query. Is a man who writes for publication in a Mathematical Magazine a "Note on Helmholtz's use of the terms 'Surface' and 'Space' as identical in meaning", properly to be considered sane?

Again when he asks "Does the 'immortal' Helmholtz in his Lectures on the—'Origin and Significance of Geometrical Axioms'—use the terms 'surface' and 'space' as identical in meaning?" since Helmholtz never delivered any lectures under this title, would it be sane to attempt to answer?

G. B. HALSTED.

The equation from Bell's Algebra, quoted by Mr. Greenwood, (MONTHLY, Vol., p. 372) is consistent if the radical be given the double sign. The equation should be

$$\pm\sqrt{2x^2-2}=3x-5.$$

The value $x=3$ belongs to the upper sign, $x=\frac{9}{7}$ to the lower.

WM. E. HEAL.

The answer to query (Monthly, Vol. II., p. 247) is not satisfactory. It is true "We *have* no method of finding the cube root by means of a compass" [and rule] but that does not prove the *impossibility* of a solution. What I wish, is a rigorous proof of the impossibility of expressing the roots of a cubic equation by a geometrical construction.

WM. E. HEAL.

Concerning the value of *factorial zero*, Chrystal says (Text Book of Algebra, Part II., page 4) "Strictly speaking 0! has no meaning. It is convenient, however, to use it, with the understanding that its value is 1; by so doing we avoid the exceptional treatment of initial terms in many series."

WM. E. HEAL.

IS THERE MORE THAN ONE ILLIMITABLE SPACE?

The Metageometers assume without proof that there are many varieties of space, differing in curvature, in the number of dimensions and in extent. Is their assumption axiomatic or does it need proof? Is it not really inconsistent with the hypothesis that space is everywhere and illimitable?

The Metageometers concede that the space that contains our Universe may for aught they know to the contrary, be trinally extended, i. e., through any point of it, whatever, three straight lines may be drawn mutually at right angles to each other. Notwithstanding this concession, they assume that there are two varieties of space at least, the number of whose dimensions is less than three.

They call a surface a variety of space that has *two* dimensions, and a line a variety of space that has *one* dimension.

The Euclidian geometers locate all their lines and surfaces in the one, trinally extended, illimitable space. They do not regard these lines and surfaces as distinct varieties of space that may be classed under an n-fold species.

Some of the Metageometers call a line one dimensional space, and a surface two dimensional space, apparently with the expectation that this ambiguous use of the word space will somehow assist them in ascending from our tridimensional space to a hypothetical one of four dimensions, and from that to one of five dimensions, and so on. This is certainly a most hazardous enterprise that they have undertaken. They are attempting to scale the transcendental heights of Hyper-space with an analogical ladder constructed out of defective timber. The two bottom rounds—one dimensional space and two dimensional space—are unable to endure the strain put upon them. We do not mount to trinally extended space from surfaces, nor to surfaces from lines. But we start with trinally extended space and in it locate surfaces and lines.

Successful ascent cannot be made from tridimensional space to fourdimensional space.

1st.—Because no one knows or can know the direction from 3-fold space to 4-fold, even if the latter exists.

2nd.—Because no one knows or can know that 4-fold space exists for the reason that the fundamental laws of thought are violated in every effort of the mind to cognize it. Legitimate thinking cannot proceed in violation of logical law, but stultification may do so. The so-called "generalized space" of the Metageometers is believed to be the joint product of pseudo-generalization, pseudo-analogical reasoning, and pseudo-analytical interpretation.

JOHN N. LYLE.

BOOKS AND PERIODICALS.

Trigonometry for Schools and Colleges. By Frederick Anderegg, A. M., Professor of Mathematics, and Edward Drake Roe, Jr., A. M., Associate Professor of Mathematics in Oberlin College. 8vo. Cloth, 108 pp. Boston: Ginn & Co.

This little work is a decided improvement over most modern treatises on trigonometry. It treats the subject with clearness and accuracy and leads the student to an easy acquaintance with modern higher mathematics. A number of new features are introduced. This is the first book we have yet seen in which it is shown that Plane Trigonometry is a special case of Spherical Trigonometry. Many other subjects of equal interest and importance are discussed. The authors deserve much credit for this original and unique work.

B. F. F.

An Elementary Treatise on Rigid Dynamics. By W. J. Loudon, B. A., Demonstrator in Physics in the University of Toronto. 8vo. Cloth, 236 pp. Price, $2.25. New York: Macmillan & Co.

This is a most excellent treatise on Rigid Dynamics. The subjects treated are made very clear and the student is still further aided in grasping those complex and difficult principles by very beautiful and accurate diagrams. Any student who has mastered the calculus can take up this work without any difficulty. At the close of each subject is a list of problems. The book closes with 306 problems all of which are very interesting to the student of dynamics. Some of these excellent problems will appear in future numbers of the MONTHLY.

B. F. F.

Notations de Logique Mathématique. Par G. Peano, Professeur d'Analyse infinitésimale à l'Université de Turin. Introduction au Formulaire de Mathématique Publie par la *Revista di Matematico*, Turin. Pamphlet, 52 pages.

A very interesting and valuable treatment of the notations of mathematical logic.

B. F. F.

Periodico di Mathematica. By L'Insegnamento Secondario. Pubblicato per cura di Aurelio Lugli, Professor di matematica nel R. Istituto tecnico di Roma.

The January-February number of this magazine contains a number of important papers and the solutions of 7 problems. B. F. F.

El Progreso Mathemático Periodico de Mathemáticas Puras y Aplicadas. Director D. Zoel G. de Galdeano, Catedrático de Geometria Analica en la Universidad de Zaragoza.

In this journal are published problems which are proposed by the best mathematicians in the world. The solutions are illustrated by beautiful diagrams. B. F. F.

Annals of Mathematics. Ormond Stone, Editor, Office of Publication, University of Virginia. Bi-monthly. Price, $2.00.

The September (1895) number contains the following articles: On the Improbability of Finding Shoals in the Open Sea by Sailing over the Geographical Positions in which they are Charted. By Mr. G. W. Littlehale. Note on the Congruence $2^{4n}\equiv(-)^n (2n)!/(n!)^2$, where $2n+1$ is a prime. By Prof. Frank Morley. Equations and Variables Associated with the Linear Differential Equation. By Dr. Geo. F. Metzler. The Calculus of Variations. By Dr. Harris Hancock. B. F. F.

March Monthly Magazine Number of The Outlook. Price, $1. per year in advance. The Outlook Company, 13 Astor Place, New York.

The illustrated monthly "Magazine Number of *The Outlook* for March has nearly fifty pages of reading matter, and more illustrations than any of the previous issues. Dr. R. L. Dickinson writes as an expert on hygienic and practical aspects of "Bicycling for Women," with cuts showing just what is right and wrong about women's riding; Edward Everett Hale tells of the "Higher Life of Boston;" there is a pleasant "Spectator" talk about picturesque New Orleans; Charleston of to-day is compared with its ante-bellum life in Mr. W. J. Abbot's "From Atlanta to the Sea;" Martin Luther is the subject of a fine article by professor Harnack, the great German theologian; and Mr. A R. Kimball has a readable article about Penzance and the Newlyn school of artists. All these articles are fully illustrated. Ian Maclaren's novel gains in interest and humor.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year. Single number, 10 cents. Irvington-on-the-Hudson, New York.

The General of the Army, the General commanding the U. S. Corps of Engineers, Vice-Pres. Webb of the New York Central, and John Jacob Astor, compose *The Cosmopolitan Magazine's* Board of Judges to decide the merits of the Horseless Carriages which will be entered in the May trials, for which the *The Cosmopolitan* offers $3000 in prizes. This committee is undoubtedly the most distinguished that has ever consented to act upon the occasion of the trial of a new and useful invention. The interest which these gentlemen have shown in accepting places upon the committee is indicative of the importance of the subject, and that the contest itself will be watched with marked interest on both sides of the Atlantic. Frank Stockton's new story, "Mrs. Cliff's Yacht," which begins in the April *Cosmopolitan*, promises to be one of the most interesting ever written by that fascinating story-teller. Readers of "The Adventures of Captain Horn" will find in "Mrs. Cliff's Yacht" something that they have been waiting for.

JOHN NEWTON LYLE.

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BIOGRAPHY.

JOHN NEWTON LYLE.

BY F. P. MATZ, SC. D., PH. D., PROFESSOR OF MATHEMATICS AND ASTRONOMY IN IRVING COLLEGE, MECHANICSBURG, PENNSYLVANIA.

JOHN NEWTON LYLE was born in Ralls County, Missouri, March 5, 1836.

"The space in which this county is located is trinally extended, and therefore objective. It has no curvature, either positive or negative. Here planes are flat, and perpendiculars to a transversal are equidistant.

If Lobatschewsky had been born and raised in Ralls County, he would perhaps never have doubted *that two straight lines equidistant from each other may be drawn in the same plane*, nor written a theory of parallels in which this postulate of sound geometry is discredited. The hills, rocks, streams, trees, and hard-pan of Ralls County exist in tridimensional space,—objective to the minds of the inhabitants who till the soil, feed the herds, quarry the rock, fell the trees, and hunt the wild game. These physical objects are realities having an *objective* existence; and the space occupied by them, and that in which they are contained, is also an objective *entity* although not a *material* one. It is believed that Helmholtz, was right in maintaining against Kant the objectivity of space, but wrong in regarding it as a *physical thing* to be moulded like potter's clay. Sir Isaac Newton held the opinion that space was immaterial, immovable, and unalterable, as well as trinally extended, continuous, and unbounded.

Immanuel Kant professed to cognize a real, objective, *extended* world as existing in *a space that*, according to his philosophy *had no existence outside of his own mind*. It would seem that if there is an extended world, there must be an extended spatial entity to contain it. If space is not *extended*, and therefore not objective, there can be no real, extended, objective, material world.

If Immanuel Kant's experiences in early life had been those of a pioneer's son in Ralls County, Missouri, he would not in all probability have undertaken in his riper years the contract of building a *real world* in a *non-existent* place.

If Fichte, or Hegel, had ever galloped after a wild steer for half a day through a Ralls County forest or been thrown from a bucking mustang on the phenomenal hard-pan of northeastern Missouri, they doubtless would not have felt inclined to regard this real matter-of-fact world as *an idealistic dream.*"

The ancestors of John Newton Lyle, on the paternal side, came from northern Ireland; and on the maternal side, from England and Wales. They settled in Berkeley County, Virginia, in the last century; and many of their descendants are still to be found there.

Samuel Oldham Lyle, the father of the subject of this sketch, emigrated from Berkeley County, Virginia, in 1832, to Ralls County, Missouri, where he purchased a farm; married, Ann Rebecca, the daughter of William Gerard, and reared his family. This pioneer couple were intellectual in their tastes, great readers, ambitious to make a pleasant home for their children, and give each of them an education.

Ann Rebecca's father, William Gerard, emigrated from Berkeley County, Virginia, to Kentucky, during the last decade of the last century, where he learned the printing business, and edited and published for many years the *Argus*, a newspaper, at Frankfort, Kentucky. He was a man of affairs, as well as of extensive reading; and he was also a practical politician intimately associated with the statesmen of his adopted State. He came with his family to northeastern Missouri, in 1830.

John Newton Lyle, in his early boyhood, was thrown in his Grandfather Gerard's society a great deal, and received from him powerful impulses towards intellectual pursuits. The venerable man treated his grandson more as a companion than as a boy needing a rod for his misdemeanors, aroused his curiosity by well-directed questions, corrected his mistakes, and entertained him with anecdotes about Amos Kendall, the elder Blair, and Henry Clay.

Samuel Oldham Lyle was an enterprising, independent, and fearless pioneer, passionately fond of the chase and free life in the wild west; but, at the same time, he was diligent in his farming and stock-raising. He was a man of quick intelligence, unfailing memory, and sound judgment, who appreciated the value and importance of education; and he gave to his children the best school advantages that his circumstances would allow.

Young Lyle, at six years of age, was placed in a district school; and here he remained until he reached the age of twelve. In November, 1848, he entered a classical school taught by the Rev. William T. Dickson, at West Ely, Marion County, three miles from the home of the young pupil. He studied the rudiments of the Latin and Greek languages, Euclid's Elements, and Day's Algebra.

Mr. Dickson was a native of the State of Maine, and came West with Dr.

Ezra Styles Ely, to attend Marion College, some time in the Thirties. "He was an enthusiastic and successful instructor in the branches of learning that he professed to teach. He did not tell his scholars anything about differential coefficients, integrals, or Cartesian co-ordinates. He was silent as to determinants, trilinears, and Non-Euclidean Geometry. He did understand Euclid's Elements, however ; and he taught the science, clearly, thoroughly, and ably. With him, straight lines were never *flexed* or *curved*. Tangents to circumferences were never confounded with the curves to which they were tangent. Planes were flat superficies ; and, in no instance, were they spherical or pseudo-spherical. He showed the *meaning* of demonstration, by *demonstrating* theorems. He illustrated by practical examples, the difference between direct and indirect proof. He was a true teacher, and succeeded well in imparting to his pupils something of his own appreciation and admiration of the enduring work of Alexandria's immortal geometer."

In the fall of 1851, within three miles of Samuel Oldham Lyle's farm, Van Rensselaer Academy was founded, at the head of which was the Rev. J. P. Finley, afterwards a professor in Westminster College, and the founder of a classical institution at Brookfield, Missouri. John Newton Lyle entered this Academy, in October, 1851 ; and he was a student there three successive winters, farming during the summer. Here he continued his studies in Latin and Greek, reviewed Euclid, then took up Davies' Legendre and Robinson's Algebra. At this time, he, also, studied Trigonometry and Surveying. Mr. Finley's tastes were classical, rather than mathematical ; and his pupil, J. N. Lyle, whilst at the Academy, devoted his energies almost exclusively to mastering the Latin and Greek texts put into his hands.

Before he was nineteen years of age he took charge of his first school in Monroe County, early in September, 1854 ; thus he began his long career as a teacher, which he has continued almost uninterruptedly until the present time. He worked with the definite plan of preparing for College and earning the funds necessary for securing a collegiate education. He taught two years in the public schools of Monroe County, spending his evenings and Saturdays in study.

During these years he plodded without assistance, through Davies' Analytical Geometry. Having finished this self-imposed task, a strong desire took possession of him to advance farther and investigate Davies' Differential and Integral Calculus. Accordingly one sultry day in August, 1856, he rode from his father's farm to Hannibal, purchased the book, and on returning home immediately sought a secluded spót in the forest and began the study of the first differential coefficient as explained by Charles Davies. He was *thoroughly disgusted* that hot August afternoon, with Davies' description of differentials as the "traces" of vanishing increments. He persevered, however, notwithstanding his dissatisfaction with the author's theory of differentials and differential coefficients. A copy of Loomis's Calculus, which came in his way, was eagerly studied. Loomis's theory of differentials as *rates of variation* had the advantage of being intelligible, and certainly offered something more substantial to be grasped by the mind than a mere "trace" of a vanishing increment or the "ghost of a departed quantity."

"Rates of variation are *finite* quantities. If differentials are rates of variation, then, of necessity, they must be definite quantities. The Leibnitzian hypothhesis that differentials are *infinitely small quantities* contradicts the hypothesis that they are rates of variation." During the fall of 1856, he studied both Loomis and Davies on the Calculus. This work was done entirely without the instruction of a teacher; because there was no one within reach, who had studied these branches, to whom he could apply for aid. "*This method of study*, whilst laborious and beset with many inconveniences, *was conducive to independence of thought and action, and the formation of the habit of self-reliance.*"

The first part of the year 1857, John Newton Lyle taught mathematics in Bethel College, a Baptist Institution located at Palmyra, Missouri. The opportunity of attending Marietta College, for which he had long planned and toiled, now presented itself. On examination he entered the Junior Class in Marietta College, the fall of 1857; and he continued in that Institution, until his graduation in 1859.

President Israel Ward Andrews conducted the examination in Mathematics, and expressed himself as highly gratified with the candidate's proficiency; and on making inquiry as to who taught him Analytical Geometry, seemed amused when informed that his only instructor was the youthful pedagogue before him seeking admittance to the privileges of the College. Dr. Andrews was his warm and steadfast friend, from the date of that morning's interview on Mathematics.

J. N. Lyle, in College, sought to utilize the advantages of the library and his literary society as well as those of the recitation-room and the laboratory. His special delight was to participate in the Saturday-morning debates held in the hall of the Alpha Kappa Society. The enjoyableness of the excitement far outweighed the unpleasantness of the collisions incident to such exercises.

He lost no time in obtaining from the College Library De Morgan's Differential and Integral Calculus, in order to learn that author's opinions respecting the principles of the science. "He was interested in noting that De Morgan employed variables that increased, and decreased, indefinitely without limit, instead of the hierarchy of infinitely great, and infinitely small, quantities of the Leibnitzian hypothesis. Whilst no lost value was attributed to these variables, every value that they did have, was *finite*. The hypothesis of increasing, and decreasing, variables having finite values not only works well in practice, but has the advantage over the hypothesis of Leibnitz in that it is intelligible and does not involve contradiction. It also harmonizes well with the view that differentials are rates of variation. Further, in considering a limit, we note that the interval between a limit and the variable that approaches it, is itself a variable that decreases without limit. From this point of view, the absurdity of regarding a variable that increases without limit as having a limit appropriately symbolized by ∞, is quite evident.

No benefit accrues to the Science of the Calculus, from De Morgan's hypothesis that there are *two* kinds of zeros—*the absolute zero*, and *the indifinitely small quotient*. The absolute zero is destitute of all value ; in fact, it is the negation of quantity,—and hence can not be treated as quantity, without violating the logical law of Non-Contradiction. A quotient may become indefinitely small, but can not become so small as not to be quantity. To name a quotient zero, is manifestly a misnomer. Mathematical and logical confusion is liable to result from the ambiguous use of the symbol 0. Treating quantity as no quantity, or no quantity as quantity, is a procedure which may be profitably dispensed with in Mathematics."

E. W. Evans, of Yale, came to Marietta College, as Professor of Mathematics, at the same time that J. N. Lyl8 entered as a student. The young professor seemed very lonesome as his wife remained in the East that fall. He would come over to Lyle's room of evenings and remain for hours. His conversation which took a wide range was quite instructive to his western pupil. Mathematics was discussed a great deal, but not exclusively. He believed most religiously that "brevity is the soul of wit." He once said : "Lyle, the longer I live the more I like 'short things'." His pupil furnished his share of the intellectual picnic with anecdotes and experiences respecting that portion of the West where Mark Twain was born, Tom Sawyer flourished, and Captain Sellers bored with a big auger.

The two years immediately after graduation he spent in teaching in Pettis and Morgan Counties, Missouri. His leisure hours were occupied in reading law-books. In the spring of 1862, he was offered the chair of Mathematics and Natural Science in Westminster College, a position he held until 1865, when he went to Carondelet, a suburb of St. Louis, where he taught a Grammar School ; but in the fall of the same year, he accepted the position of Acting Professor of Mathematics and Natural Science in his *Alma Mater*, Marietta College. He continued there three years ; at the expiration of which time, he returned to Fulton, as Professor of Natural Science in Westminster College. Here he has since remained. First and last, as the exigencies of College-work might require, he has taught branches in nearly every department of the Institution.

He is an active member of *The Missouri Teacher's Academy*. To educational journals he has contributed hundreds of articles principally on Mathematical Philosophy. During the last three years preceding 1890, he had published in the *Missouri School Journal* not less than sixty-one articles. He has written an unpublished manuscript on the Differential and Integral Calculus.

The degree of Ph. D., was conferred on him by Marietta College, in 1881. In 1868, Professor Lyle was married to Miss Margaret T. Hays, daughter of John B. Hays, M. D., of Marion County, Missouri, who until her death, December 26, 1882, in spite of ill health and great suffering, led such a life of unselfish devotion to husband, children, and friends, as called forth constant admiration of the talent, energy, and piety, that enabled her to accomplish so much. Three of the five children of this couple are living, two daughters and a son, Rev. Edward

Hays Lyle, an alumnus of Westminster College, a Theological student of Princeton Seminary for two years, and at present a minister in charge of a church at La Junta, Colorado.

Dr. Lyle, in 1884, married his second wife, Miss Mattie E. Grant, a scholarly and cultured lady, of Bardstown, Kentucky.

Dr. Lyle has been for many years an Elder in the Presbyterian Church, the church of his ancestors for, at least, the century and a half that have elapsed since his Great Grandfather emigrated from the northern part of Ireland to Berkeley County, Virginia.

THE CENTROID OF AREAS AND VOLUMES.

By G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

[Concluded.]

We will now find the centroid of the eighth part of the surface

$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$, I, when $c=b$, II, $c=a$.

We have $\bar{x}=\dfrac{\int xds}{\int ds}$, $\bar{y}=\dfrac{\int yds}{\int ds}$, $\bar{z}=\dfrac{\int zds}{\int ds}$.

$$\text{I.}\quad s=\frac{b}{a}\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\left\{\frac{a^4-(a^2-b^2)x^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}}dxdy$$

$$=\frac{\pi b}{2a^2}\int_0^a\sqrt{a^4-(a^2-b^2)x^2}\,dx=\tfrac{1}{4}\pi b\left(b+\frac{a}{e}\sin^{-1}e\right).$$

$$s.\bar{x}=\int xds=\frac{b}{a}\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\left\{\frac{a^4-(a^2-b^2)x^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}}xdxdy$$

$$=\frac{\pi b}{2a^2}\int_0^a \sqrt{a^4-(a^2-b^2)x^2}\,xdx = \frac{\pi ab(a^2+ab+b^2)}{6(a+b)}$$

$$\therefore \bar{x} = \frac{2a(a^2+ab+b^2)}{3(a+b)(b+\frac{a}{e}\sin^{-1}e)}.$$

$$s.\bar{y} = s.\bar{z} = \int y ds = \frac{b}{a}\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \left\{\frac{a^4-(a^2-b^2)x^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}} ydxdy$$

$$\frac{b^2}{a^2}\int_0^a \sqrt{(a^2-x^2)(a^2-e^2x^2)}dx = ab^2\int_0^{\frac{1}{2}\pi} \sqrt{1-e^2\sin^2\theta}\cos^2\theta d\theta,\ x=a\sin\theta$$

$$=\frac{ab^2}{3e^2}\left\{(1+e^2)E(e,\frac{\pi}{2})-(1-e^2)F(e,\frac{\pi}{2})\right\}.$$

$$\therefore \bar{y} = \bar{z} = \frac{4ab\left\{(1+e^2)E(e,\frac{\pi}{2})-(1-e^2)F(e,\frac{\pi}{2})\right\}}{3\pi e^2(b+\frac{a}{e}\sin^{-1}e)}.$$

II. $$s = \frac{a}{b}\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}} \left\{\frac{b^4+(a^2-b^2)y^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}} dydx$$

$$=\frac{\pi a}{2b^2}\int_0^b \sqrt{b^4+(a^2-b^2)y^2}\,dy = \frac{\pi a^2}{4}\left\{1+\frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right\}.$$

$$s.\bar{x} = s.\bar{z} = \int x ds = \frac{a}{b}\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}} \left\{\frac{b^4+(a^2-b^2)y^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}} xdydx.$$

$$s.\bar{x} = s.\bar{z} = \frac{a^2}{b^3}\int_0^b \sqrt{(b^2-y^2)(b^4+a^2e^2y^2)}\,dy$$

$$= a^2\int_0^{\frac{1}{2}\pi} \sqrt{b^2+a^2e^2\cos^2\theta}\,\sin^2\theta d\theta,\ y=b\cos\theta$$

$$= a^3\int_0^{\frac{1}{2}\pi} \sqrt{1-e^2\sin^2\theta}\,\sin^2\theta d\theta$$

$$= \frac{a^3}{3e^2}\left\{(1-e^2)F(e,\frac{\pi}{2})-(1-2e^2)E(e,\frac{\pi}{2})\right\}.$$

$$\therefore\ \bar{x}=\bar{z}=\frac{4a\left\{(1-e^2)F(e,\frac{\pi}{2})-(1-2e^2)E(e,\frac{\pi}{2})\right\}}{3\pi e^2(1+\frac{1-e^2}{2e}\log\frac{1+e}{1-e})}.$$

$$s.\bar{y}=\frac{a}{b}\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}}\left\{\frac{b^4+(a^2-b^2)y^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}}ydydx=\int yds$$

$$=\frac{\pi a}{2b^2}\int_0^b\sqrt{b^4+a^2e^2y^2}\,ydy=\frac{\pi ab(a^2+ab+b^2)}{6(a+b)}.$$

$$\therefore\ \bar{y}=\frac{2b(a^2+ab+b^2)}{3a(a+b)(1+\frac{1-e^2}{2e}\log\frac{1+e}{1-e})}.$$

Since the limit of $\frac{\sin^{-1}e}{e}$ and $\frac{\log\frac{1+e}{1-e}}{2e}$ is 1 when $e=0$ we have, in eith case, when $a=b$, $\bar{x}=\bar{y}=\bar{z}=\frac{1}{2}a$. The surface of the fourth part of the parabolo $x^2+y^2=2a^2z$, for $z=h$.

$$s=\int\int\sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dz}\right)^2}\,dzdx=\int\int\sqrt{1+\frac{x^2}{y^2}+\frac{a^4}{y^2}}\,dxdz.$$

$$\therefore\ s=a\int_0^h\int_0^{a\sqrt{2z}}\sqrt{\frac{a^2+2z}{2a^2z-x^2}}\,dzdx=\frac{\pi a}{2}\int_0^h\sqrt{a^2+2z}\,dz$$

$$=\frac{\pi a}{6}\left\{(a^2+2h)^{\frac{3}{2}}-a^3\right\}.$$

$$s.\bar{x}=s.\bar{y}=\int yds=a\int_0^h\int_0^{a\sqrt{2z}}\sqrt{a^2+2z}\,dzdx=a^2\int_0^h\sqrt{(a^2+2z)2z}\,dz$$

$$=\frac{a^2}{16}\left\{2(a^2+4h)\sqrt{2a^2h+4h^2}-a^4\log\left(\frac{a^2+4h+\sqrt{2a^2h+4h^2}}{a^2}\right)\right\}.$$

$$\therefore\ \bar{x} = \bar{y} = \frac{3a\left\{2(a^2+4h)\sqrt{2a^2h+4h^2} - a^4\log\left(\frac{a^2+4h+2\sqrt{2a^2h+4h^2}}{a^2}\right)\right\}}{8\pi\{(a^2+2h)^{\frac{3}{2}} - a^3\}}.$$

$$s.\bar{z} = \int z ds = a\int_o^b\int_o^{a\sqrt{2z}}\sqrt{\frac{a^2+2z}{2a^2z-x^2}}\, zdzbx$$

$$= \frac{\pi a}{2}\int_o^h\sqrt{a^2+2z}\, zdz = \frac{\pi a}{30}\left\{(3h-a^2)(a^2+2h)^{\frac{3}{2}} + a^5\right\}.$$

$$\therefore \bar{z} = \frac{(3h-a^2)(a^2+2h)^{\frac{3}{2}} + a^5}{5\{(a^2+2h)^{\frac{3}{2}} - a^3\}}.$$

The surface of the fourth part of the cone $x^2+y^2=a^2z^2$, for $z=h$.

$$s = \int\int\sqrt{1+\frac{x^2}{y^2}+\frac{a^4z^2}{y^2}}\, dzdx = a\sqrt{1+a^2}\int_o^h\int_o^{az}\frac{zdzdx}{\sqrt{a^2z^2-x^2}}$$

$$= \frac{\pi a\sqrt{1+a^2}}{2}\int_o^b zdz = \frac{\pi ah^2\sqrt{1+a^2}}{4}.$$

$$s.\bar{x} = s.\bar{y} = \int yds = a\sqrt{1+a^2}\int_o^b\int_o^{az} zdzdx = a^2\sqrt{1+a^2}\int_o^h z^2 dx = \frac{a^2h^3\sqrt{1+a^2}}{3}.$$

$$\therefore\ \bar{x} = \bar{y} = \frac{4ah}{3\pi}.$$

$$s.\bar{z} = \int zds = a\sqrt{1+a^2}\int_o^h\int_o^{az}\frac{z^2 dzdx}{\sqrt{a^2z^2-x^2}} = \frac{\pi a\sqrt{1+a^2}}{2}\int_o^h z^2 dz = \frac{\pi ah^3\sqrt{1+a^2}}{6}.$$

$$\therefore\ \bar{z} = \frac{2h}{3}.$$

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from March Number.]

Having disposed of the regular primitive groups we turn now to those whose order exceeds their degree. We have proved that all of these involve n conjugate subgroups whose degree is at most equal to $n-1$. Suppose the degree of these subgroups were $n-2$. Without loss of generality we may then assume that the following identities are satisfied :

$$G_1 \equiv G_2,\ G_3 \equiv G_4,\ \ldots\ldots\ G_{n-1} \equiv G_n.$$

If g_1 represents the order of G_1 we see that $2g_1$ substitutions of G transform G_1 into itself, viz., those which replace a_1 by itself and those which replace a_1 by a_2. All of the g_1 substitutions which replace a_1 by a_2 must therefore also replace a_2 by a_1, *i. e.*, contain the cycle a_1a_2. Similar remarks apply to the other couples $a_3a_4, \ldots\ldots a_{n-1}a_n$.

We inquire whether these couples may be used as systems of non-primitivity. We have already proved that every substitution that replaces one letter of a couple by the other contains the couple as a distinct cycle. It remains to show that the couples are interchanged as units by the substitutions of G. Suppose one of these substitutions t replaces a_1 by a_3. Then will

$$t\,G_3 t^{-1} = G_1$$

t must therefore replace a_2 by a_4. Since similar remarks apply to the other couples we have proved that the couples can be used as systems of non-primitivity.

In an exactly similar way we can prove the general case that if the degree of the conjugate subgroups is $n-\alpha$ $(\alpha > 1)$ then systems of α letters each may be used as systems of non-primitivity. Hence the

THEOREM. *Whenever a transitive group contains a subgroup whose degree is less than* $n-1$ *and which involves all the substitutions that do not contain a given letter it must be non-primitive.*

Having developed some of the most important properties of the subgroups of G which do not contain a given letter we proceed to inquire into their substitutions. Suppose that among the substitutions of a transitive group G

$$a_1 a_\alpha$$

has only one solution ; *i. e.*, there is only one cycle of this type in the group which contains a_1. Then there can be only one value of γ for each β in

$$a_\beta\, a_\gamma \qquad (\beta = 1,\ 2,\ \ldots\ldots n)$$

since any a can be transformed into a_1. All the conjugates of $a_1 a_\alpha$ are therefore distinct and may be used as systems of non-primitivity of the given transitive group.

More generally speaking we may say that if G contains a subgroup G' whose degree n' is less than the degree of G and if any given letter of G (a_1) is found in only one of the transforms of G' with respect to G, then will these transforms

$$G',\ G'',\ \ldots\ldots,\ G^{n}$$

constitute systems of non-primitivity of G.

For if G^{α} and G^{β} had a common letter then would the substitution of G which transforms this common letter into a_1 lead to two such groups both of which would involve a_1. This is contrary to the hypothesis. These conjugate subgroups must therefore involve distinct sets of letters which may be regarded the systems of non-primitivity of G. Hence the

THEOREM. *If a primitive group contains a subgroup whose degree is less than the degree of the group it must also contain a substitution which transforms this subgroup into one which contains any one of its letters together with at least one new letter.*

From this theorem it follows that if a primitive group whose degree exceeds 2 contains the cycle $a_1 a_2$ it must also contain $a_1 a_3$ (a_3 representing any suitable letter, different from a_1 and a_2) and therefore the symmetric group of these three letters $(a_1 a_2 a_3)$all.

If a primitive group whose degree exceeds three contains $(a_1 a_2 a_3)$all it must, according to the given theorem, also contain $(a_1 a_\alpha a_\beta)$all where at least one of the two subscripts α, β exceeds 3. Representing this by 4 we can easily show that the group must contain at least all the substitutions of

$$(a_1 a_2 a_3 a_4)\text{all}$$

whose degree does not exceed 3. For if any such substitution is given we can find some substitution of $(a_1 a_2 a_3)$all which is either the same or differs from it only in having another letter a_α where the given substitution has a_4. The transform of this substitution with respect to $a_\alpha a_4$ (which is known to be in the group) will be the given substitution. Since every substitution of the fourth degree is the product of two substitutions of a lower degree the given primitive group must contain

$$(a_1 a_2 a_3 a_4)\text{all}.$$

In general, if a primitive group whose degree exceeds m contains

$$(a_1 a_2 \ldots\ldots a_m)\text{all}$$

it must also contain

$$(a_1 a_\alpha \ldots\ldots a_\mu)\text{all}$$

where the number of subscripts 1, α, ...μ is m and at least one of them exceeds m. Representing this by $m+1$ we see that G must contain

$$a_\alpha\, a_{m+1} \qquad (\alpha = 1, 2, \ldots\ldots a_m).$$

We consider now any substitution of

$$(a_1 a_2 \ldots\ldots a_{m+1})\text{all}$$

whose degree does not exceed m. We can find some substitutions in

$$(a_1 a_2 \ldots\ldots a_m)\text{all}$$

which is either the same or differs from it only in having a_α where this has a_{m+1}. In this case the transform with respect to $a_\alpha\, a_{m+1}$ will be the given substitution. Since a substitution of the $m+1$ degree ($m \overline{\gtrless} 2$) may be regarded as the product of two substitutions of a lower degree the given primitive group must contain

$$(a_1 a_2 \ldots\ldots a_{m+1})\text{all}.$$

Calling $m+1$ m' we can prove in the same way that G contains the symmetric group of $m'+1 = m+2$ letters, etc. Hence the

THEOREM. *Whenever a primitive group contains a symmetric subgroup of a lower degree it must be a symmetric group.*

COROLLARY. *If a primitive group contains a substitution of the form $a_1 a_2$ it is symmetric.*

We will now suppose that the primitive group contains

$$a_1 a_2 a_3.$$

If its degree exceeds 3 it must also contain

$$a_1 a_\alpha\, a_\beta$$

where at least one of the two letters, say α, is greater than 3. We shall represent this by 4, G then contains the two substitutions

$$a_1 a_2 a_3 \text{ and } a_1 a_4 a_\beta$$

and therefore

$$(a_1 a_2 a_3 a_4)\text{pos}.$$

In general, if a primitive group whose degree exceeds m contains

$$(a_1 a_2 \ldots\ldots a_m)\text{pos}$$

it must also contain

$$(a_1 a_\alpha \ldots\ldots a_\mu)\text{pos}$$

where the number of subscripts 1, α, ...μ is m and at least one of them exceeds m. Representing this by $m+1$ we see that G contains

$$a_\alpha\, a_{m+1} a_\beta \qquad (\alpha = 1, 2, \ldots\ldots m).$$

It must therefore contain at least all of the substitutions of

$$(a_1 a_2 \ldots\ldots a_{m+1})\text{pos}$$

whose degree does not exceed m. For if s is any such substitution containing a_{m+1} there is some substitution s_1 in

$$(a_1 a_2 \ldots\ldots a_m)\text{pos}$$

which differs from s only in having a_δ where s has a_{m+1}. If β exceeds m we make $\alpha = \delta$ then will $a_\alpha\, a_{m+1} a_\beta$ transform s_1 into s. If $\beta \overline{\gtrless} m$ we transform the substitution

$$a_\alpha\, a_{m+1} a_\beta$$

with respect to some substitution of $(a_1 a_2 \ldots\ldots a_m)$pos. So that in place of a_β we may have a letter not found in s. Let this transform be

$$a_\gamma\, a_{m+1} a_\varepsilon \qquad (\gamma,\ \varepsilon \overline{\gtrless} m).$$

We now take from the substitutions of $(a_1 a_2 \ldots\ldots a_m)$pos the one s_2 which differs from s only in having a_γ where s has a_{m+1} if s does not contain a_γ, and the one s_2 which differs from s only in having a_ε, a_γ where s has a_γ, a_{m+1} if s contains a_γ. The transform of these with respect to

$$a_\gamma\, a_{m+1} a_\varepsilon$$

will be the required substitution s.

This proves that G contains all the substitutions of $(a_1 a_2 \ldots\ldots a_{m+1})$pos whose degree is equal to or less than m. These generate $(a_1 a_2 \ldots\ldots a_{m+1})$pos, for any positive substitution of the $(m+1)^{th}$ degree $(m > 2)$ may be considered as the product of two positive substitutions of a lower degree. [Let $s = \ldots a_x a_y \ldots$ be any positive substitution of the $(m+1)^{th}$ degree and $s_1 \ldots\ldots a_x a_y \ldots\ldots$ be any positive substitution of a lower than the $(m+1)^{th}$ degree. Then will s_2 in

$$s = s_1 s_2 \text{ or } s_2 = s_1^{-1} s$$

be also a positive substitution whose degree $\overline{\gtrless} m$].* Hence the

THEOREM. *If a primitive group contains a substitution of the form* $a_1 a_2 a_3$ *but none of the form* $a_1 a_2$ *it is the alternating group.*

We are now in possession of the following important facts in regard to any primitive group G.

(1) If $g=n$, G must be generated by a single cycle which involves a prime number of letters, and for each prime number there is one and only one such primitive group.

(2) If g does not equal n it must be a larger multiple of n and G must contain n conjugate subgroups whose degree is $n-1$ and whose order is $g \div n$.

(3) If G contains a substitution of the form $a_1 a_2$ or one of the form $a_1 a_2 a_3$ it must contain the alternating group.

(4) Both the alternating and the symmetric groups have a 1, 1 correspondence to the positive integers beginning with 2.

(5) The order of the symmetric group is $n!$ and that of the alternating group is $\frac{1}{2}n!$.

(6) The average number of letters in all the substitutions of a transitive group is $n-1$.

(7) Every transitive group contains at least $n-1$ substitutions of the n^{th} degree.

The three classes of primitive groups, regular, alternating, and symmetric, each of which contains an infinite number of members, are distinct when $n>3$. The groups that belong to these classes for any value of n are well known. It remains to determine those whose order satisfies the inequality

$$n>g>\tfrac{1}{2}n!.$$

Before pursuing the general discussion any farther we shall seek all the primitive groups whose degree does not exceed six. In doing this we shall use some methods which will be of service in the further study of this subject. Most of the methods, however, may serve as illustrations of the theorems which have been developed.

[To be Continued.]

*It can be easily proved that if a group contains

$$a_1 a_2 a_\alpha \qquad (\alpha=1, 2, \ldots\ldots n)$$

it contains the alternating group of degree n, and if it contains

$$a_1 a_\alpha \qquad (\alpha=1, 2, \ldots\ldots n)$$

it contains the symmetric group of degree n. Cole's Netto, §§ 84, 85.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from March Number.]

Corollary II. But again I am able hence to show that those two straights AX, BX, meeting which the straight $PFHD$ makes either two internal angles toward the same parts equal to two right angles, or consequently (from Eu. I. 13 and 15) alternate external or internal angles equal to one another, or again, from the same cause, an external (as suppose DHX) equal to an internal and opposite HFX; that, say I, those two straights not even in their infinite production can meet one another.

For if from any point N of AX is let fall to BX the perpendicular NR, this will be in the hypothesis of acute angle (which alone in any case can hinder us) greater (from III. Cor. I.) than the common perpendicular KL. Therefore those two straights AX, BX cannot ever meet one another.

But furthermore here thou hast demonstrated propositions 27 and 28 of the first book of Euclid, and indeed without immediate dependence from the preceding 16 and 17 of the same first book, about which difficulties could arise when the triangle should be of infinite sides on a finite base; to which sort of a triangle without doubt would refer one who believed that these two straights AX, BX met one another at least at an infinite distance, although the angles at the transversal $PFHD$ were such as we have supposed.

Moreover, on account of the demonstrated common perpendicular KL, surely those two KX, LX cannot come together toward the part of the points X, since also (from a superposition easily understood) toward the other part also would meet at the same time the remaining and themselves unterminated KA, LB. Wherefore two straights AX, BX would enclose a space; which is contrary to the nature of the straight line.

But these things are later. For in the preceding I have never applied either the 16th or 17th of the first book of Euclid, except where clearly it treats of a triangle bounded on every side, as indeed I promised I would so take care to do *in Proemio ad Lectorem*.

[To be Continued.]

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from March Number.]

V. Let ABC be $\triangle$ right-angled at C. Draw FD perpendicular to AB, meeting either leg produced. There are thus four similar right triangles.

Fig. 5.

Letting $AC=b$, $AB=c$, $BC=a$, $CD=x$, $CE=y$, $AF=s$, $EB=a-y$, $FB=c-s$, $AD=b+x$, $FE=v$, $ED=w$, $FD=v+w$, we obtain the following proportions, with their resulting equations:

(1). $b:s::c:b+x$. $\therefore b(b+x)=cs$ ....1.

(2). $b:s::a:v+w$. $\therefore b(v+w)=as$ ......2.

(3). $c:b+x::a:v+w$. $\therefore c(v+w)=a(b+x)$ ....3.

(4). $b:y::c:w$. $\therefore bw=cy$ ....4.

(5). $b:y::a:x$. $\therefore bx=ay$ ....5.

(6). $c:w::a:x$. $\therefore cx=aw$ ....6.

(7). $b:v::c:a-y$. $\therefore b(a-y)=cv$ ....7.

(8). $b:v::a:c-s$. $\therefore b(c-s)=av$. ....8.

(9). $c:a-y::a:c-s$. $\therefore c(c-s)=a(a-y)$ ....9.

(10). $s:y::b+x:w$. $\therefore sw=y(b+x)$ ....10.

(11). $s:y::v+w:x$. $\therefore sx=y(v+w)$ ....11.

(12). $b+x:w::v+w:x$. $\therefore x(b+x)=w(v+w)$ ....12.

(13). $s:v::b+x:a-y$. $\therefore s(a-y)=v(b+x)$ ....13.

(14). $s:v::v+w:c-s$. $\therefore s(c-s)=v(v+w)$ ....14.

(15). $b+x:a-y::v+w:c-s$. $\therefore (c-s)(b+x)=(a-y)(v+w)$....15.

(16). $y:v::w:a-y$. $\therefore y(a-y)=vw$ ....16.

(17). $y:v::x:c-s$. $\therefore y(c-s)=vx$ ....17.

(18). $w:a-y::x:c-s$. $\therefore w(c-s)=x(a-y)$ ....18.

We are now to find combinations of the above equations from which the elements x, y, s, v, w, can be eliminated, thus leaving us the relation existing between a, b, and c.

It is evident that from no single equation, nor from any set of two equations, can the relation be determined.

There remains three possible cases of combinations to be considered :

1. When three of the elements x, y, z, v, w, are involved.
2. When four.
3. When five, or all.

FIRST CASE. Of this case there are $\frac{5.4.3}{\lfloor 3}=10$ possible combinations of e unknown elements : $v, w, x; v, w, y;$ and so on.

Before taking up these in detail, we note that by inspection of the propor- , it easily may be seen that the following eighteen sets of equations each prise dependent equations :

1, 2, 3 ; 4, 5, 6 ; 7, 8, 9 ; 10, 11, 12 ; 13, 14, 15 ; 16, 17, 18 ; 1, 4, 10 ; 1, , 5, 11 ; 2, 8, 14 ; 3, 6, 12 ; 3, 9, 15 ; 4, 7, 16 ; 5, 18, 17 ; 6, 9, 18 ; 10, 11, 14, 17 ; 12, 15, 18.

Hence, in our search for possible combinations, all such must be rejected tain any of these sets.

There are three equations involving v, w, x: 3, 6, 12. But this combi- on must be rejected for the reason just given. For the same reason, or use there is wanting a sufficient number of equations involving the three un- wn elements, the other nine combinations must be rejected, except the com- tion x, y, z, which elements are involved in equations 1, 5, 9. If we elimin- x, y, z from these equations, we obtain the desired relation, $c^2=a^2+b^2$.

It should be observed, in passing, that future combinations including 1, 5, nust also be rejected.

SECOND CASE. Of this case there are $\frac{5.4.3.2}{\lfloor 4}=5$ possible combinations our unknown elements ; and, besides, the exceptional combination, $v+w$, x, , $v+w$ being regarded as a single unknown.

Before proceeding to investigate this case, it is necessary to call attention sets of four dependent equations. Take, for example, the set 1, 2, 6, 12. m 1 and 2, 3 is obtained. But 3 with 6 and 12 gives a set of three depend- equations ; hence the set 1, 2, 6, 12 must be rejected. A little stndy of the nteen sets given in Case 1, will disclose forty-five sets of four dependent ations.

The equations involving the unknown elements v, w, x, y, are 3, 4, 5, 6, 2, 16. Out of these seven equations, there are $\frac{7.6.5.4}{\lfloor 4} = 35$ combinations, ing four at a time. Of these thirty-five sets, fourteen are to be rejected, for ons previously stated. The remaining twenty-one sets, of which 7, 5, 4, 3, type, and to which the other twenty easily can be reduced, give, after the nown elements have been eliminated, the desired relation between a, nd c.

Similarly, we find twenty-one sets each of four equations, involving v, x, z) and (v, w, y, z), and seventeen each involving (v, x, y, z), (w, x, y, z), and $(v+w, x, y, z)$, thus making in all 114 proofs for this case.

THIRD CASE. Of this case, there are $\frac{18.17.16.15.14}{\lfloor 5}=8568$ sets of the eighteen equations, taking five at a time.

To determine how many of this number must be rejected, proceed as follows. Begin with the list of sets of dependent equations found in Case 1. Notice that there are $\frac{15.14}{\lfloor 2}=105$ sets of the eighteen equations taking five at a time, each containing equations 1, 2, 3; the same number containing equations 4, 5, 6; and so on, till we come to 1, 4, 10; for while there are 105 sets containing equations 1, 4, 10, three of them have already been counted out. So proceed, with the entire list of sets of dependent equations in Case 1, then with the set 1, 5, 9, following this with the sets of Case 2. We thus find that there are 3746 sets of five to be rejected, either because they contain sub-sets of dependent equations or sub-sets of equations from which the desired relation between a, b, c, is obtained.

One more class must be rejected: sets of five dependent equations. For example, 10, 9, 7, 6, 3, which is a type of all the others—72 in number—and from which the 72 can easily be deduced.

Deducting from 8568, 3746+73, we have remaining 4749 sets of five, from which can be derived the identity $c^2=a^2+b^2$.

$\therefore$ 1+114+4749=4864, the number of proofs by this method.

EXAMPLES:

1. $cv+cw-ax=ab$ .... 3.

$bw=cy$ .... 4.

$aw=cx$ .... 6.

$cv+by=ab$ .... 7.

4 in 6, $bx=ay$ .... 5.

4, 5 and 7 in 3, $ab-by+\frac{c^2y}{b}-\frac{a^2y}{b}=ab$. $\therefore c^2=a^2+b^2$.

2. $cz-bx=b^2$ .... 1.

$bv+bw-az=0$ .... 2.

$bw=cy$ .... 4.

$bx=ay$ .... 5.

$cv+by=ab$ .... 7.

1 in 2, $cv+cw-ax=ab$ .... 3.

4, 5, and 7 in 3, same as in 1st example.

VI. Let ABC be $\triangle$ right-angled at C. Produce AC to some point as D. Draw DF perpendicular to AB, produced, and meeting CB, produced.

Employing notation similar to that used in V., and proceeding somewhat in the same manner, we find that this method also yields a large number of proofs, in fact the same number that we found in V.

[To be Continued.]

Fig. 6.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

56. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

A, B, and C can walk at the rate of $a=3$, $b=4$, and $c=5$ miles, per hour. They start from Washington, at $m=1$, $n=2$, and $p=3$ o'clock, P. M., respectively. When B overtakes A, he is ordered (by A) back to C. When will B and C meet? Suppose B had ordered A back to C, when would A and C meet? In case all three continue walking ahead, at what time will they meet?

Solution by P. S. BERG, Larimore, North Dakota.

Since B gains 1 mile in 1 hour on A, to gain 3 miles will require 3 hours, or it will be 5 o'clock and 12 miles from starting point when B and A meet. C has traveled 10 miles. Since B and C travel 9 miles in 1 hour, they will travel 2 miles in $\frac{2}{9}$ hour, hence they will meet at $5\frac{2}{9}$ o'clock. Since A and C travel 8 miles in 1 hour, they will travel 2 miles in $\frac{1}{4}$ hour, hence they will meet at $5\frac{1}{4}$ o'clock.

In case all three continue walking ahead, as stated above A and B will meet at 5 o'clock. Since C gains 2 miles on A in 1 hour, to gain 6 miles will require 3 hours. Hence they will meet at 6 o'clock. Since C gains 1 mile on B in 1 hour, to gain 4 miles will require 4 hours. Hence it will be 7 o'clock when they meet.

Also solved by *B. F. YANNEY* and *H. C. WILKES*.

57. Proposed by L. B. FRAKER, Weston, Ohio.

Suppose that in a meadow the grass is of uniform quality and growth and that 6 oxen or 10 colts could eat up 3 acres of the pasture in $\frac{11}{12}$ of the time in which 10 oxen and 6 colts could eat up 8 acres; or that 600 sheep would require $2\frac{4}{7}$ weeks longer than 660 sheep to eat up 9 acres.

In what time would an ox, a colt, and a sheep together eat up an acre of the pasture on the supposition that 589 sheep eat as much in a week as 6 oxen and 11 colts? By Arithmetic, if possible.—Hunter's Arithmetic. (Unsolved in *School Visitor*.)

Solution by B. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio.

1. By first condition, the eating capacity of a colt is to that of an ox as 6 : 10.

∴ By last condition, the eating capacity of a colt is to that of a sheep as 589 : 21.

∴ The eating capacity of a colt is to that of a sheep, an ox, and a colt together, as 1767 : 4775.

2. ∴ The first two conditions of the problem may be stated as follows:

10 colts could eat up 3 acres of the pasture in $\frac{18}{25}$ of the time in which 17 colts could eat up 6 acres, or 1400 colts would require $2\frac{6}{7}$ weeks longer than 1540 colts to eat up 589 acres.

3. Let $42u$ be the amount of grass consumed each week by a colt.

4. Suppose the time it takes 10 colts to eat up 3 acres is 18 weeks; then, the time it takes 17 colts to eat up 6 acres would be 25 weeks.

5. ∴ $(10\times18\times42u)\div3=2520u$, total amount of pasture eaten from 1 acre in 18 weeks; and $(17\times25\times42u)\div6=2975u$, total amount of pasture eaten from 1 acre in 25 weeks.

6. ∴ $(2975u-2520u)\div(25-18)=65u$, obviously the amount of growth on 1 acre in 1 week, and the same result that would be obtained whatever the time supposed in (4).

7. ∴ $2520u-18\times65u=1350u$, amount of pasture originally on 1 acre.

8. ∴ $(589\times1350u)\div(1400\times42u-589\times65u)=\frac{159030}{4103}$, the number of weeks it would take 1400 colts to eat of 589 acres of pasture; similarly, the time required for 1540 colts is found to be $\frac{159030}{5279}$ weeks. Now, the difference between these two numbers, $\frac{159030\times1176}{4103\times5279}$ weeks : $2\frac{6}{7}$ weeks, the true difference :: 18 weeks, the supposed time : the true time.

9. ∴ Since the only number that needs correcting, to enable us to complete the solution, is $1350u$, the amount of pasture originally on 1 acre, the time required for an ox, a colt, and a sheep together to eat up 1 acre, is

$$\left(\frac{20}{7}\times\frac{4103\times5279}{159030\times1176}\times1350u\right)\div\left(\frac{4775}{1767}\times42u-65u\right)=9\frac{2482499}{11757354}\text{ weeks. Answer.}$$

H. C. Wilkes gets 142.25+ days.

NOTE. This problem appeared a few years ago in the *School Visitor*. With no little difficulty, we obtained a solution by Algebra. The solution was not published because of the difficult composition. It is strange that such a problem should appear in an arithmetic which is to be used by boys and girls 12 years old and upwards. EDITOR.

PROBLEMS.

58. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Two men, A and B, in Boston, hire a carriage for \$25, to go to Concord, N. H., and back, the distance being 72 miles, with the privilege of taking in three more persons. Having gone 20 miles, they take in C; at Concord, they take in D; and when within 30 miles of Boston, they take in E. How much shall each man pay? [From *Greenleaf's National Arithmetic.*]

59. Proposed by ISAAC L. BEVERAGE, Monterey, Virginia.

A broker charges me $1\frac{1}{4}$ per cent. brokerage for buying some uncurrent bank bills at 20 per cent. discount. Of these bills 4 of \$50. each become worthless, but the remainder I dispose of at par, and make by the operation \$364. What was the face amount? [Which answer is correct, \$3000, or $\$3048\frac{24}{27}$?]

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

56. Proposed by CHAS. E. MYERS, Canton, Ohio, and Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

(a) How much can be paid for a bond, bearing 5 per cent. interest, and having ten years to run, so as to realise 3 per cent. on the investment? (b) At what price must the government sell 5 per cent. \$100 bonds to run ten years, interest payable annually, to make them the same to the buyer as 3 per cent. bonds at par, to run ten years, interest payable annually, provided the buyer can invest all interest received at 4 per cent. interest, payable annually?

Solution by J. K. ELWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

Let x=price, a=face, n=number of periods, R=rate bond bears, r=rate to be realized, r'=rate on interest.

The interest on bond is an annuity at compound interest whose final value $=\frac{Ra}{r'}[(1+r')^n-1]$, which added to the *face* value of bond must equal the compound amount of the *price* for n periods, or $x(1+r)^n$.

$\therefore x=\dfrac{a+Ra[(1+r')^n-1]}{(1+r)^n}$. For ($a$), $a=100$, $n=10$, $R=.05$, $r=.03$, $r'=.03$.

$$\therefore x=\frac{100+\frac{.05}{.03}(1.03^{10}-1)}{1.03^{10}}=\$117.0604.$$

For (b), $a=100$, $n=10$, $R=.05$, $r=.03$, $r'=.04$.

$$\therefore x=\frac{100+\frac{.05}{.04}(1.04^{10}-1)}{1.03^{10}}=\$119.0777.$$

If in (a) interest were payable semi-annually, we should have $a=100$, $n=20$, $R=.025$, $r=.015$, $r'=.015$, and $x=\$117.168+$, or \$117.17 as given in the tables of bond values used by brokers and bankers.

Also solved by *E. W. MORRELL*, *B. F. YANCEY* and *G. B. M. ZERR*. Prof. Morrell obtained as results \$118.256 and \$117.661; and Proposer, to last part, \$117.89.

57. Proposed by J. C. CORBIN, Pine Bluff, Arkansas.

Find the quotient of

$$\begin{vmatrix}(s-a_1)^2 & a_1^2 & a_1^2\ldots\ldots a_1^2\\ a_2^2 & (s-a_2)^2 & a_2^2\ldots\ldots a_2^2\\ a_3^2 & a_3^2 & (s-a_3)^2\ldots\ldots a_3^2\\ \ldots\ldots & \ldots\ldots & \ldots\ldots\\ a_n^2 & a_n^2 & a_n^2\ldots s-a_n^2\end{vmatrix}\div\begin{vmatrix}s-a_1 & a_1 & a_1\ldots\ldots\ldots a_1\\ a_2 & s-a_2 & a_2\ldots\ldots\ldots a_2\\ a_3 & a_3 & s-a_3\ldots\ldots\ldots a_3\\ \ldots\ldots & \ldots\ldots & \ldots\ldots\\ a_n & a_n & a_n\ldots\ldots s-a_n\end{vmatrix}$$

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Let Q=the quotient and as we can exchange row for column without altering the value, we get

$$Q=\begin{vmatrix}(s-a_1)^2 & a_2^2 & a_3^2\ldots a_n^2\\ a_1^2 & (s-a_2)^2 & a_3^2\ldots a_n^2\\ a_1^2 & a_2^2 & (s-a_3)^2\ldots a_n^2\\ \ldots\ldots & \ldots\ldots & \ldots\ldots\\ a_1^2 & a_2^2 & a_3^2.(s-a_n)^2\end{vmatrix}\div\begin{vmatrix}s-a_1 & a_2 & a_3\ldots\ldots a_n\\ a_1 & s-a_2 & a_3\ldots\ldots a_n\\ a_1 & a_2 & s-a_3\ldots\ldots a_n\\ \ldots\ldots & \ldots\ldots & \ldots\ldots\\ a_1 & a_2 & a_3\ldots s-a_n\end{vmatrix}$$

All the elements in the i^{th} column of the numerator being a_i^2, of the denominator a_i, except in the i^{th} row which is $(s-a_i)^2$ for numerator, and $s-a_i$ for denominator. Hence, we have

$$Q=\begin{vmatrix}1, & 0, & 0, & 0,\ldots\ldots\\ 1, & (s-a_1)^2, & a_2^2, & a_3^2,\ \ldots\\ 1, & a_1^2, & (s-a_2^2,) & a_3^2,\ \ldots\\ 1, & a_1^2, & a_2^2, & (s-a_3)^2,\ldots\\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\end{vmatrix}\div\begin{vmatrix}1, & 0, & 0, & 0,\ldots\ldots\\ 1, & s-a_1, & a_2, & a_3,\ldots\ldots\\ 1, & a_1, & s-a_2, & a_3,\ldots\ldots\\ 1, & a_1, & a_2, & s-a_3,\ldots\\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\end{vmatrix}$$

Multiply first column of numerator by a_i^2, of the denominator by a_i and subtract from the i^{th} column; do this for each column and the value is unaltered.

$$Q=\begin{vmatrix}1, & -a_1^2, & -a_2^2, & -a_3^2, \ldots \\ 1, & s(s-2a_1), & 0, & 0, \ldots \\ 1, & 0, & s(s-2a_2), & 0, \ldots \\ 1, & 0, & 0, & s(s-2a_3), \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}+\begin{vmatrix}1, & -a_1, & -a_2, & -a_3, \ldots \\ 1, & s-2a_1, & 0, & 0, \ldots \\ 1, & 0, & s-2a_2, & 0, \ldots \\ 1, & 0, & 0, & s-2a_3, \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

Let $u=(s-2a_1)(s-2a_2)(s-2a_3)\ldots\ldots(s-2a_n)$.

$$\sum\frac{a_i^2}{s-2a_i}=\frac{a_1^2}{s-2a_1}+\frac{a_2^2}{s-2a_2}+\frac{a_3^2}{s-2a_3}+\ldots\ldots$$

$$\therefore\ Q=\frac{s^{n-1}u\left\{s+\sum\frac{a_i^2}{s-2a_i}\right\}}{u\left\{1+\sum\frac{a_i}{s-2a_i}\right\}}=\frac{s^{n-1}\left\{s+\sum\frac{a_i^2}{s-2a_i}\right\}}{\left\{1+\sum\frac{a_i}{s-2a_i}\right\}}.$$

ERRATA. On page 52 of last issue, line 3 from bottom, read = before and in the denominator read $\sqrt{a^2-x^2}$ for "$\sqrt{a^2+x^2}$"; on page 53, line 15, end the radical sign over a^2-x^2 and b^2-x^2, in the numerators.

PROBLEMS.

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64. Proposed by A. H. BELL, Box 184, Hillsboro, Illinois.

Solve the equations:

$$a^2x=(2x^2-a^2)\sqrt{x^2+y^2}\ldots\ldots(1).$$
$$b^2y=(2y^2-b^2)\sqrt{x^2+y^2}\ldots\ldots(2).$$

65. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knox-Tennessee.

Prove that $\cos\frac{n\pi}{7}+\cos\frac{3n\pi}{7}+\cos\frac{5n\pi}{7}=\frac{1}{2}$ or $-\frac{1}{2}$, according as n is *odd* even.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

50. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Divide a triangle into the ratio of m to n by a line perpendicular to the base.

Solution by J. C. GREGG, Superintendent of Schools, Brazil, Indiana; E. W. MORRELL, Professor of Mathematics, Montpelier Seminary, Montpelier, Vermont; and the PROPOSER.

Let ABC be the triangle. Draw the altitude BD. Divide the base AC at E so that $AE:EC=m:n$. Draw the line BE.

Then $\triangle ABE:\triangle EBC=AE:EC=m:n$.....(1).

Take AF a mean proportional between AE and AD, then draw GF parallel to BD.

Then $\triangle AFG:\triangle ADB=AF^2:AD^2$.

But $AF^2=AE\times AD$.

$\therefore \triangle AFG:\triangle ADB=AE\times AD:AD^2=AE:AD=\triangle ABE:\triangle ADB$.

$\therefore \triangle AFG=\triangle ABE$ and $\triangle EBC=FGBC$.

Hence, using in (1), we have $\triangle AFG:FGBC=m:n$. Q. E. D.

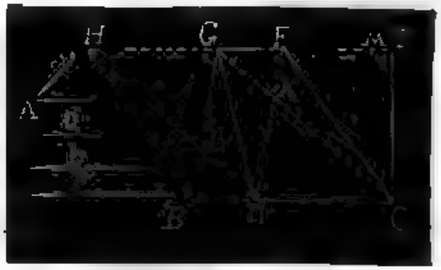
Also solved in various ways by *G. B. M. ZERR, B. F. YANNEY, J. SCHEFFER, A. H. BELL, F. R. HONEY, O. W. ANTHONY, H. J. GAERTNER, G. I HOPKINS, J. M. COLAW, J. O. MAHONEY.*

51. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Construct a tropezoid, given the bases, the altitude, and the angle formed by the intersection of the diagonals.

Solution by J. OWEN MAHONEY, B. E., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tennessee; FREDERICK R. HONEY, A. B., New Haven, Connecticut; J. SCHEFFER, Hagerstown, Maryland; B. F. SINE, Principal of High School, Rock Enon Springs, Virginia; and PROPOSER.

Let a and b be the bases, p the perpendiculars, and A the angle between the diagonals.



Take $BC=a+b$ and describe upon BC a segment to contain an angle $=$ to A. The problem is possible when p is less than the greater segment of the diameter perpendicular to BC. Take $CE=p$ and perpendicular to BC. Draw EH parallel to BC cutting the circle in M and G. Draw BG and GC. Also draw DF parallel to BG and DH parallel to GC.

Then is $DCFG$ or $BDGH$ the required trapezoid. For $BD=GF=b$, $DC=HG=a$, $\angle DKC=\angle BLD=\angle BGC=A$, and $CE=p$. By treating the point m as we did G we get two other trapezoids answering all conditions.

This problem was solved in a similar manner by *COOPER D. SCHMITT*, *A. H. BELL*, *J. SCHEFFER*, *B. F. SINE*, *J. M. COLAW*, *P. S. BERG*, *O. W. ANTHONY*, *E. W. MORRELL*, *J. C. GREGG*, and *H. J. GAERTNER*.

PROBLEMS.

56. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The locus of the centers of the isogonal transformations of all the diameters of the circumcircle of any triangle is the nine-points circle. *Brocard.*

57. Proposed by J. OWEN MAHONEY, B. E., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tennessee.

Show that pairs of points, on a straight line may be so related harmonically that a pair of real points will be harmonic with regard to a pair of imaginary points, and by this means prove that there are an indefinite number of conjugate pairs of imaginary points on a real line.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

46. Proposed by GEORGE LILLEY, Ph. D., LL. D., Principal of Park School, 394 Hall Street, Portland, Oregon.

A fly starts from a point in the circumference of a table, 3 feet in diameter, and travels uniformly along the diameter to a point in the circumference of the table directly opposite the starting point. The table moves uniformly to the right about a center axis in such manner that it makes one complete revolution while the fly passes over its diameter. Find the absolute path described by the fly and the ratio of rates of movement of the table and the fly.

I. Solution by the PROPOSER.

The curve described by the fly is the spiral of Archimedes. Its equation is $r=a\theta$.

$$S=\int_0^\pi\left(\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\right)d\theta,=\frac{a\pi\sqrt{1+\pi^2}}{2}+\frac{a}{2}\log(\pi+\sqrt{1+\pi^2}).$$

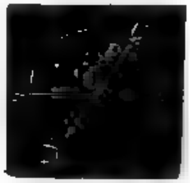
Hence, $2S$, or the absolute path described by the fly, is 63.994+ inches.

If we take the Naperian logarithm of $(\pi+\sqrt{1+\pi^2})$ the result is 69.6+ inches.

The ratio of rates$=\frac{2\pi r}{2r}=\pi$. The ratio of rates in *space*$=\frac{2\pi r}{63.994}=1.76+$.

II. Solution by G. B. M. ZERR, Ph. D., Professor of Mathematics in Texarkana College, Texarkana, Arkansas-Texas; and Professor J. SCHEFFER, A. M., Hagerstown, Maryland.

Let P be the position of the fly when A has moved to C, and let A move m times as fast as P. Let $OA=r$, $OP=\rho$, $\angle COA=\theta$.
Then $mPC=AC$. $\therefore m(r-\rho)=r\theta$.



$$\therefore \rho=\frac{r(m-\theta)}{m}=\frac{r(\pi-\theta)}{\pi}, \text{ since } m=\pi.$$ This

is the equation to the fly's path.

$$\therefore S=\int_0^{2\pi}\frac{r}{\pi}\sqrt{1+(\pi-\theta)^2}\,d\theta$$

$$=r\sqrt{1+\pi^2}+\frac{r}{\pi}\log(\pi+\sqrt{1+\pi^2}).$$

$$\therefore S=\tfrac{1}{2}\left\{\sqrt{1+\pi^2}+\frac{1}{\pi}\log(\pi+\sqrt{1+\pi^2})\right\}=5.835 \text{ feet}.$$

$$\frac{3\pi}{S}=\frac{1885}{1167}=\frac{13}{9}\text{ nearly}.$$

III. Solution by Prof. J. M. BANDY, A. M., Old Trinity College, North Carolina, and J. C. GREGG, Superintendent of City Schools, Brazil, Indiana.

Let (ρ, θ) denote the co-ordinates of P, and since AR and RP are in a constant ratio, ρ and θ are in the same ratio, which denote by c.

Hence, $\theta=-\rho c$ [Archimedean spiral]......(1).
By theory of curves,

$$S=\int\left(\rho^2+\frac{d\rho^2}{d\theta^2}\right)^{\frac{1}{2}}d\theta\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1), $\frac{d\rho^2}{d\theta^2}=\frac{1}{c^2}$, and $\rho=\frac{\theta^2}{c^2}$. Substituting

these values in (2), $S=\frac{1}{c}\int_0^{2\pi}(1+\theta^2)^{\frac{1}{2}}\,d\theta$ ....(3).

Integrating (3) by formula for reducing $p=1/2$,

$$S=\left[\frac{\theta(1+\theta^2)^{\frac{1}{2}}}{2c}\right]_0^{2\pi}+\frac{1}{2c}\log\left[\theta+\sqrt{1+\theta^2}\right]_0^{2\pi}\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

But $c=\dfrac{\text{arc } AR}{RP}=\dfrac{\text{circumference}}{\text{diameter}}=\dfrac{\pi}{1}$. Substituting in (4), and reducing,

$S=6.4533+$ feet.

The movement of the fly in its path is the resultant of the motion of the fly along the diameter and the motion of the table to the right about its axis. The rate of motion of the fly in its path is variable, and is measured at any instant by the measuring circle given by any particular value of ρ. So that the ratio of the motion of the table to that of the fly can be found for any particular value of ρ.

Also solved by *O. W. ANTHONY* and *E. L. SHERWOOD*. Prof. Sherwood's solution will be published under problem No. 69.

66. Proposed by H. C. WHITAKER, M. E., Sc. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

There are four points, A, B, C, and D in space. Point D remains fixed with its co-ordinates (1, 2, 2) feet. At a given time A is at (2, 3, 4) feet, is moving in a straight line at the rate of 3 feet per minute, and has passed through (5, 9, 10) feet; B is at (1, 4, 2) feet, moves in a straight line at the rate of 7 feet per minute, and will pass through (—2, 2, 8) feet; C is at the origin and moves along the axis of X in the direction of x positive at the rate of 6 feet per minute.

The motion of the points being continuous before and after the given time, required the times when the volume of the tetrahedron whose edges are the lines joining these points will be 108 cubic inches.

Solution by the PROPOSER.

The length of a base edge [from (x_1, y_1, z_1) to (x_2, y_2, z_2)] is well known to be

$$\sqrt{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}^2+\begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix}^2+\begin{vmatrix} z_1 & 1 \\ z_2 & 1 \end{vmatrix}^2}$$

Finding the distance from $(x_3y_3z_3)$ to this edge, multiplying this distance by the length of the edge just given, the area of the base is

$$\frac{1}{|\underline{2}}\sqrt{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2+\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}^2+\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2}$$

Finding the distance from $(x_4y_4z_4)$ to this base, multiply this distance by the area of the base just given, the volume of the tetrahedron is found to be *

$$\frac{1}{|\underline{3}}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

[*The extension of each of these values in n dimentioned co-ordinates is obvious, as is also the sursolidity of a figure in four dimensioned space bounded by five tetrahedra; and so on.]

Substituting the given values of the co-ordinates, we have

$$\frac{1}{6}\begin{vmatrix} 2-t & 3-2t & 4-2t & 1 \\ 1-3t & 4-2t & 2+6t & 1 \\ 6t & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 \end{vmatrix}=V.$$

This reduces to $16t^3-14t^2+2t=\pm.0625$; whence by solving, $t=-.026$, .045, .125, .217, .684, and .705 seconds, respectively.

Also solved by *J. SCHEFFER* and *G. B. M. ZERR*.

PROBLEMS.

53. Proposed by O. D. SMITH, A. M., Professor of Mathematics, Alabama Polytechnic Institute, Auburn, Alabama.

Solve the differential equation, $dy/dx=y(x-y)/x(x+y)$; and show that $x=y\log(xy)$.

54. Proposed by Prof. J. SCHEFFER, A. M., Hagerstown, Maryland.

A certain solid has a square, side$=a$, for its base, and all parallel sections are squares, the two sections through the middle points of the opposite sides of the square are semi-circles, however. Find surface, volume, and the centers of gravity of each.

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

PLAYFAIR'S PSEUDO-PROOF OF THE ANGLE-SUM.

BY GEORGE BRUCE HALSTED.

The living person who has most capital invested in Playfair's fallacious demonstration reproduced in the March number of THE AMERICAN MATHEMATICAL MONTHLY, pp. 77—79, is Professor George C. Edwards of the University of California, who unfortunately gives it as the basis for his treatment of parallels in §16 of his Elements of Geometry, Macmillan, 1895.

His §16 is Playfair's Proposition I "All the exterior angles equal four right angles," with Playfair's fallacious proof. Then his §17 is "THEOREM. If two straight lines make equal angles with a third straight line intersecting them, they will make equal angles with *any* straight line intersecting them," in proving which he twice cites §16. Then as Exercise 1 under §17 he has "Establish the theorem when the fourth line passes through *B*." But this very special case of his §17 he assumes in his §16, thus making his treatment of parallels a simple *rumentum in circulo.* I wrote this to Professor Edwards and he wrote in answer at clearly seemed an explicit acknowledgment of it. But it was so unlike a radoxer to acknowledge a fallacy, that in wonder I wrote again, "You mean to te that in your proof of the theorem §16 of your book, you do assume (without ting the assumption) your Exercise 1 under your §17. Am I right in this unrstanding of your letter?" And strange as it may seem he wrote March 7th, 96, "You are practically right in your understanding of my letter of February nd."

I have given three different exposures of Playfair's fallacy in the fourth ition of my Bolyai pp. 65—71.

THEORY AND PRACTICE COMBINED.

BY WARREN HOLDEN, GIRARD COLLEGE, PHILADELPHIA, PENNSYLVANIA.

Common experience, applied to Mechanical and Engineering problems, s always been in harmony with the principles of Euclidean Geometry. With overthrow of these principles we might expect chaos to come again. And if thematics has not yet demonstrated all of these principles, so much the worse Mathematics. Let its Professors try again. Their failure in any particular e does not establish the opposite.

Abstract studies in Philosophy, unmodified and unillustrated by human perience, have often led to bewildering vagaries. Does not a similar fate, from rresponding causes, impend over Non-Euclidean Geometry? Theory and actice should go hand in hand.

All mathematical instruments in use, whether in the department of echanics, Physics or Engineering, are constructed upon the basis of Euclidean eometry. Where are the instruments of precision which serve to illustrate and pply the principles of Non-Euclidean Geometry?

QUERIES.

1. Please give me address of publishing house that publishes the most reliable works n How to Calculate Timber on the Stump, also names of most reliable works on same.

JOHN BRIDGES.

EDITORIALS.

Prof. C. A. Waldo is now Professor of Mathematics in Purdue University, afayette, Indiana.

Science, March 27, contains an able article, The Essence of Number, by r. George Bruce Halsted.

Prof. J. A. Calderhead has been elected Professor of Mathematics in the rry University, Pittsburg, Pennsylvania.

Dr. Byerly's Fourier's Series and Spherical Harmonics, we are informed by the Publishers, is gaining an international reputation.

Dr. E. H. Moore has been promoted to Head Professor of Mathematics in the University of Chicago. This is a merited recognition.

Professor J. J. Sylvester, formerly of the Johns Hopkins University, has just been made a Foreign Member of the Turin Royal Academy of Science.

Our subscribers will do us a kindness by sending us the names of persons who are likely to subscribe for the MONTHLY, as we would be pleased to send such persons sample copies.

A few of our former subscribers who are in arrears have asked us to discontinue the MONTHLY to their address. In no case will we discontinue to send the MONTHLY until the amount due us is paid.

Mr. W. J. C. Miller, who is editor of the Mathematical Department of the *Educational Times*, London, England, says, "THE AMERICAN MATHEMATICAL MONTHLY is one of the best magazines that I receive." Mr. Miller has edited the Mathematical Department of the *Educational Times* for over 30 years.

M. A. Gruber, of Washington, D. C., writes: You will please find enclosed a Money Order of $3.00 as my subscription to THE AMERICAN MATHEMATICAL MONTHLY for 1896. It is a magazine worthy of long life; if the additional mite is any assistance in putting it upon a paying basis, I shall always remain among your best friends.

We have on hand a few bound copies of Volumes I and II which we will sell at $2.75 each. By special arrangements with the binders we can have volumes of the MONTHLY bound for 75 cents. If any of our subscribers wish to avail themselves of this opportunity to have their volumes of the MONTHLY bound, they may send them to B. F. Finkel, Springfield, Mo.

Philadelphia Summer Meeting will hold its fourth session, July 6—31, 1896, in the buildings of the University of Pennsylvania, under the auspices of the American Society for the Extension of University Teaching. Department E—Mathematics: I. Methods of Teaching Mathematics; II. Plane and Solid Geometry; III. Algebra (Elementary Course); IV. Algebra (Advanced Course); V. Trigonometry; VI. Analytical Geometry; VII. Differential and Integral Calculus; VIII. Theory of Equations and Determinants; IX. Differential Equations; X. Theory of Functions.

The lecturers are I. J. Schwatt, Ph. D., and G. H. Hallett, M. A., of the University of Pennsylvania. On Wednesday evening, July 8, Dr. Schwatt will deliver to the students of all departments of the Summer Meeting an address on the Philosophy and Utility of the Calculus.

We are sorry to announce the death of one of our valued contributors, T. P. Stowell, of Rochester, N. Y., which occurred February 29th, 1896. Mr.

Stowell's name has been closely associated with nearly all the mathematical journals published in this country within the last fifty years. The following sketch is taken from *The Union and Advertiser*, Rochester, New York:

Thomas P. Stowell, of No. 29 Atkinson street, died Saturday at the home of the family, aged 77 years. Mr. Stowell, who had resided in the city since April 1, 1864, at the residence now occupied by the family, was born September 5, 1819, and was widely known, respected and esteemed, not only in Rochester but throughout the entire country. He graduated from the well-known Hallowell University of Virginia, and was considered one of the ablest mathematicians in the United States. He retired from business in 1895, in the enjoyment of robust health, having apparently the strength and certainly the appearance of a middle-aged man.

Mr. Stowell had been a member of St. Luke's Church during the entire period of his residence in Rochester. He leaves a wife and five children, Miss Anna Stowell, Miss M. Louise Stowell, Dr. Henry F. Stowell, and C. L. Stowell, all of this city, and Charles F. Stowell of Albany.

BOOKS AND PERIODICALS.

Syllabus of Geometry. By G. A. Wentworth, A. M., Author of a Series of Text-books in Mathematics. Pamphlet form. 50 pages. Boston and Chicago: Ginn & Co.

This pamphlet contains the enunciations of the propositions and corollaries of the author's text-book in Geometry, numbered as they are in the text-book. B. F. F.

Rational Mathematics. By Charles De Medici.

Under the above title the author is publishing a work—The New Geometry and Commensurational Arithmetic—which is divided into three sections: A, B, C. In Section A, Part I, the first principles and primary elements of Geometry are taught; Part II. First principles of Commensuration, founded on the Natural Division and Inherent Dimensions of Geometric Elements are taught; Part III. Classification of Geometric Figures and Forms. Section B, Geometry Study and Practice. The work is published by A. Lovell & Co., New York. B. F. F.

Elementary Treatise on Electricity and Magnetism Founded on Joubert's Traité Élémentaire D'Éléctricité. By G. C. Foster, F. R. S., Quain Professor of Physics in University College, London, and E. Atkinson, Ph. D., formerly Professor of Experimental Science in the Staff College. 8vo. Cloth, 552 pp. Introduction price, $1.80. New York: Longmans, Green & Co.

This treatise on Electricity and Magnetism is confined to facts, hypotheses being studiously avoided. The treatment of each subject is clear, simple, direct, and exhaustive. Whenever necessary, the higher mathematics are used in computations and the establishment of electrical laws. It is the best treatise on Electricity and Magnetism that we have yet seen and we heartily commend it to any person desiring a good work on these important subjects. B. F. F.

Elementary Algebra. By J. A. Gillett, Professor in the New York Normal College. 8vo. Half Leather Back. 412 pp. New York: Henry Holt & Co.

Among other commendable features of this book may be mentioned, (1) the prominence given to problems and the consequent introduction of the equation, (2) the attention given to negative quantities, (3) the attention given to the formal laws of Algebra,—the Commutative, the Associative and the Distributive laws, and (4) the simplicity, clearness, and logical arrangement of the matter. The book is beautifully printed and handsomely bound, and presents a most attractive appearance.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Albert Shaw. Price, $2.50 per year. Single number 25 cents. The Review of Reviews Co., New York City.

The *Review of Reviews* is almost indispensable to the general reader who wishes to keep abreast of the rapidly developing international questions of the day. In the April number there is a full and able editorial discussion of the complicated African situation, which is described as "the drama of 'Europe in Africa.'" The mixed interests and motives of England, Russia, Italy and France in the Dark Continent are clearly set forth. Russia's general attitude toward the European powers is also discussed, and the editor comments briefly on America's relations with Spain, our interests in the Cuban revolution, and the present status of the Venezuelan boundary dispute. In addition to this editorial treatment (in the department entitled "The Progress of the World") the *Review* presents a remarkably complete survey of the Cuban situation by Murat Halstead, a summary of the best current thought in England on the subject of international arbitration, and a vivid account of the relief work now going on in Armenia. In short the *Review of Reviews* records a month's activities in both hemispheres.

April Monthly Magazine Number of the Outlook. Price, $1.00 per year in advance. The Outlook Company, 13 Astor Place, New York.

In the April Magazine Number of *The Outlook* there will appear an article on William H. Prescott, by Kenyon West. It will be in commemoration of the centenary of the great American historian, who was born May 4, 1796. The article will be enriched by numerous portraits and other illustrations contirbuted from the private collection of members of the Prescott family, who have been interested in Kenyon West's tribute to Prescott. Among these are Mr. Arthur Dexter, of Boston, the nephew of the historian; Mrs. Roger Wolcott, Prescotts grand-daughter, who lives also in Boston; and Mr. Linsee Prescott of Greenwich, Conn., who is the son of Prescott's eldest son.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hudson, New York.

The April *Cosmopolitan* contains the following: A word about Golf, Golfers, and Golf-links in England and Scotland, by Price Collier; Vicissitudes of the Dead, by Eleanor Lewis; Development of the Overland Mail Service, by Thomas L. James; The Lyceum, by James B. Pond; Mrs. Cliff's Yacht, by Frank R. Stockton; The Bargain of Faust (Poem) by Alice W. Rollins; Hilda Stafford, by Beatrice Harraden. Each of these articles are beautifully illustrated.

The following periodicals have been received: Journal de Mathématiques Élémentaires, (15 Mars 1896); American Journal of Mathematics, (April, 1896); The Mathematical Gazette, (October, 1895); L'Intermédiaire des Mathématiciens, (Mars, 1896); El Progreso Matemático, (Tomo V. Ano 1895); Notes and Queries, (April, 1896); The Kansas University Quarterly, (January, 1896); Popular Astronomy, (June, 1895); The Monist, (April, 1896); Bulletin of the American Mathematical Society, (March, 1896); The Educational Times, (March, 1896).

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A SPECIAL COMPLEX OF THE SECOND DEGREE AND ITS RELATION WITH THE PENCILS OF CIRCLES.

By DR. ARNOLD EMCH, University of Kansas, Lawrence, Kansas.

1. Before entering upon the treatment of this problem I will make a few preliminary remarks which, although well known to the reader, may give a clearer conception of what follows.

If a straight line is given we can write the equations of its projections upon the co-ordinate planes in the following form :*

$$(1)\qquad \begin{aligned} L&=yZ-zY,\\ M&=zX-xZ,\\ N&=xY-yX, \end{aligned}$$

where x, y, z designate the current co-ordinates. The six constants L, M, N, X, Y, Z can be considered as the co-ordinates of the straight line and satisfy the relation

$$(2)\qquad LX+MY+NZ=0.$$

An algebraic complex of straight lines of the n^{th} degree is defined by an equation of the form

$$(3)\qquad F(L,\ M,\ N,\ X,\ Y,\ Z)=0,$$

F being a polynom of the n^{th} degree and homogeneous in L, M, N, X, Y, Z.

Through every point in spaces passes an infinite number of lines belonging to the complex; they form a cone of the n^{th} order, and in every plane lies an infinite number of lines belonging to the complex and enveloping a curve of the

*We use the designation of M. Picard in his *Traite d'Analyse*, Vol. I, p. 312.

n^{th} class. Thus, a special kind of a complex of the n^{th} degree may be obtained by all the tangents of a surface of the n^{th} order, or the secants of a curve in space of the n^{th} order.

2. In our problem we define as a special complex P of the second degree the system of secants passing through a fixed conic. Every point in space determines a cone of the second order, whose elements belong to the complex and every plane intersects the conic in two points which represent a degenerated curve of the second class, whose tangents belong to the complex.

The conic itself we will describe in the following manner:

Through any two fixed points A and B of the xy-plane draw the two circles

$$(4)\qquad U_1=(x-a_1)^2+(y-b_1)^2-r_1{}^2=0,$$

$$(5)\qquad U_2=(x-a_2)^2+(y-b_2)^2-r_2{}^2=0,$$

and form the pencil of circles

$$(6)\qquad U_1-\lambda U_2=0$$

passing through A and B. At the center of every circle of the pencil erect a perpendicular to the xy-plane and equal to the radius of the circle above and below the xy-plane. The extremities of these perpendiculars lie in an equilateral hyperbola H whose plane passes through the central line of the pencil of circles and is perpendicular to tne xy-plane. The vertices of the hyperbola are equal distant from the xy-plane and lie in a perpendicular through the center of the circle with the sect AB as a diameter.*

To every point of the equilateral hyperbola belongs a circle of the pencil (6), which with the point determines a right cone whose elements all include angles of 45° with the xy-plane. We may ask what is the character of the system of lines R passing through the equilateral hyperbola and including angles of 45° with the xy-plane. For this purpose intersect the *cone-director* of these lines with the plane at infinity and establish the new complex Q consisting of all the lines passing through the intersection. As the intersection is a circle I, the complex is of the second degree and contains all the lines including angles of 45° with the plane xy.

Evidently the system R is the common solution of the complex P and Q and is therefore a *congruence*. The degree of this congruence is 6, since the degrees of P and Q are 2, and since the hyperbola H and the circle I have two points in common. Through each point in space pass two lines, and in each plane lie four lines belonging to the congruence. It is therefore of the second order and of the fourth class. Since the equilateral hyperbola H is symmetrical in regard to the xy-plane, it is easily seen that the complex and the congruence connected with it are symmetrical to the xy-plane, in other words they are reflected into themselves. It is known that through every generatrix of a congruence of straight lines pass two developable surfaces whose elements belong to

*The thought to represent points in space by circles in a plane originated with Prof. W. Fiedler, of Zurich, who applied it in his beautiful treatise on "*Cyclographie*," Teubner, Leipsig.

the congruence. In our congruence R the developable surfaces through a generatrix D are the right cone having its vertex in the hyperbola H and the hyperbolic cylinder passing through H. The focal surface of the congruence degenerates into the hyperbola H and the plane at infinity. If we designate the representation of a pencil of circles by Fiedler's method, and the complex and congruence of rays connected with it as *cyclographic*, we may now state the theorem:

The theory of the pencils of circles is identical with the theory of the cyclographic congruence.

3. To the first pencil of circles through A and B, or cyclographic congruence, we add another pencil of circles through the points C and D, or cyclographic congruence. It may be determined by two circles

$$(7) \qquad V_1=(x-c_1)^2+(y-d_1)^2-s_1^2=0,$$

$$(8) \qquad V_2=(x-c_2)^2+(y-d_2)^2-s_2^2=0,$$

passing through C and D and assumes the form

$$(9) \qquad V_1-\mu V_2=0.$$

The corresponding congruence is obtained as in the first pencil. Designating this congruence by S and the complex through the hyperbola G which represents the pencil (9) by T we have to solve the problem to find the common part of the congruences R and S, or as these have the circle I at infinity in common, to find the common figure of the complexes P, Q, and T. Each of the hyperbolas H and G intersect the circle I in two points and as the complexes are all of the second degree, they have a ruled surface in common whose degree according to the rules of algebra is

$$2\times2\times2\times2-2\times2-2\times2=8.$$

To a generatrix in this ruled surface of the eighth order can be found one in the same surface symmetrical to the first in regard to the xy-plane. Hence the whole surface is symmetrical to the xy-plane and as it contains two double generatrices through the circular points of the circle I, it intersects the xy-plane in a bicircular curve of the fourth order. Every generatrix of the surface intersects the xy-plane in a point of the curve and includes an angle of 45° with the xy-plane.

Through each generatrix pass four developable surfaces, two hyperbolic cylinders and two cones of the second order. These cones are tangent to each other and intersect the xy-plane in two tangent circles. As these circles always pass through A, B and C, D and as their point of tangency lies in the above curve we have the theorem:

The locus of the points of tangency of each two tangent circles of two pencils of circles is a bicircular curve of the fourth order.

Figure 1 will show the relation of these pencils in the case that each two circles are tangent.

4. We will now take another view of the problem. For fixed values of λ and μ the equations of two circles respectively belonging to the pencil (6) and (9) may be written

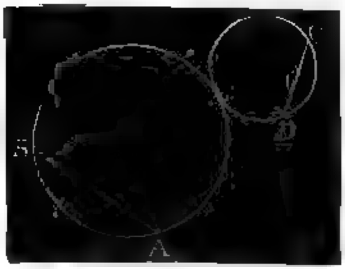

Fig. 1.

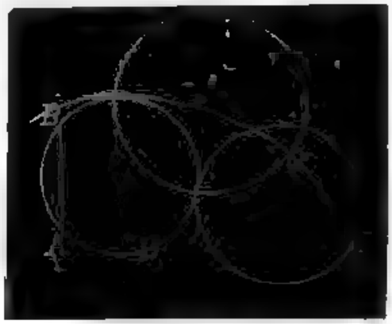

Fig. 2.

$$(10) \qquad x^2+y^2-2\frac{a_1-\lambda a_2}{1-\lambda}x-2\frac{b_1-\lambda b_2}{1-\lambda}y+\frac{M_1-\lambda M_2}{1-\lambda}=0,$$

$$(11) \qquad x^2+y^2-2\frac{c_1-\mu c_2}{1-\mu}-2\frac{d_1-\mu d_2}{1-\lambda}y+\frac{N_1-\mu N_2}{1-\mu}=0,$$

where $$M_1=a_1^2+b_1^2-r_1^2,\ M_2=a_2^2+b_2^2-r_2^2,$$

$$N_1=c_1^2+d_1^2-s_1^2,\ N_2=c_2^2+d_2^2-s_2^2.$$

The condition that the circle (10) is orthogonal to the circle (11) is

$$2\frac{a_1-\lambda a_2}{1-\lambda}\cdot\frac{c_1-\mu c_2}{1-\mu}+2\frac{b_1-\lambda b_2}{1-\lambda}\cdot\frac{d_1-\mu d_2}{1-\mu}-\frac{M_1-\lambda M_2}{1-\lambda}-\frac{N_1-\mu N_2}{1-\mu}=0.$$

It is now possible to determine the co-efficients of this equation such that for variable parameters the pencils (10) and (11) are projective. In this case each two corresponding circles are orthogonal.

Evidently we have to put $\mu=\lambda$, which after some reductions gives for the equation of condition

$$[2a_1c_1+2b_1d_1-M_1-N_1]-\lambda[2a_2c_1+2b_2d_1-M_2-N_1+2a_1c_2+2b_1d_2-M_1-N_2]$$
$$+\lambda^2[2a_2c_2+2b_2d_2-M_2-N_2]=0. \qquad (12)$$

This indicates that in the first place the circles (4) and (7), and (5) and ust be orthogonal. Secondly, for every value of λ there must be

$$2a_2c_1+2b_2d_1-M_2-N_1+2a_1c_2+2b_1d_2-M_1-N_2=0.$$

This equation is satisfied if the circles (5) and (7), and (4) and (8) are or- al. In this case the pencils (6) and (9) are said to be conjugate pencils of Every circle of the one pencil is orthogonal to every other circle of the t in equation (12) we do not desire to change the points A, B, C, D. equation can be satisfied by changing the radii of the circles (4), (5), which gives for the solutions of M_1, M_2, N_1, N_2 three equations with own quantities. We can however fix one of the circles without alter- al result. This implies the theorem :

o pencils of circles can be made projective in one and only one way such responding circles in the projectivity are orthogonal.

The product of these projective pencils is a bicircular curve of the fourth r, as it is well known. In figure 2, we consider the two pencils of circles ugh A and B, and C' and D', where C' and D' are assumed to be imagin- and on the line l and in these pencils two orthogonal circles U and V' inter- ing each other in two points J and J'. In these points draw tangent circles l having their centers on l. These circles are orthogonal to V' and intersect other in two fixed points C and D, *i. e.*, they belong to the conjugate pen- f circles of the pencil through C' and D'. Whence the general theorem :

The locus of the points of tangency of each two tangent-circles of two pencils of les is a bicircular curve of the fourth order. The same curve is also produced by of the pencils and the projective conjugate pencil of the other pencil.

Under the given conditions the equation of the curve may be written $V_2'-U_2V_1'=0$.

It is easily seen that this curve passes through the four points A, B, C, D as stated in the theorem contains the circular points at infinity as double- nts.

5. Without entering into further details on the nature of this curve ay be mentioned that there exists an interesting connection between this re and the circular curves of the third order if these are considered as locii of nts from which two sects AB and CD appear under the same angle. An logon exists in space, the discussion of which however goes over the limits of article. A paper on this subject by the author was read in the January ses- of the Kansas Academy of Science and will appear in the next volume of the nsactions of this Academy.

PROBLEMS.

1. Given n straight lines in a plane. Another straight line in this plane volves about a fixed point and in every position intersects the n lines

in n points. These points determine n "sects" measured from the fixed point and their algebraic sum represents a point on the revolving line. What is the curve which this point describes?

2. Find a geometrical construction for the following problem: Given the distances AO, BO, CO of the points A, B, C of an equilateral triangle from a fixed point O. Construct the equilateral triangle, or triangles satisfying these conditions.

3. What is the locus of the points from which any two sects in space AB and CD (not in the same plane) appear under the same constant angle?

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from April Number.]

Proposition XXIV. *The same hypothesis remaining: I say the four angles together (Fig. 27.) of the quadrilateral KDHK nearer the base AB are less (in hypothesis of acute angle) than the four angles together of the quadrilateral KHLK more remote from the same base; and indeed this is so, whether those two AX, BX somewhere at a finite distance meet toward the parts of the point X; or never meet one another; but toward those parts either ever more mutually approach each other, or somewhere receive a common perpendicular, after which of course (in accordance with Cor. II. of the preceding proposition) toward the same parts they begin mutually to separate.*

Fig. 27.

Proof. Here however we suppose the portions KK assumed to be mutually equal. Since therefore (from the preceding) the side DK is greater than the side HK, and similarly HK greater than the side LK, the portion MK in HK is assumed equal to LK, and in DK the portion NK equal to HK; and MN, MK, LK are joined, truly the intermediate point K with the point L, and the point K near to the point B with the point M.

Now I proceed thus.

Since indeed the sides of the triangle KKL (I make beginning always from the point K nearer the point B) are equal to the sides of the triangle KKM, and the included angles equal, as being right, equal also will be (from Eu. I. 4) the bases LK, MK, and likewise equal the angles which correspond mutually, ıese bases, indeed the angle KLK to the angle KMK, and the angle LKK to angle MKK. Therefore equal also are the remainders NKM and HKL. erefore, since the sides NK, KM of the triangle NKM are equal in the same to the sides HK, KL of the triangle HKL, equal also will be (from the same I. 4) the bases NM, HL, the angles KNM, KHL, and finally the angles $'N$, KLH. But in the preceding triangles are already proved equal the angles K, KMK. Therefore the whole angle NMK is equal to the whole angle HLK.

Wherefore, since all angles at the points K are right, it follows manifestly four angles together of the quadrilateral $KNMK$ are equal to all four angles ther of the quadrilateral $KHLK$.

But since the two angles together at the points N and M in the quadrilat- $KNMK$ are greater, in hypothesis of acute angle, than the two angles ther (from Cor. after P. XVI) at the points D and H in the quadrilateral HM, or the quadrilateral $KDHK$, the consequence thence is, that (the com- ı right angles at the points K being added) the four angles together of the drilateral $KNMK$, or the quadrilateral $KHLK$ are greater (in hypothesis of e angle) than the four angles together of the quadrilateral $KDHK$.

Quod erat demonstrandum.

COROLLARY.

But it ought here opportunely to be observed, nothing will fail in the ar- ıent made, although the angle at the point L is assumed right, together with othesis of acute angle. For still that common perpendicular LK would be (from Cor. I. after II of this) than the other perpendicular HK, from which ore still a portion MK could be assumed equal to the aforesaid LK. Which standing, it follows that no hindrance can intervene.

[To be Continued.]

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from April Number.]

PRIMITIVE GROUPS OF TWO, THREE, AND FOUR, LETTERS.

Since all of these must contain substitutions of the form a_1a_2 or of form $a_1a_2a_3$ they must all contain the symmetric group. The following is efore a complete list:

Degree.	Order.	Group.
2	2	(ab)
3	3	(abc)
	6	(abc)all
4	12	$(abcd)$pos
	24	$(abcd)$all.

Primitive Groups of Five Letters.

All the transitive groups of this degree must be primitive and there must be one regular group, viz:

(1) $$(abcde).$$

The lowest order of any other possible primitive group is 10. Such a primitive group would contain five subgroups of the form $(ab.cd)$ and therefore four substitutions of degree 5. All the substitutions of degree 5 whose powers do not contain a substitution of the type a_1a_2 are of the type $a_1a_2a_3a_4a_5$. Hence all the substitutions of degree 5 of a primitive group which is not the symmetric group must be of the given type.

If a primitive group of order 10 exists we may therefore assume that it contains

$$(abcde)$$

and some substitutions of the form $ab.cd$. These substitutions are all equal to

$$(abcde)S$$

where S is any one among them. They therefore transform the substitutions of $(abcde)$ into the same power. This cannot be the first power for a substitution consisting of a single cycle can be transformed into the first power only by its own powers. If we represent this power by α and observe that the product of two of these supstitutions is equal to some substitution in $(abcde)$ we have*

$$\alpha^2 \equiv 1 \pmod 5),\ (1 < \alpha < 5).$$

Since this has only one solution it follows that there is only one group of order 10. We may find the substitutions by writing the fourth power of $abcde$ under $abcde$, thus,

*If s_1 and s_2 transform s into s^α we have

$$s_1^{-1}ss_1 = s^\alpha$$
$$s_2^{-1}s_1^{-1}ss_1s_2 = s_2^{-1}s^\alpha s_2 = s_2^{-1}ss_2 . s_2^{-1}ss_2 \ \ldots\ldots\ldots\ \alpha \text{ times}$$
$$= s^\alpha . s^\alpha . s^\alpha \ \ldots\ldots\ldots\ \alpha \text{ times}$$
$$= s^{\alpha^2}$$

abcde	*abcde*	*abcde*	*abcde*	*abcde*
aedcb	*baedc*	*cbaed*	*dcbae*	*edcba*.

The required substitutions are

be.cd *ab.ce* *ac.de* *ad.bc* *ae.bd*

[We might evidently have obtained all of these by multiplying one into (*abcde*)]. Hence the group of order 10 is

(2) $$(abcde)(ab.ce)=(abcde)_{10}.$$

All the substitutions that transform (*abcde*) into itself form a group. There are five substitutions that transform the substitutions of (*abcde*) into their first power, therefore there must be five that transform them into each of their other powers. We thus obtain a group of order 20 which is generated by *abcde* and some substitution *bced* which transforms this into its second power. We have therefore

(3) $$(abcde)(bced)=(abcde)_{20}.$$

There cannot be more than one of this order because each would have to contain five conjugate subgroups of one of the two types

$$(abcd)_4, (abcd)$$

and therefore only one subgroup (necessarily self conjugate) of the type

$$(abcde).$$

This may be supposed to be the same in all of the groups; but there is only one set of twenty substitutions that transform this into itself. The groups are therefore identical.

For all the other possible orders the subgroups of degree 4 would contain either a substitution of the type *ab* or one of the type *abc*. Hence all the other primitive groups are the alternating and the symmetric group. The following is a complete list of the primitive groups of degree five.

Order.	Group.
5	$(abcde)$.
10	$(abcde)_{10}$
20	$(abcde)_{20}$
60	$(abcde)$pos
120	$(abcde)$all

These groups could also have been found in the following manner, without employing the groups of a lower degree. We know that there is one group of each of the three classes—regular, alternating and symmetric. We know also that the order of each of the other primitive groups exceeds five and that they do not contain any substitutions of either of the two types

$$ab \qquad abc$$

Hence they can contain only substitutions of the fourth and fifth degrees together with unity.

Since the average number of letters in all the substitutions of these groups must be four each group can contain only four substitutions of the fifth degree. The only type of substitutions of the fifth degree which can be used is

$$abcde.$$

All these primitive groups may therefore be supposed to contain

$$(abcde)$$

as a self-conjugate subgroup and to be subgroups of the group of order 20 which contains all the substitutions that transform (*abcde*) into itself.

Any negative substitution of this group together with (*abcde*) generates the entire group, the only subgroup besides the group itself and (*abcde*) must therefore consist of the positive substitutions of the group. Hence there are only two primitive groups of degree five in addition to the regular, alternating, and symmetric groups. The generating substitutions of these groups are evident.

Primitive Groups of Six Letters.

There is no regular group. If there were a group of order 30 it would contain 24 substitutions of the type *abcde* and five substitutions of degree six. These five substitutions would generate a regular group; for only one of them could replace a given letter by any required letter since there are four of the form *abcde* which perform this operation, and therefore the product of any two must be of degree six or it must be unity.

[To be Continued.]

SIMULTANEOUS QUADRATIC EQUATIONS.

By I. H. BRYANT, Instructor of Mathematics, Waco High School, Waco, Texas.

This discussion is restricted to the special cases of simultaneous quadratic equations of n variables which always admit of solution. It is assumed that solutions are always possible :

(1) When there is one equation of the second degree and one variable.

(2) When all equations except one are of the first degree.

Let q, q_1 .... q_n=terms of the second degree.

Let p, p_1 .... p_n=terms of the first degree.

Let k, k_1 .... k_n=absolute terms.

Let l, l_1 .... l_n=absolute terms.

Let m=a constant factor.

Let x, x_1 .... x_n=the variables.

Let v_1, v_2 .... v_n=the variables when the equations are transformed.

CASE 1. When one equation is general, and the rest are of the first degree, or reducible to the first degree; *i. e.* when they assume any of the following forms :

(1) $(p+k)^n=0.$

(2) $(p+k)(p_1+k_1)\ldots\ldots(p_n+k_n)=0.$

(3) $(p+k)^n+m(p_1+k_1)^n=0.$

(4) $(p+k)^{2n}+m(p+k)^n+l=0.$

In the next four cases one or more of the equations may assume the above forms instead of the forms of these cases.

CASE 2. When each equation can be resolved into two factors of the first degree and an absolute term and when one of these factors is common to all equations.

$$\left\{\begin{array}{c}(p+k)(p_1+k_1)+l_1=0.\\(p+k)(p_2+k_2)+l_2=0.\\\ldots\ldots\ldots\ldots\ldots\ldots\\(p+k)(p_n+k_n)+l_n=0.\end{array}\right\}$$

Eliminate the common factor. There are now $n-1$ equations of the first degree.

CASE 3. When each equation can be resolved into two factors of the first degree and an absolute term and when each factor occurs in two equations.

As in the previous case, $n-1$ equations of the first degree can be obtained.

$$\left\{\begin{array}{l}(p_1+k_1)(p_2+k_2)+l_1=0.\\(p_2+k_2)(p_3+k_3)+l_2=0.\\\dots\dots\dots\dots\dots\dots\\(p_n+k_n)(p_1+k_1)+l_n=0.\end{array}\right\}$$

CASE 4. When like terms save the terms in which one variable occurs, are equal, or can be made equal in all equations. The terms can be made equal when the co-efficients of like terms are proportional in all equations, or when they are all similar and but one occurs in each equation. By eliminating these equal terms we can obtain $n-1$ equations in which the same variable will occur in each term. By dividing by this variable, we can reduce each of these equations to the first degree.

CASE 5. When like terms of the second degree are equal, or can be made equal, in all equations. Like terms can be made equal when they meet with the requirements indicated in Case 4. Eliminate the terms of the second degree. There are now $n-1$ equations of the first degree.

CASE 6. When the equations are homogeneous and like terms save those in which one variable occurs and the absolute term, are equal, or can be made equal in all equations. The requirements for making like terms equal are given in Case 4.

$$\left\{\begin{array}{l}p_1x+q+k_1=0.\\p_2x+q+k_2=0.\\\dots\dots\dots\dots\\p_nx+q+k_n=0.\end{array}\right\}$$

Let $x_1=v_1x$, $x_2=v_2x \dots\dots x_n=v_nx$. Eliminate x^2. We now have $n-1$ equations with $n-1$ variables which meet with the requirements of Case 5.

CASE 7. When two equations are homogeneous and the rest are of the first degree, or reducible to the first degree, with no absolute term. Eliminate all except two variables from the two homogeneous equations by means of the equations of the first degree. These two equations will then be homogeneous and will fall under Case 6.

CASE 8. When the terms containing one variable are equal, or can be made equal, in all equations and the remaining terms meet with the requirements of Case 6. Eliminate the terms containing this variable. We now have $n-1$ equations and $n-1$ variables which fall under Case 6.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

57. Proposed by L. B. FRAKER, Weston, Ohio.

Suppose that in a meadow the grass is of uniform quality and growth and that 6 oxen or 10 colts could eat up 3 acres of the pasture in $\frac{12}{13}$ of the time in which 10 oxen and 6 colts could eat up 8 acres; or that 600 sheep would require $2\frac{2}{7}$ weeks longer than 660 sheep to eat up 9 acres.

In what time would an ox, a colt and a sheep together eat up an acre of the pasture on the supposition that 589 sheep eat as much in a week as 6 oxen and 11 colts? By Arithmetic, if possible.—Hunter's Arithmetic. (Unsolved in *School Visitor*.)

II. Solution by Henry Heaton, M. S., Atlantic, Iowa.

Since 6 oxen=10 colts, 1 ox=$1\frac{2}{3}$ colts, and 6 oxen and 11 colts=21 colts =589 sheep. ∴ 1 colt=$28\frac{1}{21}$ sheep and 1 ox=$1\frac{2}{3}\times 28\frac{1}{21}$ sheep=$46\frac{47}{63}$ sheep.

10 oxen and 6 colts=$22\frac{2}{3}$ colts, eat 8 acres of grass in the same time that $\frac{1}{8}$ of $22\frac{2}{3}$ colts or $2\frac{5}{6}$ colts eat 1 acre, and $3\frac{1}{3}$ colts eat an acre in the same time that 10 colts eat 3 acres. Hence $3\frac{1}{3}$ colts eat an acre in $\frac{12}{13}$ the time that $2\frac{5}{6}$ colts eat it. In $\frac{12}{13}$ the time $3\frac{1}{3}$ colts eat as much grass as $\frac{12}{13}$ of $3\frac{1}{3}$ colts or $2\frac{5}{6}$ colts would eat it in the full time. The difference between $2\frac{5}{6}$ colts and $3\frac{1}{3}$ colts is $\frac{19}{78}$ of a colt. The difference in the grass eaten by them is $\frac{1}{13}$ of the growth. Hence $\frac{19}{78}$ of a colt eats $\frac{1}{13}$ of the growth. Hence to eat all the growth will require $\frac{13}{1}$ of $\frac{19}{78}$ of a colt or $\frac{19}{6}$ of a colt=$\frac{19}{6}$ of $28\frac{1}{21}$ sheep=$43\frac{3}{8}\frac{4}{5}$ sheep. To eat the growth on 9 acres will require 9 times $43\frac{3}{8}\frac{4}{5}$ sheep=$390\frac{5}{6}$ sheep. $600-390\frac{5}{6}=209\frac{1}{6}$. $660-390\frac{5}{6}=269\frac{1}{6}$. Hence it will require $209\frac{1}{6}$ sheep $2\frac{2}{7}$ weeks longer to eat the original grass on 9 acres than it will $269\frac{1}{6}$ sheep to eat the same. Hence $209\frac{1}{6}$ sheep eat in the $2\frac{2}{7}$ weeks what the 60 other sheep eat in the first part of the time. Hence this time is $209\frac{1}{6}\times 2\frac{2}{7}$ weeks÷60=$9\frac{1}{6}\frac{5}{6}\frac{3}{8}$ weeks. Hence it will take $269\frac{1}{6}$ sheep $9\frac{1}{6}\frac{5}{6}\frac{3}{8}$ weeks to eat the original grass on 9 acres. To eat 1 acre will require them $1\frac{1}{18}\frac{9}{9}\frac{3}{5}$ weeks.

An ox, a colt, and a sheep=$75\frac{5}{6}\frac{3}{5}$ sheep.

If $75\frac{5}{6}\frac{3}{5}$ sheep were eating on one acre, $43\frac{3}{8}\frac{4}{5}$ sheep would eat the growth leaving $32\frac{3}{8}\frac{7}{8}\frac{1}{5}$ sheep to eat the original grass. If it require $269\frac{1}{6}$ sheep $1\frac{1}{18}\frac{9}{9}\frac{3}{5}$ weeks to do this, it will require $32\frac{3}{8}\frac{7}{8}\frac{1}{5}$ sheep $(269\frac{1}{6}\div 32\frac{3}{8}\frac{7}{8}\frac{1}{5})\times 1\frac{1}{18}\frac{9}{9}\frac{3}{5}$ weeks= $9\frac{3}{17}\frac{5}{8}\frac{9}{8}\frac{9}{4}\frac{8}{5}\frac{1}{5}\frac{3}{8}$ weeks.

58. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Two men, A and B, in Boston, hire a carriage for $25, to go to Concord, N. H., and back, the distance being 72 miles, with the privilege of taking in three more persons. Having gone 20 miles, they take in C; at Concord, they take in D; and when within 30 miles of Boston, they take in E. How much shall each man pay? [From *Greenleaf's National Arithmetic.*]

I. Solution by H. C. WHITAKER, A. M., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

If we denote taking one person one mile by a person-mile, then the total person-miles was 514 and the cost of each of them was 4.8638 cents; the cost of taking A and B 144 miles was $7 each; the cost of taking C 124 miles was $6.03; the cost of taking D 72 miles was $3.50, and the cost of taking E 30 miles was $1.46.

II. Solution by F. M. McGAW, Bordentown, New Jersey; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas, and H. C. WILKES, Skull Run, West Virginia.

Five men ride 30 miles; four, 42 miles; three, 52 miles; and two, 20 miles.

∴ E pays for $\frac{1}{5}$ of 30=6 miles.

D pays for $\frac{1}{5}$ of 30+$\frac{1}{4}$ of 42=16$\frac{1}{2}$ miles.

C pays for $\frac{1}{5}$ of 30+$\frac{1}{4}$ of 42+$\frac{1}{3}$ of 52=33$\frac{5}{6}$ miles.

B pays for $\frac{1}{5}$ of 30+$\frac{1}{4}$ of 42+$\frac{1}{3}$ of 52+$\frac{1}{2}$ of 20=43$\frac{5}{6}$ miles.

A pays for $\frac{1}{5}$ of 30+$\frac{1}{4}$ of 42+$\frac{1}{3}$ of 52+$\frac{1}{2}$ of 20=43$\frac{5}{6}$ miles.

144:43$\frac{5}{6}$=$25:$7.609$\frac{103}{108}$, share of each A and B.

144:33$\frac{5}{6}$=$25:$5.873$\frac{91}{108}$, share of C.

144:16$\frac{1}{2}$=$25:$2.864$\frac{7}{12}$, share of D.

144:6=$25:$1.041$\frac{2}{3}$, share of E.

III. Solution by A. P. REED, A. M., Clarence, Missouri, and J. C. CORBIN, Pine Bluff, Arkansas.

144 miles=distance A rides, 144 miles=distance B rides, 124 miles=distance C rides, 72 miles=distance D rides, and 30 miles=distance E rides.

They should each pay in proportion to the distance each rides. Hence

$\frac{144}{514}$ of $25=$7.00$\frac{100}{257}$=amount A should pay.

$\frac{144}{514}$ of $25=$7.00$\frac{100}{257}$=amount B should pay.

$\frac{124}{514}$ of $25=$6.03$\frac{28}{257}$=amount C should pay.

$\frac{72}{514}$ of $25=$3.50$\frac{50}{257}$=amount D should pay.

$\frac{30}{514}$ of $25=$1.45$\frac{235}{257}$=amount E should pay.

[NOTE. Greenleaf gives the answers as obtained in the second solution. But we think it is best to solve the problem on the principle that each pay in proportion to the distance he rides. This principle prevails in practice at the present time and is just in its application. EDITOR.]

PROBLEMS.

60. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

A pipe 1 foot long and $\frac{27}{32}$ inch in diameter has a half-inch orifice and weighs 1$\frac{3}{4}$ pounds. What is the diameter of a pipe of the same length and orifice, but weighing 41 ounces?

61. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Insured my store for a/bth=$\frac{4}{5}$th part of its value, at r=1$\frac{1}{4}$ per cent.; but soon afterward the store was burned down, and my loss over the insurance was $$L$=$4150. What was the valne of my store?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

58. Proposed by D. G. DORRANCE, Jr., Camden, Oneida County, New York.

Sum the series 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, etc., to n terms; also what is the n^{th} term?

Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee, and Prof. P. S. BERG, Larimore, North Dakota.

The series is evidently made up as follows from the different rows in Pascal's Triangle, beginning three farther to the right every time; thus,

a.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	......
b.				1	2	3	4	5	6	7	8	9	10	11	12	13	......
c.							1	3	6	10	15	21	28	36	45	55	
d.										1	4	10	20	35	56	84	
e.													1	5	15	35	
f.																1	
	1,	1,	1,	2,	3,	4	6,	9,	13,	19,	28,	41,	60,	88,	129,	189,	etc.

The n^{th} term of (a) is 1; the $(n-3)^{th}$ term of (b) is $n-3$; the $(n-6)^{th}$ term of (c) is $\dfrac{(n-5)(n-6)}{\underline{|2}}$; the $(n-9)^{th}$ term of (d) is $\dfrac{(n-7)(n-8)(n-9)}{\underline{|3}}$; and the $(n-12)^{th}$ term of (e) is $\dfrac{(n-9)(n-10)(n-11)(n-12)}{\underline{|4}}$; and so on. Hence the n^{th} term of the original series is composed of the sum of the above different terms; *i. e.*

$$1+(n-3)+\frac{(n-5)(n-6)}{\underline{|2}}+\frac{(n-7)(n-8)(n-9)}{\underline{|3}}+\frac{(n-9)(-10)(n-11)(n-12)}{\underline{|4}}$$

$+\ldots\ldots$ Also, the sum of n terms of (a) is n; of $(n-3)$ terms of (b) is $\dfrac{(n-3)(n-2)}{\underline{|2}}$; of $(n-6)$ terms of (c) is $\dfrac{(n-6)(n-5)(n-4)}{\underline{|3}}$; and the sum of $(n-9)$ terms of (d) is $\dfrac{(n-9)(n-8)(n-7)(n-6)}{\underline{|4}}\ldots\ldots$ and hence $S=n+$

$$\frac{(n-3)(n-2)}{\underline{|2}}+\frac{(n-6)(n-5)(n-4)}{\underline{|3}}+\frac{(n-9)(n-8)(n-7)(n-6)}{\underline{|4}}+\ldots\ldots$$

Also solved by *B. F. YANNEY* and *G. B. M. ZERR*.

59. Proposed by DAVID EUGENE SMITH, Ph. D., Professor of Mathematics in Michigan State Normal School, Ypsilanti, Michigan.

Prove that the product of the n n^{th} roots of 1 is +1 or −1 according as n is odd or even. Prove and generalize, for the n n^{th} roots of m.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

I. $(1)^{\frac{1}{n}}=\cos\frac{2m\pi}{n}+\sqrt{-1}\sin\frac{2m\pi}{n}$.

The several roots are: $\varepsilon^{0\sqrt{-1}}$, $\varepsilon^{(2\pi/n)\sqrt{-1}}$, $\varepsilon^{(4\pi/n)\sqrt{-1}}$, ...$\varepsilon^{[2(n-1)\pi/n]\sqrt{-1}}$.

$\therefore$ Product $=\varepsilon^{(n-1)\pi\sqrt{-1}}=\cos(n-1)\pi+\sqrt{-1}\sin(n-1)\pi=\pm1$.

If n is even product is negative; if n is odd product is positive.

II. Let $m=x+y\sqrt{-1}$.

Then $(x+y\sqrt{-1})^{\frac{1}{n}}=(\sqrt{x^2+y^2})^{\frac{1}{n}}\left[\cos\left(\frac{2m\pi+\theta}{n}\right)+\sqrt{-1}\sin\left(\frac{2m\pi+\theta}{n}\right)\right]$,

where $\theta=\tan^{-1}\frac{x}{y}$.

$$R_1=\left(\sqrt{x^2+y^2}\right)^{\frac{1}{n}}\varepsilon^{(\theta/n)\sqrt{-1}},$$

$$R_2=\left(\sqrt{x^2+y^2}\right)^{\frac{1}{n}}\varepsilon^{[(2\pi+\theta)/n]\sqrt{-1}},$$

$$R_3=\left(\sqrt{x^2+y^2}\right)^{\frac{1}{n}}\varepsilon^{[(4\pi+\theta)/n]\sqrt{-1}},$$

....

$$R_n=\left(\sqrt{x^2+y^2}\right)^{\frac{1}{n}}\varepsilon^{\{[2(n-1)\pi+\theta]/n\}\sqrt{-1}}.$$

$$P=\sqrt{x^2+y^2}\,\varepsilon^{[(n-1)\pi+\theta]\sqrt{-1}}=\left[\cos\left((n-1)\pi+\theta\right)+\sqrt{-1}\sin[(n-1)\pi+\theta]\right]$$

$$\sqrt{x^2+y^2}=\pm\sqrt{x^2+y^2}\left[\cos\theta+\sqrt{-1}\sin\theta\right].$$

II. Solution by B. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

I. $x^n-1=0$ is the equation from which are derived the n n^{th} roots of 1. Now, since 1 in the equation is negative there is one positive real root and $(n-1)$ if any, imaginary roots, if n is odd; and there is one positive real root, one negative real root, and $(n-2)$, if any, imaginary roots, if n is even. $\therefore$ Since ginary roots occur in conjugate pairs, and since the product of any two conte imaginaries is a positive real number, the sign of the product of the n n^{th} s of 1 when n is odd, is $+$; and when n is even, $-$.

Furthermore, since the successive powers of the first imaginary root of 1, the 1st to the n^{th}, give us all the n^{th} roots of 1, therefore, if we denote the imaginary root by ω, we shall have as the product of the n n^{th} roots,

$$\omega^2.\omega^3\ldots\ldots\omega^n=\omega^{n\frac{n+1}{2}}.\quad \text{But } \omega^n=1.\quad \therefore\ \omega^{n\frac{n+1}{2}}=+1 \text{ when } n \text{ is odd; and}$$

when n is even. But of these last two signs, $-$ must be chosen, for reasons gned in the preceding paragraph.

II. That the theorem is true in general for the n n^{th} roots of m, is made lent when we remember that the n n^{th} roots of any number may be found by tiplying any one of the n^{th} roots of such number by the different n^{th} roots of For then, we would have $m^{\frac{1}{n}}\times\omega.m^{\frac{1}{n}}\times\omega^2\ldots\ldots m^{\frac{1}{n}}.\omega^n=m\omega^{n\frac{n+1}{2}}=+m$ or according as n is odd or even, as shown above.

Also solved by *COOPER D. SCHMITT.*

ERRATA. In numerator of the expression, in next to last line, on page of last issue, for "Ra" read $\frac{Ra}{r'}$; on page 117, line 4, for "$s(s-2a_2)$" read $-2a_2)$; and in "Errata," for "last issue" read February issue. Also probs numbered 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, should be Nos. 58, 59, 60, 62, 63, 64, 65, 66, 67, respectively.

PROBLEMS.

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68. Proposed by ROBERT J. ALEY, A. M., Professor of Mathematics in Indiana University, Bloomington, Indiana.

Sum to n terms the series, $n\cos\theta+(n-1)\cos 2\theta+(n-2)\cos 3\theta+$, etc.

[*Chrystal's Algebra.*]

69. Proposed by Prof. C. E. WHITE, A. M., Trafalgar, Indiana.

Prove that $x^n\pm x^{n-1}+x^{n-2}\pm\ldots\ldots+(\pm1)^{n-1}x+(\pm1)^n=(x\mp1)^n\pm$ $(x\mp1)^{n-1}+B(x\mp1)^{n-2}\pm\ldots\ldots+(\pm1)^n x$, where A, B, C, ...are the binomial coefficients of the $(n+1)^{th}$ order.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to h

SOLUTIONS OF PROBLEMS.

63. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving Col Mechanicsburg, Pennsylvania.

If the *center* of a rolling ellipse *move in a horizontal line*, determine the surface which the ellipse rolls.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let BPA be a quadrant of the ellipse semi-axes AC, and BC, O the p tion of the center when BC coincides with OY, and $\angle BCP=\theta$. Then

$$PC=y=\frac{ab}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}=\frac{b}{\sqrt{1-e^2\sin^2\theta}}.$$

$\therefore$ The ellipse rolls on the inner surface of the cylinder

$$y^2+z^2=\frac{b^2}{1-e^2\sin^2\theta}.$$

When $e=0$, this becomes $y^2+z^2=b^2$.

To find the abscissa of the point of contact, we have, since arc $PB=$ arc l

$$ds=\sqrt{r^2d\theta^2+dr^2}=\sqrt{y^2d\theta^2+dy^2} \text{ since } PC=r=y;$$

$$\text{also } ds=\sqrt{dx^2+dy^2}.$$

$$\therefore \sqrt{dx^2+dy^2}=\sqrt{y^2d\theta^2+dy^2}.$$

$$\therefore dx=yd\theta, \text{ or } x=\int yd\theta=\int\frac{bd\theta}{\sqrt{1-e^2\sin^2\theta}}=bF(e,\ \theta).$$

When $e=0$, $x=b\theta$.

[No other solution of this problem was received. Editor.]

53. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Spr field, Missouri.

A pole, a certain length of whose top is painted white, is standing on the side hill. A person at A observes that the white part of the pole subtends an angle equal t and on walking to B, a distance a, directly down the hill towards the foot of the pole the white part subtends the same angle. What is the length of the white part, if the point B is at a distance b from the foot of the pole?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let DE be the length painted white; then a circle will pass through A, B, D, E. Let $\angle EAD=\angle EBD=\alpha$, $AB=a$, $BC=b$, $\angle DAB=\angle DEB=\theta$, $\angle ABE=\angle ADE=\varphi$, $DC=y$, and $DE=x$.

Then $(x+y)y=(a+b)b$ ....(1).

$AE:a=\sin\varphi:\sin(\alpha+\theta+\varphi)$, $x:AE=\sin\alpha:\sin\varphi$.

$$\therefore x=\frac{a\sin\alpha}{\sin(\alpha+\theta+\varphi)} \quad \text{......................(2)},$$

$b:x+y=\sin\theta:\sin(\alpha+\varphi)$ ....(3),

$(x+y):a+b=\sin(\alpha+\theta):\sin(\alpha+\varphi)$ ....(4).

Eliminating θ between (3) and (4),

$$\left\{\frac{(x+y)^4}{(a+b)^2}-\frac{2b(x+y)^2\cos\alpha}{a+b}+b^2\right\}\sin^2(\alpha+\varphi)=(x+y)^2\sin^2\alpha \quad \text{........(5).}$$

Eliminating θ between (2) and (3),

$$[\{b^2x^2-x^2(x+y)^2\}^2+4a^2b^2x^2(x+y)^2\sin^2\alpha]\sin^4(\alpha+\varphi)$$
$$-2a^2\sin^2\alpha(x+y)^2\{b^2x^2+x^2(x+y)^2\}$$
$$\sin^2(\alpha+\varphi)+a^4(x+y)^4\sin^4\alpha=0 \quad \text{................(6).}$$

Eliminating $\sin(\alpha+\varphi)$ between (5) and (6) we get an equation in x and y which with (1) gives us the value of x.

Solved with result in terms of BC by *A. H. HOLMES*, and *FREDERICK R. HONEY*.

PROBLEMS.

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56. Proposed by I. J. SCHWATT, Ph. D., Instructor in Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania.

1. The point of intersection K_a' of the tangent drawn to the circumcircle about the triangle ABC at A and the side BC is harmonic conjugate to K_a with respect to BC. (K_a is the point where the symmedian line through A of the triangle ABC meets the side BC.)

2. The point K_a' is the center of the Apollonius circle passing through A of the triangle ABC.

3. Grebes point is on the line joining the middle point of any side of a triangle with the middle point of the altitude to this side.

50. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Show that the tangent plane at any point of the surface $a^2x^2+b^2y^2+c^2z^2=2bcyz+2aczx+2abxy$ intersects the surface $ayz+bzx+cxy=0$ in two straight lines at right angles to one another.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

47. Proposed by Prof. J. SCHEFFER, A. M., Hagerstown, Maryland.

The floor of a vault forms a square, and all sections parallel to it are squares. The two vertical sections through the middle points of the opposite sides of the floor are equal semi-circles. Find the convex surface and the volume of the vault.

I. Solution by C. W. M. BLACK, Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts; O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland; A. H. HOLMES, Brunswick, Maine, and the PROPOSER.

Let $ABCD$ represent the base square, side$=2a$, and KEI and GFH the two equal semi-circles, radius$=a$. Let $LMNO$ be another square parallel to the base square, and at the distance $PE=x$ from it. The area of $LMNO$ is$=4(a^2-x^2)$,

$$\therefore\ \text{Vol.}=4\int_0^a(a^2-x^2)dx=\tfrac{8}{3}a^3.$$

Denoting $\angle PEQ$ by θ, we have for the surface

$$\int_0^{\frac{1}{2}\pi}8a\cos\theta d(a\theta)=8a^2.$$ Or for the volume, $dV=4a^2\cos^2\theta dx$, where x is the vertical distance. $x=a\sin\theta$; $dx=a\cos\theta d\theta$.

$$\therefore\ V=4a^3\int_0^{\frac{1}{2}\pi}\cos^3\theta d\theta,=a^3\int_0^{\frac{1}{2}\pi}(\cos3\theta+3\cos\theta)d\theta=\frac{8a^3}{3}.$$

II. Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

The convex surface of the vault is equivalent to the surface of a right circular cylinder intercepted by another right circular cylinder, their axes intersecting at right angles, the two cylinders being equal, and the diameter of each equal to that of the vertical sections of the vault.

$$\therefore \text{ Letting the radius}=a,\ S=8a\int_0^a\int_0^{\sqrt{a^2-x^2}}\frac{dxdy}{\sqrt{a^2-x^2}}=8a^2,\ \text{the equa-}$$

tions of the cylinders being $x^2+z^2=a^2$, and $x^2+y^2=a^2$.

The volume is equivalent to that of four wedges cut from the cylinder, $x^2+y^2=a^2$, by the planes, $z=0$, and $z=x$.

$$\therefore V=8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^x dxdydz=\frac{8a^3}{3}.$$

Also solved by *E. L. SHERWOOD* and *G. B. M. ZERR.*

40. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

I have a circular section basin 12 inches in perpendicular height; the diameters are as follows: At base, 2 inches; one inch perpendicular height, 6 inches; two inches perpendicular height, 18 inches; three inches perpendicular height, 54 inches; and so on, the diameter being trebled for every inch in height. After a rain the water in the basin is six inches deep, what was the rainfall?

I. Solution by E. L. SHERWOOD, A. M., Professor of Mathematics, Mississippi Normal College, Houston, Mississippi; C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts, and the PROPOSER.

The basin is generated by revolving the curve $x=3^y$ about the axis of y.

$$\therefore \text{ Volume of water}=\pi\int_0^6 x^2dy=\pi\int_0^6 3^{2y}dy.$$

$$\therefore V=\pi\frac{3^{12}-1}{2\log 3}=\frac{531440\pi}{2\log 3}.$$

Let x=depth of rain-fall, then since radius of top of basin$=3^{12}$, $V=\pi 3^{24}x$.

$$\therefore x=\frac{565720}{282429536481\log 3}=.00000036 \text{ inches.}$$

II. Solution by G. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

Call x the length of any radius, and y the vertical distance, y being 1 at the bottom of the basin. Then the equation of side of basin is $x=3^{y-1}$,

$$V=\pi x^2dy,\ V=\pi\int_1^y 3^{2y-2}dy=\frac{\pi[3^{12}-1]}{3\log_e 3}.$$

The radius of upper base$=3^{12}$. Call R the rainfall, then

$$\pi 3.^{24} R = \frac{\pi[3^{12}-1]}{2\log_e 3}. \quad R = \frac{3^{12}-1}{2.3.^{24}\log_e 3}.$$

Also solved by *A. H. HOLMES*, *J. SCHEFFER*, and *B. F. YANNEY*.

ERRATA. In last issue, page 120, line 4 from bottom, for "$\rho=\frac{\theta^2}{c^2}$" read, $\rho^2=\frac{\theta^2}{c^2}$.

PROBLEMS.

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55. Proposed by GEORGE LILLEY, Ph. D., LL. D., Principal of Park School, 394 Hall Street, Portland, Oregon.

A horse is tethered by a rope, a feet long, fastened to a post in a circular fence enclosing a circular piece of ground b feet in diameter. If the horse is outside of the fence over how much ground can he feed? If he is inside the fence over how much ground can he feed? $b>a$ in each case.

56. Proposed by Prof. B. F. BURLESON, Oneida Castle, New York.

Find (1) the length s of the closed curve of the cardioid; (2) its area A; (3) if made to revolve about its axis $2a$, find the maximum longitudinal circumference C of the solid generated; (4) find the surface K of the same; (5) its volume V; (6) the distance x_0 of the center of gravity of the solid from the origin O; and (7) the distance g_0 of the center of gravity of the plane curve from the origin O.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

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31. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

A perfectly elastic, but perfectly rough mass M, and radius R, rotating in a vertical plane with an angular velocity ω, is let fall from a height, a, upon a perfectly elastic but perfectly rough horizontal plane. Determine the motion of the body after striking the plane. What will be its ultimate motion?

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let V be the vertical velocity of the center just before impact; u, v, the horizontal and vertical velocities of the center just after the first impact; ω, the ılar velocity after first impact; u' the velocity of the center just before the nd impact; u_1, ω_2 the values of u, ω_1 just after the second impact, k the ra- ı of gyration.

The equations of motion for first impact are

$$(v+V)(k^2+R^2)=2V(k^2+R^2)\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$u(k^2+R^2)=\omega Rk^2\ \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The geometrical condition for no sliding is

$$u-R\omega_1=0\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$V=\sqrt{2ag}$, $k^2=\frac{2}{5}R^2$.

$\therefore\ v=\sqrt{2ag},\ u=\frac{2}{7}R\omega,\ \omega_1=\frac{2}{7}\omega,\ u'=\sqrt{v^2+u^2}=\frac{1}{7}\sqrt{4\omega^2R^2+98ag}.$

If β be the angle the center of the sphere makes with the plane just after pact we easily get

$$\cos\beta=\frac{u}{\sqrt{u^2+v^2}}=\frac{u}{u'},=\frac{2R\omega}{\sqrt{4\omega^2R^2+98ag}}.$$

Thus the motion is determined after striking the plane. Let F be the im- lse arising from friction, then the equations of motion for second impact are,

$$Mu_1=Mu'\cos\beta+F\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$\tfrac{2}{5}MR^2\omega_2=\tfrac{2}{5}MR^2\omega_1-RF\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

d the geometrical condition $u_1-R\omega_2=0\ldots\ldots\ldots\ldots\ldots\ldots(6).$

$\therefore\ F/M=-\frac{2}{7}(u'\cos\beta-R\omega_1),\ u_1=R\omega_2=\frac{5}{7}u'\cos\beta+\frac{2}{7}R\omega_1,$

t $u'\cos\beta=R\omega_1$, $\therefore\ F/M=0$, and no impulsive friction is called into play af- the first impact. Hence the center of the sphere describes the same parabola er each impact and the ultimate motion is the same as that after striking the ıne.

III. Solution by the PROPOSER.

Each motion of the sphere may be considered, in its reactionary effect, ıarately. The motion of translation will cause the sphere to rebound after ‌h impact to its original altitude. The time taken to attain the altitude a will

$t=\sqrt{\frac{2a}{g}}.$

The effect of the motion of rotation may be considered in this way: Let rotating sphere be brought into contact with a plane slowly. The sphere will, course, roll along the plane. The energy of translation and rotation being ıual to the original energy, E, we shall have the same result in the case under consideration, that is, we shall have the same velocity parallel to the plane, and the same angular velocity as if the sphere were in contact *with the* plane, because there is no slipping at the instance of contact.

Let v_1=velocity parallel to the plane. Then $\frac{v_1}{R}$=new angular velocity $=\omega_1$.

$$E=\tfrac{1}{5}MR^2\omega^2.$$

$$\text{Energy of translation}=\tfrac{1}{2}Mv_1{}^2.$$

$$\text{New energy of rotation}=\tfrac{1}{5}MR^2\omega_1{}^2=\tfrac{1}{5}Mv_1{}^2.$$

$$\therefore\ \tfrac{1}{5}MR^2\omega^2=\tfrac{1}{2}Mv_1{}^2+\tfrac{1}{5}Mv_1{}^2.\quad \text{Whence,}$$

$$v_1{}^2=\tfrac{2}{7}R^2\omega^2.$$

$$\therefore\ v_1=\sqrt{\tfrac{2}{7}}R\omega, \text{ and}$$

$$\omega_1=\sqrt{\tfrac{2}{7}}\omega.$$

The distance which the sphere will move parallel to the plane while it is attaining its highest altitude will be $=tv_1=2\sqrt{\frac{a}{7g}}R\omega$.

From these data, knowing that the curve will be a parabola, we obtain

$$y^2=\frac{4R^2\omega^2}{7g}x,$$

the highest point in the origin. The distance between first and second impact is $4\sqrt{a/7g}\,R\omega$. As to the subsequent motion, we have the equation of energy

$$\tfrac{1}{2}Mv_1{}^2+\tfrac{1}{5}Mv_1{}^2=\tfrac{1}{2}Mv_2{}^2+\tfrac{1}{5}Mv_2{}^2,\ \text{or } v_2=v$$

and the subsequent parabola will be the same as the first.

PROBLEMS.

37. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

A thin board of which the elements are given is balanced at the center but inclined at an angle. A sphere of known dimensions is put directly above the point of suspension and liberated. Find the motion of the system. That is, find (a) the time until the sphere leaves the board, (b) the ultimate angular velocity of the board.

38. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

A prolate spheriod of revolution is fixed at its focus; a blow is given it at the extremity of the axis minor in a line tangent to the direction perpendicular to the axis major. Find the axis about which the body begins to rotate. [From *Loudon's Rigid Dynamics.*]

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

37. Proposed by A. H. BELL, Box 184, Hillsboro, Illinois.

Find the first four integral values of n in $\frac{n(5n-3)}{2}=\square$.

I. Solution by the PROPOSER, and Prof. J. SCHEFFER, A. M., Hagerstown, Maryland.

Let the heptagonal numbers $\frac{n(5n-3)}{2}=\square=y^2$. Clearing of fractions, ıen multiplying by 20 and adding 9 to both sides, $(10n-3)^2=40y^2+9=\square=x^2$. . $n=(x+3)/10$........(1). Let $x^2-40y^2=9$ be written $3^2x_1{}^2-40.3^2y_1{}^2=3^2$. ividing by 3^2 and solving $x_1{}^2-40y_1{}^2=1$, the convergent of $\sqrt{40}$ is $19/3$. . $x_1=19$; by the general formula $x_{n+1}=2x_1\times x_n-x_{n-1}$, we have $x_1=1$, 19, 721, 7379, 1039681, 39, 480, 499, etc. As $x=3x_1$ and as integral values for n can nly be obtained by the numbers ending in 9, then in (1) $n=1$, 6, 8214, nd 11844150.

II. Solution by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

The expression readily reduces to $10n^2-6n=\square$......(1). It is readily een that $n=1$ satisfies this equation. Take $n=m+1$, substitute it in (1), reduce nd we have $10m^2+14m+4=\square=$(say) $(pm-2)^2$, from which we obtain $m=(4p+14)/(p^2-10)$. Take $p=4$ and we have $m=5$, and $n=6$, the second value. Now take $n=m+6$, substitute in (1) and reduce as before and we find, $m=43$, and $n=49$, the third value. In $(4p+14)/(p^2-10)$ take $p=19/6$, $p^2-10=1/36$ and we have $m=960$, and $n=961$, the fourth value.

III. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

If we put the expression equal x^2 and reduce, we readily obtain $10n=3\pm\sqrt{40x^2+9}$. Putting $x=1$, 2, 9, 40 and 77, respectively, I find the first four integral values of n to be, respectively, ±1, 6, -25, and 49.

38. Proposed by H. C. WILKES, Skull Run, West Virginia.

Let n be any number and let $n^3+1=x$.

Then $x^3+(2x-3)^3+(nx-3n)^3=n^3x^3$. Demonstrate.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

The simplest way is to substitute the value of x and expand. An identity the result.

II. Solution by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Substituting n^3+1 for x, $(n^3+1)^3+(2n^3-1)^3+(n^4-2n)^3=(n^4+n)^3$, which, if we put c for n, is the same as equation (12) on page 155 of Vol. II., No. 9 of the *Mathematical Magazine*, and is an identity as will be found by performing the indicated operations and adding.

III. Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

Suppose the statement true: $x^3+(2x-3)^3+(nx-3n)^3=n^3x^3$.

Then, $x^3+8x^3-36x^2+54x-27+n^3x^3-9n^3x^2+27n^3x-27n^3=n^3x^3$.

Whence, $(x-1)(x^2-3x+3)-n^3(x^2-3x+3)=0$. Whence, $x-1-n^3=0$.

Whence, $n^3+1=x$, which is the hypothesis. ∴ The above supposition is true.

IV. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics in the University of Tennessee, Knoxville, Tennessee.

Performing the operations we have

$$x^3+8x^3-36x^2+54x-27+n^3x^3-9n^3x^2+27n^3x-27n^3, \text{ or}$$

$$9x^3-36x^2+54x-27+n^3x^3-9n^3x^2+27n^3(x-1), \text{ but } n^3=x-1;$$

hence $9x^3-36x^2+54x-27+n^3x^3-9n^3x^2+27(x-1)^2$, which upon reduction gives

$$9x^3-9x^2-9n^3x^2+n^3x^3,=9x^3-9x^2-9x^2(x-1)+n^3x^3,$$

$$=9x^3-9x^2-9x^3+9x^2+n^3x_3,=n_3x_3.$$

Also solved by *J. H. DRUMMOND, M. A. GRUBER, J. SCHEFFER*, and *G. B. M. ZERR*.

39. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

The m^{th} root of the n^{th} power of an *integral* number is a perfect p^{th} power. What is the number?

Solutions by J. H. DRUMMOND, LL. D., Portland, Maine; M. A. GRUBER, A. M., Washington, D. C., and J. SCHEFFER, A. M., Hagerstown, Maryland.

Let $x^{1+m}=a^p$, then $x=a^{pm}$, in which a may be any integral number: for the n^{th} power of a^{pm} must also be a p^{th} power. [J. H. DRUMMOND.]

Manifestly any whole number raised to the mp^{th} power.

[J. SCHEFFER.]

Let x=the integral number. Then $x^{n+m}=a^p$. Raising to m^{th} power and extracting n^{th} root, we obtain $x=a^{mp+n}$, or $\sqrt[n]{a^{mp}}$. ∴ The required integral number is the n^{th} root of the mp^{th} power of any integer, mp being a multiple of n.

[M. A. GRUBER.]

Also solved by *G. B. M. ZERR*.

PROBLEMS.

45. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the first six sets of values in which the sum of two consecutive integral squares equals a square.

46. Proposed by B. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

If any positive integral number N be divided by another positive integral number D, leaving a remainder of 1, then any positive integral power of N, divided by D, will leave a remainder of 1.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

28. Proposed by B. F. FINKEL, A. M., Professor of Mathematics, Drury College, Springfield, Missouri.

What is the average area of all triangles having a given base, b, and a given vertical angle, α?

Solution by the PROPOSER.

Let ABC be a triangle whose base $AC=b$ and vertical angle $ABC=\alpha$. Let $BC=x$, $\angle BAC=\theta$, and Δ average area required.

Then $x=\dfrac{b}{\sin\alpha}\sin\theta$; and $BD=x\sin\angle BCA$

$=\dfrac{b}{\sin\alpha}\sin\theta\sin(\theta+\alpha)$.

$\therefore$ Area of the triangle $=\dfrac{b^2}{2\sin\alpha}\sin\theta\sin(\theta+\alpha)$.

The limits of θ are 0 and $\pi-\alpha$.

$$\Delta=\frac{\displaystyle\int_0^{\pi-\alpha}\frac{b^2}{2\sin\alpha}\sin\theta\sin(\theta+\alpha)d\theta}{\displaystyle\int_0^{\pi-\alpha}d\theta}=\frac{b^2}{2(\pi-\alpha)\sin\alpha}\int_0^{\pi-\alpha}\sin\theta\sin(\theta+\alpha)d\theta$$

$$=\frac{b^2}{2(\pi-\alpha)\sin\alpha}\Big[(-\tfrac{1}{2}\sin\theta\cos\theta+\tfrac{1}{2}\theta(\cos\alpha+\tfrac{1}{2}\sin^2\theta\sin\alpha\Big]_0^{\pi-\alpha}$$

$$=\frac{b^2}{4(\pi-\alpha)}\Big\{(\pi-\alpha)\cot\alpha+1\Big\}.$$

COROLLARY. Let $\alpha=\frac{1}{2}\pi$; then $\Delta=\frac{b^2}{2\pi}$, the same as problem 26.

[NOTE.—By mistake in numbering the problems in this department, number 28 was omitted. The above problem and solution are inserted that problems be numbered consecutively. EDITOR.]

29. Proposed by JOHN DOLMAN, Jr., Philadelphia, Pennsylvania.

Neglecting perturbations, what is the average distance of the earth from the sun?

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The focus being the pole, the polar equation to the ellipse is

$$r=\frac{a(1-e^2)}{1-e\cos\theta}\quad\ldots\ldots(1).$$

I. The radii vectores being drawn at equal angular intervals,

$$m'=\frac{\int r d\theta}{\int d\theta}=a(1-e^2)\frac{\int_0^\pi \frac{d\theta}{1-e\cos\theta}}{\int_0^\pi d\theta}=a\sqrt{1-e^2}=b.$$

II. If x be the abscissa of any point on the curve, the focal distance is

$$r=a-ex\quad\ldots\ldots(2),$$

$$\text{and } m''=\frac{\int_{-a}^{+a}(a-ex)dx}{\int_{-a}^{+a}dx}=a,$$

the points on the curve being so taken that their abscissas increase uniformly.

III. If the number of radii vectores depends upon the length of the curve,

$$m'''=\frac{\int r ds}{\int ds},$$

ds being an element of the curve.

Also solved as I. above by *Profs. F. P. MATZ*, and *O. W. ANTHONY*, and as III. by *Prof. G. B. M. ZERR*.

PROBLEMS.

37. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

Required the average area of all triangles two of whose sides are a and b.

30. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Two arrows are sticking in a circular target: show that the chance that their distance is greater than the radius of the target is $3\sqrt{3}/4\pi$. [From *Todhunter's Integral Calculus*, page 35.]

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

33. Proposed by Prof. ALEXANDER ROSS, C. E., Sebastopol, California.

From a point P without a square field ABC, the distances PA, PB, and PC measured to the corners are, respectively, 70, 40, and 60 chains. What is the area of the field?

I. Solution by A. H. BELL, Hillsboro, Illinois, and A. H. HOLMES, Brunswick, Maine.

Let $a>b>c$ equal the distances 70, 60, and 40, and let $x=a$ side of the square field. Then $\cos A=\dfrac{a^2+x^2-c^2}{2ax}$, and this multiplied by $a=AF=\dfrac{a^2+x^2-c^2}{2x}$. $AF-AB=BF=EP=\dfrac{a^2-c^2-x^2}{2x}$; then, also, $BE=\dfrac{b^2-c^2-x^2}{2x}$.

$$\therefore \frac{(a^2-c^2-x^2)^2+(b^2-c^2-x^2)^2}{4x^2}=c^2 \quad\ldots\ldots\ldots\ldots\ldots(1).$$

$$x^4-(a^2+b^2)x^2=c^2(a^2+b^2)-\frac{a^4+b^4+2c^4}{2} \quad\ldots\ldots\ldots(2).$$

$$\text{Area of square}=x^2=\tfrac{1}{2}\left[a^2+b^2\pm\sqrt{4c^2(a^2+b^2-c^2)-(a^2-b^2)^2}\right] \ldots(3).$$

Then area required$=(8500\pm6516.901)\div2=750.84\frac{1}{2}$ or 99.155 acres.

The second is the value required; the other is for point within the field.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let $ABCD$ be the square, $OA=70=a$, $OB=40=c$, $OC=60=b$, O the origin, (x, y) co-ordinates of A, (u, v) co-ordinates of C, $\angle ABE=\theta$, $\angle EBC=\frac{\pi}{2}-\theta$.

$$\therefore (x-c)^2+y^2=(u-c)^2+v^2,\ x^2+y^2=a^2,\ u^2+v^2=b^2 \ldots\ \ldots\ldots(1, 2, 3).$$

$\tan\theta=\frac{y}{x-c}$, $\cot\theta=\frac{v}{u-c}$, $\therefore\frac{y}{x-c}=\frac{u-c}{v}$. $\therefore yv=(x-c)(u-c)$..........(4.)

(2) and (3) in (1) and (4) gives, $2c(x-u)=a^2-b^2$, ...(5),

$(a^2-x^2)(b^2-u^2)=(x-c)^2(u-c)^2$......................(6). (5) in (6) gives

$\{4a^2c^2-(a^2-b^2)^2-4cu(a^2-b^2)-4c^2u^2\}(b^2-u^2)=(a^2-b^2+2uc-2c^2)(u-c)^2$.

$\therefore (74175-520u-16u^2)(3600-u^2)=(4u-95)^2(u-40)^2$.

$\therefore 32u^3-2840u^2+825u+3157375=0$. Let $u=z+\frac{355}{12}$.

$\therefore z^3-\frac{249575}{96}z+\frac{164676875}{3456}=0$.

This equation has three roots.

$\therefore z_1=23.02208$, $z_2=36.23197$, $z_3=-58.43863$.

$\therefore u_1=52.60541$, $u_2=65.81530$, $u_3=-28.85530$.

$\therefore x_1=69.35541$, $x_2=82.56530$, $x_3=-12.10530$.

$\therefore y_1=9.47772$, $y_3=68.94535$.

The first values satisfy the problem in question; the second must be rejected as not admissible; while the third values satisfy the problem for the point within the field.

$\therefore$ area $ABCD=(x-c)^2+y^2=951.5672$ square chains $=95.15672$ acres.

When the point is within field, area $=(x-c)^2+y^2=7468.424$ square chains $=746.8424$ acres.

Also solved by *O. W. ANTHONY*.

84. Proposed by THOS. U. TAYLOR, C. E., M. C., Department of Engineering, University of Texas, Austin, Texas.

Given a variable parallelogram $ABCP$, where P remains fixed. A moves on an irregular plane curve (closed) and C moves on another irregular plane curve (closed) whose plane is parallel to the plane of (A) curve. The generator PC moves completely around and returns to its initial position, AB always moving parallel to PC, and, of course, returns to its initial position. If distance between planes (A) and (C)$=h$, show by elementary mathematics and without using theorem of Koppe that volume of solid generated by variable parallelogram $ABCP=\frac{1}{3}h$ (area generated by AP+area generated by BC).

No solution of this problem has been received.

PROBLEMS.

88. Proposed by F. M. PRIEST, Mona House, St. Louis, Missouri.

Suppose two cylindrical iron shafts, each 6 inches in diameter and respectively, 20 and 40 feet in height, are both standing perpendicular at the sea level. They start to fall in still air, how long will it require each one to fall to a horizontal position ?

39. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, California.

A straight inflexible bar of uniform weight and thickness, length m is suspended at the two ends by a string without weight, length $l>m$ passing freely over a peg driven in a perpendicular wall. Describe and analyse the curve traced on the wall by the ends of the hanging bar.

NOTE.—Problem No. 43, Calculus, should read as suggested by Dr. E. A. Bowser on page 60 of February number. Prof. Black had noted the correct form in his copy of Williamson, and so sent it to the MONTHLY, but an error was made in printing the expression. A letter from Dr. Williamson, Trinity College, Dublin, Ireland, acknowledges the error in his work, and says it will be corrected in the forthcoming new edition of his book.

WANTED.—Some one to give a list, partial or complete, of the curves of the fourth degree that have received particular names, such as the "Lemniscate," "Cocked Hat," "Devil's Walking Stick," "Conchoid," etc.

COOPER D. SCHMITT.

BOOKS.

Warren Colburn's First Lessons: Intellectual Arithmetic upon the Inductive Method of Instruction (1891).

H. N. Wheeler's Second Lessons (1893). Boston, New York, and Chicago: Houghton, Mifflin and Company.

First Lessons, which has been famous for three-fourths of a century, contains, besides the four fundamental operations, little but fractions and denominate numbers. It has no rules and but little written work. It follows the inductive method—the method of "Practice before Theory"—which is based on the soundest psychological principles. This book should be in the hands of every teacher, whether used as a class book or not.

Wheeler's Second Lessons is intended as a continuation of Colburn's First Lessons, and is well adapted for that purpose.

J. M. C.

Logarithmic Tables. By Professor George William Jones, of Cornell University. Sixth Edition. Royal 8vo. Cloth. 160 pages. Price, $1.00. Published by the author.

These are the best tables that we have yet seen. Eighteen tables (four-place, six-place, ten-place) with full explanation for their use, for use in the class-room, laboratory, and the office. The tables of Mathematical Constants, Chemistry, Engineering, and Physics deserve special mention. Also Table IX which gives the prime factors of composite numbers less than 20000, and Tables X and XI which give the squares and cubes of all three figure numbers in full. If you want a complete and valuable set of tables buy a copy of Prof. Jones, and you will need none other. B. F. F.

Mathematical Papers Read at the International Mathematical Congress held in Connection with the Columbian Exposition, Chicago, 1893. Edited by the Committee of the Congress, E. Hastings Moore, Oskar Bolza, Heinrich Maschke, Henry S. White. Large 8vo. Cloth, 412 pages. Price, $4.00, New York: Macmillan & Co.

This important collection of important mathematical papers is given to the mathematicians of all time at no small amount of labor at the hands of the editors.

It is especially fitting that these papers, many of which indicate the high-water mark of the development of mathematics at the present time, should be collected and bound for the benefit of the mathematicians of the centuries yet to be.

Neither the management of the Exposition nor the government of the United States had made any provisions for the publication of the proceedings of any of the Chicago Congresses. No publisher was found willing to issue the papers at his own risk.

At last a guarantee fund of one thousand dollars in all was subscribed, six hundred dollars by the American Mathematical Society, and four hundred dollars by members of the Society and other mathematicians. On the basis of this guaranty fund the publication of the volume of the papers was made possible, the American Mathematical Society assuming the financial, and the Chicago Committee the editorial responsibility. *Preface.* B. F. F.

NOTES.

Dr. William B. Smith, of the Tulane University of Louisiana, has in press the first volume of his Infinitisimal Analysis.

The June number of the MONTHLY will be mailed about the 16th of the month. In this issue will appear the biography of Mr. W. J. C. Miller.

Dr. George Bruce Halsted, of the University of Texas, and Dr. David E. Smith, of the Michigan State Normal School, will spend the summer in Europe. Dr. Halsted will visit Paris, Genoa, Buda Pest, Moskow, Kazan, etc.

ERRATA. In Prof. G. B. M. Zerr's paper, "The Centroid of Areas and Volumes," in value of $\bar{x}$, bottom of page 73, in numerator read $\frac{1}{2}(2p+1)$ for "$\frac{1}{2})2p+1)$," and in denominator read $\frac{k}{2}(2n+1)$ for "$\frac{k}{2}(2n-1)$" and $\frac{h}{2}(2m+1)$ for "$\frac{h}{3}(2m+1)$. Page 75, line 8 read $\left(\frac{z}{c}\right)$ for "$\frac{z}{c}$." Page 102, last line, read $-a^4\log\left(\frac{a^2+4h+2\sqrt{2a^2h+4h^2}}{a^2}\right)$ for "$-a^4\log\left(\frac{a^2+4h+\sqrt{2a^2h+4h^2}}{a^2}\right)$" Page 103, first line, last expression in numerator read $\sqrt{2a^2h+4h^2}$ for "$\sqrt{2a^2h+4h^2}$" and in second line read dx for "bx."

W J C MILLER.

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BIOGRAPHY.

MR. W. J. C. MILLER.

BY B. F. FINKEL.

THE native place of Mr. Miller is in one of the most beautiful parts of the South coast of England. Of this district he has given a sketch in an article (in Nature-Notes for August, 1894) entitled a "Devonian Headland," which he describes as lying deep within the great West Bay of Dorset and evon, and to which sea-birds have always flocked as to a chosen retreat. The pland chalk downs end in lofty cliffs that run sheer to the water's edge: and lose by, both east and west, clear brooks, which spring from the underlying reen-sand, have worn out charming little valleys that bear the Celtic name of ombes. The headland itself bears a Norse name, derived from a village that ies in the eastern valley,—it was a little way off on the shores of the same great ay, that the Norsemen had their first historic conflict with the English—but the illage itself might well bear a similar place-name with its western neighbor, and be called more appropriately, Chalcombe.

The district was a perfect paradise of birds, with which he became perfectly familiar as a boy, and on which, in later life, he loved to write articles describing their various habits. The mobbing, by a mingled flock of rooks and jackdaws, of a pair of ancient ravens that had built for ages in a neighboring cliff, till, at last, the powerful ravens, worn out by numbers, would find shelter from their tormentors in some wood, or cleft, or cave: the motions and chirpings of the stone-chats and the win-chats in the furze-bushes: the swift and dazzling flight of the king-fisher: the finding of the habitat of the dipper or water-ouzel——a song-bird that dives, and wades, and swims—watching its motions under water, and finding its nest year after year in the same stream: and the delightful turr-turr of the turtle-doves in the woods, reminding him that, for six months, he might say, with Virgil's Melibœus,

"Nec gemere aeria cessabit turtur ab ulmo":

these and many such sights and sounds he was ever delighted to recall and record.

In such pleasant regions as these, Mr. Miller was born, on August 31, 1832; here he roamed as a boy, always fond of books; and amid these scenes he acquired that love of Nature, and especially of bird-life, that never afterwards deserted him. From the village-school he went to the Independent College of Taunton in Somerset, where he had for a time, the teaching of a distinguished Mathematician; and from there he matriculated with mathematical honors at the University of London. Then came the great disappointment of his life. He was desirous to enter the great mathematical University of Cambridge; but his parents belonged to the sect that had trampled down King, Church, and Aristocracy, one after the other; that had formed an army that had never met, either in the British Islands or on the Continent, an enemy that could stand its onset; and had sent across the Atlantic a band which, fleeing from persecution, had founded the third home of the great English race. Thus they could not endure that a son of theirs should submit to the tests then imposed in the University; so he had to give it up. Years afterward, he learned from eminent mathematicians, that the best of all science was learned by one's own self, and never derived from any Professors at College or University. But he would then have gladly submitted to any test, if he had been allowed.

So he turned to study and instruction in mathematics; and after teaching at various Institutions, became finally Professor at Huddersfield College in Yorkshire, where he remained many years, till he took the post that he now holds. There it was that he devised, and, after many trials, got a Publisher to undertake, the series of Volumes that he has edited ever since, and of which he is now engaged upon the sixty-fifth Volume. It was in 1861 that he conceived the idea of devising some plan whereby the contributions to the mathematical columns of ths *Educational Times*, which had been for some years under his Editorship, might be presented, apart from other matter, in a more convenient form than could be furnished by the pages of the Journal; and, after ascertaining the views of his contributors, and obtaining promises of support, the mathematical solutions that appeared in each number were, from Midsummer, 1863, printed off, in the narrow columns then in use, from the journalistic types; and at the end of a year the collection was, in July, 1864, issued as the first of the series. By and by the narrow columns were altered to wider columns; and then the contributors were not content to wait a year for their articles: thus, ultimately, the issues took place at half-yearly intervals.

The series that took its rise from such small beginnings has gone on continuously from that time to this; and is going on still. After 25 years it was necessary to issue a second edition of the first volume; and this was brought out with improvements, uniform with the other Volumes, in the wider columns, in

1886. In these Volumes there have appeared, from time to time, articles in almost all branches of Mathematics, and the leading Mathematicians of all countries have continuously helped the work forward. One valued contributor, among early ones, was Dr. Hirst, F. R. S., who developed, in various articles, those elegant branches of Geometry in which all took a deep interest; and who, at last, collected and published his contributions in a separate Volume. Other important contributors to the early Volumes were Professor Cayley, to whom many articles were due; and the too-early lost Professor Clifford, who, being a fellow Devonian with the Editor, began to write when he was flying kites, and continued to furnish articles that increased in number and value through many volumes, accompanied by letters to the Editor that contained comments and developed views that were often more interesting than the articles themselves. The comparatively new theory of Local Probability was largely developed in the early Volumes by such writers as Woolhouse, Clarke, Crofton, Stephen Watson, our countryman, the late Professor E. B. Seitz who was a great master of difficult Probability Problems, and others. These contributors have all passed into the silent land. From a contributor who, it is to be feared, is getting near it, Professor Sylvester, articles followed in such quick succession that, from the very earliest times, there were but three or four numbers of the Journal, and those through the merest inadvertence, that did not contain, till the very last, at least one of his articles.

In 1876, Mr. Miller obtained the highly responsible post of Executive Officer (Registrar and Secretary) of the General Medical Council, an office in which he has remained ever since, continuing to edit, at his leisure, the mathematical periodical that has now attained to its 65th Volume. Among Editors of Mathematics this is deemed to establish for Mr. Miller what is termed "a Record": seeing that no other Mathematical Editor has ever, it is believed, gone on so long, with such laborious work as this. Always interested in Literature, no less than in Science, he edited for his Students at Huddersfield College a Magazine in which there came forward young contributors who afterward attained to eminence, whereof one has recently written an able book on the geography and resources of Africa.

During this time, he has been living in the finest of all the suburbs of London, in that Richmond whose name has been transferred to many other places, notably to that city which figured so largely in the Civil War. Under the title of "a Bird-loved Suburb of London" he has written an article to set forth its Bird-life, and its many beauties.

Here he founded in 1887, a Literary Society of which he is still President, and before which, on March 20th, one of his Mathematical contributors, Mr. George Heppel, M. A., lectured on the "Origins of European Poetry." In the course of his introductory remarks that evening, the chairman, Mr. Miller, said, "We are this evening entering upon a new departure. Hitherto, the lecturers have been members of our own society, but, in bringing in now for the first time a lecturer from outside, we are adopting a course that might hereafter be worked out with advantage by our able and energetic secretary. Mr. Heppel is a mathematician, and such men have long been found peculiarly sensitive to the influence of the sister-arts of music and poetry. The very greatest of all living mathematicians [Professor Sylvester] called the attention of the Royal Society, twenty-five years ago, to the coincidence or parallelism, which observation has long made familiar, between the mathematical and the musical ethos: music being the mathematics of the sense, mathematics the music of reason; the soul of each the same. Music the dream, mathematics the working life; each to receive its consummation from the other when the human intelligence, elevated to its perfect type, shall shine forth glorified in some future Beethoven-Gauss."

Other doings of Mr. Miller's during his life in Richmond, and his official duties at the General Medical Council, are set forth in the following article from the *Richmond and Twickenham Times* for August 17, 1889:—

"Those who attended the meetings of the Richmond Athenæum, and the far larger number who read the reports of the proceedings of that body, are familiar with the pleasant, gracefully worded, and often erudite little speeches of Mr. W. J. C. Miller, a member of the council who has always been, in a double sense, a right-hand man to the chairman, sitting upon his right on the platform, and always ready, however abtruse the subject, to save a debate from flagging by filling up the regulation ten minutes with remarks which are always appropriate, often profound, invariably couched in the happiest words, and abounding in quotations from the poets, displaying a memory which is the admiration of all. Comparatively few, however, in Richmond know of the laborious and difficult duties in the world of mathematics to which Mr. Miller has devoted himself for more than thirty years, as editor of the *Educational Times*, or of the position which he has filled for thirteen years as the sole executive officer (registrar and secretary by name) in the management of the business of the General Medical Council.

With regard to Mr. Miller's editorial duties, many eminent mathematicians have given ungrudging testimony to their value. Thus Professor Sylvester—the first of living mathematicians—speaks of him as "an excellent mathematician, extensively and critically versed in all parts of the science, a good writer and lecturer on various subjects of natural science and other parts of human knowledge lying outside his own more special pursuits, and a most able and painstaking editor. . . . His scientific attainments are of a high order; he is deeply skilled in nearly all the departments of the highest mathematics, and is a novice in none. His labour as mathematical editor of the *Educational Times*, in which his own original papers are fit company for those of our foremost analysts, is proof of that. It would be a mistake to suppose him a mere schoolmaster or a mere mathematician. He is a sound classical scholar, and an erudite man of letters." The late lamented Professor Clifford considered that the mathematical portion of the *Educational Times* "has done more to suggest and encourage original research than any other European periodical." Equally gratifying words are used by Sir Robert Ball, Professor Tait, Dr. Hirst, Rev. Dr. Booth, Professor Crofton, Colonel Clarke, Dr. S. T. Hall, Professor Townsend, Professor Young, Dr. Todhunter, Rev, George Salmon, Professor Cayley, Professor Everett, and others whose attainments have raised them to the highest eminence. It has often been said that by Mr. Miller's mathematical work, the culture and study of the science have been more advanced than by any two or three agencies put together, in any or all parts of the world. When he commenced this important work he had but what was then an utterly obscure and almost unknown journal to use as his means of intercommunication and publication. Now he has nearly five hundred vigorous contributors, from all parts of the globe. Many are educated Hindoos (professors and others); many are Americans or Australians; still more are Germans, Frenchmen, Russians, or Italians; some are Spaniards; and some write from the South American Republics.

The multifarious work of the General Medical Council has more than quadrupled since Mr. Miller took it in charge thirteen years ago. Established to carry out the voluminous Medical Acts (which cover fifty-nine pages of the *Medical Register*), the Council had to take charge, in 1878, of all the dentists in the empire, and since then of various other matters, including, quite recently, the registration of sanitary officers. Many testimonies to the appreciation of Mr. Miller's services have been given by the Council, and by the medical newspapers. Thus the *Medical Press*, in a recent article on the General Medical Council, says that—"Every session marks a distinct improvement in the business aptitude of the Council, and in the amount of work accomplished, results which may fairly be attributed in no small degree to a more vigorous presidential control, and to the efficiency of the business arrangements, which depend so much for their success on the services of a competent and attentive registrar." The *British Medical Journal*, in reviewing the *Minutes of the General Medical Council*, says—"The Volume has been edited by Mr. W. J. C. Miller, B. A., the Registrar of the General Council, with the care which he has accustomed us to expect from him." The *Report of the Statistical Committee of the General Medical* meci is another work of which Mr. Miller has charge, and in noticing this the *Medical Press* says—"We sume that its compilation is chiefly due to the energy and noted mathematical skill of Mr. Miller, the gistrar of the Council, and if we are correct in this assumption we can only remark that both the prosion and the Medical Council owe that gentleman much thanks for work which, though no doubt a lar of love, must involve great devotion of time and mental capacity." Another work of the utmost portance to the public, and for the annual publication of which Mr. Miller is responsible, is the *Medil Register*, which has now grown to a volume of 1,198 pages. In addition to this there is the *Dentist's gister* (222 pages), besides the *Medical Students' Register*, the latter alone requiring 100 pages. It would difficult to speak too highly of the care exhibited in the compilation of these important works. Refer-g to the issues for the present year, the *Medical Press* says "They display all the progressive improvent which has been manifested since Mr. Miller took them in hand."

Being an ardent lover of science and literature, Mr. Miller has all his life striven to aid others in aring their delights, by lectures, writings, and teaching. And all this work, editorial and other, has en not only unremunerative, but carried on with no little outlay. But the world's best workers have rays been the most unselfish. Mr. Miller has at least the gratification of knowing that his favourite rsuits have been greatly advanced by his efforts, and that he has earned the gratitude of many o have reaped the advantages of his self-denying work."

Mr. Miller was one of the earliest members of the London Mathematical ociety; but as he found that, with his official duties, and his Editorial work, he uld not spare time to attend the Meetings, he was reluctantly compelled to regn his membership. Since that time, he has had to devote the whole of his nall leisure to the duties of his Editorship, which goes on increasing every onth, with new contributors from foreign countries, especially India, where an larged interest is rapidly growing in all the articles that are published in his ournal. Mr. Miller is a great admirer of America and American ways of manging; he entertains a high opinion of our magazine, and says it is one of the best at comes to him. He has a large circle of friends and admirers in America, ost of whom are contributors to the Mathematical Department in the *Educaonal Times*.

HE EXPONENTAL DEVELOPMENT FOR REAL EXPONENTS.

By WILLIAM BENJAMIN SMITH, Ph. D., (Görtingen) Professor of Mathematics, Tulane University, New Orleans, Louisiana.

The Exponental Series is of such fundamental and far-reaching importnce, it is so indispensable to all higher Analysis, that it seems strange so few if y deductions of it accessible to the English reader should be carefully conduct-; not even that given by Chrystal in his superb *Treatise on Algebra* can lay aim to rigor. It may be worth while then, under no pretense of novelty, to atnpt to supply this lack in some measure.

I. We consider the expansion given by the Binomial Theorem:

$$\left(1+\frac{x}{n}\right)^n = 1 + n.\frac{x}{n} + \frac{n(n-1)}{\underline{|2}}.\frac{x^2}{n^2} + \frac{n(n-1)(n-2)}{\underline{|3}}.\frac{x^3}{n^3} + \dots\dots$$

$$=1+x+(1-\frac{1}{n}).\frac{x^2}{\underline{|2}}+(1-\frac{1}{n})(1-\frac{2}{n}).\frac{x^3}{\underline{|3}}+\ldots\ldots,$$

where x is finite and positive, while n is *positive* and *integral*, and we inq whether this series tends toward a definite form and value as n increases witl limit.

We denote the differences $1-\frac{1}{n}$, $1-\frac{2}{n}$, ... $1-\frac{k}{n}$ ..... by

d_2, ...d_k, ... and the products of these, d_1, d_1d_2, $d_1d_2d_3$, ...d ......d_k, ...by p_1, p_2, p_3, ...p_k, ...

Then plainly $1>d_1>d_2>d_3>\ldots\ldots>d_k>\ldots\ldots$

and also $1>p_1>p_2>p_3>\ldots\ldots>p_k>\ldots\ldots$

In the expansion there are $n+1$ terms, which we may write t_0, t_1, ...t_r, ...t_n. We consider the sum $t_0+t_1+\ldots\ldots+t_r$ and denote it S_r; then the sum of remaining $n-r$ terms we denote by V_r, so

$\left(1+\frac{x}{n}\right)^n=S_r+V_r$ where

$$S_r=1+x+p_1\frac{x^2}{\underline{|2}}+p_2\frac{x^3}{\underline{|3}}+\ldots\ldots+p_k.\frac{x^{k+1}}{\underline{|k+1}}+\ldots\ldots+p_{r-1}.\frac{x^r}{\underline{|r}},$$

$$V_r=p_r.\frac{x^{r+1}}{\underline{|r+1}}+p_{r+1}.\frac{x^{r+2}}{\underline{|r+2}}+\ldots\ldots+p_{n-1}.\frac{x^n}{\underline{|n}}.$$

Since n is to be taken *great at will*, r may also be taken *great at will* yet *always less than n*. We now ask, what becomes of S_r as r increases witl limit while always $r<n$? Since all the p's are <1, it is plain that

$$S_r<\{1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ \ldots\ldots+\frac{x^r}{\underline{|r}}\}.$$

However we can make *each* of the p's$>1-\sigma$, where σ is *small at will.* is enough to prove this for the *least* of the p's, p_r. We have

$$p_r=(1-\frac{1}{n})(1-\frac{2}{n})\ldots\ldots(1-\frac{r-1}{n})>(1-\frac{r-1}{n})^{r-1}.$$

Now $(1-\frac{r-1}{n})^{r-1}=1-\sigma$ if $1-\frac{r-1}{n}=(1-\sigma)^{\frac{1}{r-1}}$, or if $\frac{r-1}{n}=1-(1-\sigma)^{\frac{1}{r-1}}$,

$$n \overline{\overline{>}} \frac{r-1}{1-(1-\sigma)^{\frac{1}{r-1}}}.$$

Now for any finite value of r *however great*, and for any finite value of σ *however small*, this fraction on the right will always be finite and perfectly difinite, though perhaps very great; hence it will always be possible to choose n equal or even greater (in case the fraction be not integral in value); hence it will always be possible to make $p_r > 1-\sigma$, no matter how great r or how small σ, by merely choosing n great enough, and this we can always do, since n is quite at our will.

$$\text{Hence } S_r > (1-\sigma)\{(1+x+\frac{x^2}{\underline{|2}}+ \ldots\ldots + \frac{x^r}{\underline{|r}}\}.$$

$$\text{Hence } \{1+x+\frac{x^2}{\underline{|2}}+ \ldots\ldots + \frac{x^r}{\underline{|r}}\} > S_r > (1-\sigma)\ \{1+x+\frac{x^2}{\underline{|r}}+ \ldots\ldots + \frac{x^r}{\underline{|r}}\}.$$

Hence S_r differs from $\{\ldots.\}$ by less than $\sigma\{\ldots.\}$. Now this brace, $\{\ldots.\}$ is of course finite for all finite values of r, and it also remains finite (for x finite) even for r increasing without limit. For the ratio of two consecutive terms is $\frac{x}{k}$ and this ratio is not only finitely <1 for $k>x$ but it becomes ever smaller and smaller, sinking below every assignable degree of parvitude as k increases without limit, x of course being finite and fixed, however great. Hence $\sigma\{\ldots\ldots\}$ is small at will, since the product of a magnitude small at will multiplied by a finite number is itself small at will. Hence the two magnitudes $\{\ \ldots..\}$ and $(1-\sigma)\{\ldots..\ \}$ close down upon each other as r increases without limit, hence they close down upon S_r always between them, so that we have

$$\text{Lim.}S_r = 1+x+\frac{x^r}{\underline{|r}}+ \ldots\ldots + \frac{x^r}{\underline{|r}}.$$

for r and n increasing without limit, $n>r$. It remains to examine V_r. We may

$$\text{write } V_r = t_{r+1}(1+d_{r+1}\,\frac{x}{r+2}+d_{r+1}d_{r+2}\,\frac{x^2}{(r+2)(r+3)}+ \ldots..\ ).$$

Now $t_{r+1} < \frac{x^{r+1}}{\underline{|r+1}}$ and this we have just seen is small at will for r great at will, or $t_{r+1} < \sigma$. Also the parenthesis $(\ \ldots.\ ) < [1+\frac{x}{r+2}+\frac{x^2}{(r+2)^2}+\ldots.]$,

and this bracket $[\ldots\ldots]$ is finite for all finite values of n and r, and it has 1 for its limit as n and r increase without limit. Hence $V_r < \sigma$, or Lim. $V_r = 0$, hence

$$\text{Lim.}\,(1+\frac{x}{n})^n = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots\ldots \textit{ in infinitum.}$$

II. Thus far n has been integral at every stage of value. What if it be *fractional* or *irrational?* The preceding proof does not then apply but we shall always have n lying between two consecutive integers, or $w<n<w+1$.

Now $(1+\frac{x}{n})^n < (1+\frac{x}{w})^{w+1}$, a greater number raised to a higher power; or

$(1+\frac{x}{n})^n < (1+\frac{x}{w})^w\,(1+\frac{x}{w})$. Likewise $(1+\frac{x}{n})^n > (1+\frac{x}{w+1})^w$, a smaller

number to a lower power. Or $(1+\frac{x}{n})^n > (1+\frac{x}{w+1})^{w+1}/(1+\frac{x}{w+1})$.

Now for w and $w+1$ increasing without limit we have just proved that

$(1+\frac{x}{w})^w$ and $(1+\frac{x}{w+1})^{w+1}$ both close down upon one and the same Limit,

$$1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots$$

Also the multiplier $1+\frac{x}{w}$ and the divisor $1+\frac{x}{w+1}$ both close down upon

the same limit 1; hence the product $(1+\frac{x}{w})^w.(1+\frac{x}{w})$ and the quotient

$(1+\frac{x}{w+1})^{w+1}/(1+\frac{x}{w+1})$ both close down upon the same Limit,

$$1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots;$$

hence $(1+\frac{x}{n})^n$ lying always between this product and this quotient itself closes

down upon the same limit; hence $\text{Lim.}(1+\frac{x}{n})^n = 1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots$ for

finite positive x and for positive n increasing no matter how without limit.

III. For x *negative* we must consider $(1-\frac{x}{n})^n$. This we write $=E-O$, a sum of even powers, less a sum of odd powers, of x. Each of these we break up into two parts, S_r and V_r, and reiterate the foregoing argument with insignificant and self-evident modifications. There results

$$\text{Lim.}(1-\frac{x}{n})^n = 1 - x + \frac{x^2}{\underline{|2}} - \frac{x^3}{\underline{|3}} + \ldots\ldots$$

for finite positive x and n increasing no matter how without limit. Or

$$\text{Lim.}(1+\frac{x}{n})^n = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots\ldots$$

for any finite real x positive or negative, n increasing any way without limit.

IV. For n *negative* we have

$$\left(1-\frac{x}{n}\right)^{-n} = \left(\frac{n-x}{n}\right)^{-n} = \left(\frac{n}{n-x}\right)^{n} = \left(1+\frac{x}{n-x}\right)^{n} =$$

$$\left(1+\frac{x}{n-x}\right)^{n-x}\left(1+\frac{x}{n-x}\right)^{x} = \left(1+\frac{x}{n-x}\right)^{n-x} \Big/ \left(1-\frac{x}{n}\right)^{x}.$$

Now as n increases without limit, so also does $n-x$, for x finite no matter how great; hence $(1+\frac{x}{n-x})^{n-x}$ approaches as its limit the series $1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots$ and $(1-\frac{x}{n})^x$ approaches 1 as its limit manifestly; hence

$$\text{Lim.}(1-\frac{x}{n})^{-n} = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots\ldots$$

Hence $\text{Lim.}(1+\frac{x}{n})^n = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots\ldots$ *for all finite real values of* x, *for real* n *increasing without limit no matter how, positively or negatively.*

For $x=1$ we obtain $\text{Lim.}(1+\frac{1}{n})^n = 1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|5}}+\ldots.$

The number defined by this series and denoted by e, is one of the three irrationals (π, i, e) all-important to analysis. That e is irrational may be easily seen thus: Consider the first $(p+1)$ terms $1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots\ldots\frac{1}{\underline{|p}}$; the sum Σp is $\frac{w}{\underline{|p}}$ where w is some integer no matter what; the remainder

$$\frac{1}{\underline{|p+1}} + \frac{1}{\underline{|p+2}} + \ldots\ldots \quad \text{is} \frac{1}{\underline{|p}}\left\{\frac{1}{p+1} + \frac{1}{(p+1)(p+2)} + \ldots\ldots\right\}$$

$$< \frac{1}{\lfloor p}\left(\frac{1}{p+1} - \frac{1}{(p+1)^2} + \ldots\ldots\right) < \frac{1}{\lfloor p}. \quad \text{Hence} \frac{w}{\lfloor p} < e < \frac{w+1}{\lfloor p}.$$

Now as p increases without limit these two fractions close down upon each other and upon e always between them. It is plain that there is no fixed fraction as N/D, always between $\frac{w}{\lfloor p}$ and $\frac{w+1}{\lfloor p}$; for however great D might be, we could choose p so large, that $\lfloor p$ would include all the factors of D; hence $\frac{N}{D} = \frac{w}{\lfloor p}$, whereas $e > \frac{w}{\lfloor p}$. In fact as the two fractions $\frac{w}{\lfloor p}$ and $\frac{w+1}{\lfloor p}$ close down on each other and on their common limit e they *pass over* (either the one or the other) every assignable fraction lying between them.

The importance of e lies in the fact that the series $1+x+\frac{x^2}{\lfloor 2}+\ldots\ldots$ is expressible through it. We have

$$(1+\frac{x}{n})^n = \{(1+\frac{x}{n})^{n/x}\}^x = \{(1+\frac{1}{n/x})^{n/x}\}^x. \quad \text{Hence}$$

$$\text{Lim.}(1+\frac{x}{n})^n = \text{Lim.}\{(1+\frac{1}{n/x})^{n/x}\}^x = \{\text{Lim.}(1+\frac{1}{n/x})^{n/x}\}^x,$$

where we indeed assume that the Limit of the Power equals the Power of the Limit. But this is plainly correct, at least in the present case; for

$$(1+\frac{1}{n/x})^{n/x} = e+\sigma.$$

Hence $\text{Lim.}\{(1+\frac{1}{n/x})^{n/x}\}^x = \text{Lim.}(e+\sigma)^x = \text{Lim.}(e^x+\sigma x e^{x-1}+\ldots\ldots) = e^x$ for all finite real values of x. Hence

$$\text{Lim.}(1+\frac{x}{n})^n = e^x = 1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots\ldots \textit{ in infinitum.}$$

Herewith then the exponental development is established for all finite real values of the exponent.

Tulane University, May, 1896.

W AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from April Number.]

VII. Let ABC be a $\triangle$ right angled at C. Produce BC making $BD=BA$. DA. From E, the middle point of CD, a perpendicular meeting DA, as at F. FB. $\triangle ADC$ is similar to $\triangle BFE$.

Fig. 7.

$\therefore AC : BE :: DC : FE$.

$\therefore AC : BC+(AB-BC)\div 2 :: AB-BC$
$C\div 2$.

$\therefore \overline{AB}^2=\overline{AC}^2+\overline{BC}^2$.

NOTE.—This proof is credited to Hoffman.

VIII. Triangles BDF, BFE, and FDE, similar.

Letting $BD=BA=c$, $AC=b$, $BC=a$, $BE=\frac{a+c}{2}$, $DE=\frac{c-a}{2}$, $FE=\frac{b}{2}$, $=x$, $BF=y$, we obtain the following:

(1). $\frac{c-a}{2} : x :: x : c. \quad \therefore x^2=\frac{c(c-a)}{2}$ ....1.

(2). $\frac{c-a}{2} : x :: \frac{b}{2} : y. \quad \therefore bx=(c-a)y$ ....2.

(3). $x : c :: \frac{b}{2} : y. \quad \therefore xy=\frac{bc}{2}$ ....3.

(4). $\frac{c-a}{2} : \frac{b}{2} :: \frac{b}{2} : \frac{c+a}{2}. \quad \therefore c^2-a^2=b^2$ ....4.

(5). $\frac{c-a}{2} : \frac{b}{2} :: x : y. \quad \therefore bx=(c-a)y$ ....2.

(6). $\frac{b}{2} : \frac{c+a}{2} :: x : y. \quad \therefore by=(c+a)x$ ....5.

(7). $\frac{c+a}{2} : y :: y : c. \quad \therefore y^2=\frac{c(c+a)}{2}$ ....6.

(8). $\frac{c+a}{2} : y :: \frac{b}{2} : x. \quad \therefore by=(c+a)x$ ....5.

(9). $y : c :: \frac{b}{2} : x. \quad \therefore xy=\frac{bc}{2}$ ....3.

From 4 we get $c^2=a^2+b^2$.

The set 2 and 5 gives the same result. But equation 2 may come from proportion (2) or (5), and 5 from (6) or (8), thus making four proofs for this set.

The following sets of three equations furnish fourteen proofs, since each set can come from two or more sets of three proportions: 1, 2, 6; 1, 3, 5; 1, 3, 6; 1, 5, 6; 2, 3, 6. Total number of proofs for this method is 19.

IX. Comparing the triangles BDF, BFE, ADC, BLC, and ALF, Fig. 7, we may put the result in the following condensed form:

$$DF=x : EF=\tfrac{1}{2}b : DC=c-a : LC=z : FL=y-v$$

$$:: FB=y : EB=\tfrac{1}{2}(c+a) : AC=b : CB=a : AF=x$$

$$:: BD=c : FB=y : AD=2x\ LB=v : AL=b-z.$$

From this we easily may derive thirty different simple proportions, which give twenty-seven different equations. Some idea of the number of proofs that may be obtained from different sets of these equations, can be formed from the fact that there are 17550 sets of four equations, to say nothing of sets of three and of five. Of course, many of the sets must be rejected for reasons stated fully in V. We leave details to the reader.

X. Suppose the theorem true. Then $\overline{AB}^2=\overline{AC}^2+\overline{BC}^2$, $\overline{BC}^2=\overline{CD}^2+\overline{BD}^2$, and $\overline{AC}^2=\overline{AD}^2=\overline{CD}^2$.

Fig. 1.

$\therefore\ \overline{AB}^2=\overline{AD}^2+2\overline{CD}^2+\overline{BD}^2.$

But $\overline{CD}^2=AD\cdot BD$.

$\therefore\ \overline{AB}^2=\overline{AD}^2+2AD\cdot BD+\overline{BD}^2.$

$\therefore\ AB=AD+BD$, which is true.

$\therefore$ The supposition is true.

NOTE.—This method is credited to Hoffman.

XI. In Fig. 1, $\overline{AB}^2<, =,$ or $>\ \overline{AC}^2+\overline{BC}^2$. Suppose it less. Then, since $\overline{AB}^2=(AD+DB)^2=(\overline{CD}^2\div DB+DB)^2$, and $\overline{AC}^2=(\overline{CD\cdot BC}+DB)^2$,

$$(\overline{CD}^2\div DB+DB)^2<(\overline{CD\cdot BC}+DB)^2+\overline{BC}^2.$$

$\therefore\ (\overline{CD}^2+\overline{DB}^2)^2<\overline{BC}^2(\overline{CD}^2+\overline{DB}^2).$

$\therefore\ \overline{BC}^2>\overline{CD}^2+\overline{DB}^2$, which is absurd. For were the supposition true, we should have $\overline{BC}^2<\overline{CD}^2+\overline{DB}^2$, as can easily be shown.

Similarly the supposition that $\overline{AB}^2>\overline{AC}^2+\overline{BC}^2$ can be proven false.

$\therefore\ \overline{AB}^2=\overline{AC}^2+\overline{BC}^2.$

XII. $BC=a : EF=x : DF=y : DE=z$

$:: AC=b : AF=v : EF=x : AE=w$

$:: AB=c : AE=w : ED=z : AD=v+y$.

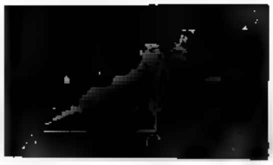

Fig. 8.

The above condensed form is self-explanatory, as are also the two following.

We leave the selection of simple proportions, the derivation and solution of consequent equations, as an exercise for the interested reader.

XIII. $BC=a : DE=x : DL=y$,

$:: AB=b : AE=z : LF=FE=v$,

$:: AB=c : AD=v+y : DF=x-v$.

Fig. 9.

Fig. 10.

XIV. $BC=a : ED=EC=x : FD=y : EF=z$,

$:: AC=b : AE=b-x : EF=z : AF=v$,

$:: AB=c : AD=v+y : ED=x : AE=b-x$.

[To be Continued.]

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from May Number.]

Each of the regular groups of degree six contains only one subgroup of the type $(abc.def)$. Since no substitution of the form $abcde$ can transform this into itself the group of order 30 is impossible.

If a group of order 60 exists it must contain six subgroups of the type $(abcde)_{10}$. We may assume that it contains $(abcde)_{10}\equiv G_f$. G_f must then contain $ac.de$ and a substitution of the type $abcde$ which contains the letters a, c, d, e, f. We may assume that this substitution is $acd_1e_1f_1$. It is then necessary that $ac.de.acd_1e_1f_1=af_1e_1d_1c.ac.de$. Hence

$$acd_1e_1f_1=acdfe \text{ or } acefd.$$

Since $acefd.adbec=bef$ every group of order 60 must contain

$$(abcde)_{10} \text{ and } acdfe.$$

These substitutions generate a group whose order $\overline{>}60$, hence only one group of order 60 is possible.

We shall prove that these substitutions generate a group of order 60 by employing a very elementary but somewhat lengthy method. Representing the substitutions of $(abcde)_{10}\equiv 1$, $abcde$, $acebd$, $adbec$, $aedcb$, $ab.ce$, $ac.de$, $ad.bc$, $ae.bd$, $be.cd$ respectively by $1=s_1, s_2, s_3, s_4, s_5, c_6, s_7, s_8, s_9, s_{10}$ and $acdfe$ by t, we form the rectangle

$$\begin{array}{llll}
1 & s_2 & s_3 & \ldots\ldots\ldots\ldots s_{10} \\
t & s_2t & s_3t & \ldots\ldots\ldots\ldots s_{10}t \\
t^2 & s_2t^2 & s_3t^2 & \ldots\ldots\ldots\ldots s_{10}t^2 \\
t^3 & s_2t^3 & s_3t^3 & \ldots\ldots\ldots\ldots s_{10}t^3 \\
t^4 & s_2t^4 & s_3t^4 & \ldots\ldots\ldots\ldots s_{10}t^4 \\
t_1 & s_2t_1 & s_3t_1 & \ldots\ldots\ldots\ldots s_{10}t_1
\end{array}$$

Where t_1 is any substitution generated by $(abcde)_{10}$ and $acdfe$ which is not found in the preceding five rows. These substitutions are all different. They form a group if $t_1{}^2$ is contained in the first five rows and

$$t^\alpha s_\beta = s_\gamma t^\delta \text{ or } s_\gamma t_1,\ t_1 s_\beta = s_\gamma t^\delta \text{ or } s_\gamma t_1$$

$$(\beta, \gamma = 1, 2, \ldots\ldots 10),\ (\alpha, \delta = 1, 2, \ldots\ldots 4).$$

Instead of allowing β to have 10 values it is clearly sufficient to assign to it only the two values of 2 and 6 since $abcde$ and $ab.ce$ generate $(abcde)_{10}$. The following shows that the necessary conditions are fulfilled:

$$\begin{array}{ll}
ts_2 = adf.bce = s_{10}t^2 & ts_6 = aeb.cdf = s_3t^2 \\
t^2s_2 = aed.bcf = t_1{}^* & t^2s_6 = adcfb = s_9t_1 \\
t^3s_2 = afdbc = s_9t^4 & t^3s_6 = afedb = s_4t \\
t^4s_2 = bc.ef = s_8t & t^4s_6 = acb.def = s_8t^4 \\
t_1{}^2 = ade.bfc = s_6t^3 & t_1s_2 = bd.cf = s_4t^3 \\
 & t_1s_3 = acf.bed = s_9t^2
\end{array}$$

There is therefore one group of order 60, viz :

(1) $$(abcde)_{10}(acdfe) = (abcdef)_{60}.$$

If there is a primitive group of order 120 it may be assumed that it con-

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*In the above rectangle f is followed by the same letter as in the corresponding t or t_1. Since it is not followed by b in t, $aed.bcf$ cannot be contained in the first five lines and may therefore be used for t_1. All these relations may be readily found if this property is observed.

tains $(abcde)_{20}$ and therefore $(abcdef)_{60}$. Since half of its substitutions must be negative it must contain $(abcdef)_{60}$ as a self-conjugate subgroup.

The order of a group which satisfies these conditions cannot be less than 120. From this we see that there cannot be more than one group of this order. That there is one follows from the facts that *acbe* belongs to $(abcde)_{20}$ and transforms *acdfe* into $acbdf=(s_2t_1)^2$=some substitution of $(abcdef)_{60}$.

The other primitive groups of degree six must contain subgroups of degree five which contain substitutions of one of the two types

ab *abc*

They must therefore be the alternating and the symmetric group. The following is therefore a complete list of these groups:

Order	Group
60	$(abcdef)_{60}$
120	$(abcdef)_{120}$
360	$(abcdef)$pos
720	$(abcdef)$*all*

REMARKS.

We have now finished the explanations of the elementary methods of group construction. By means of these we have been able to find, with a reasonable amount of labor, all the groups whose degree does not exceed six. It scarcely needs to be stated that this labor could have been considerably reduced by employing more advanced methods. In fact, we did not endeavor so much to find these groups by the least labor as to find them in such a way as to illustrate some of the most important elementary methods of group construction.

We are indebted to our honored teacher, Professor F. N. Cole not only for many of these methods but also for the fundamental ideas.

Most of the theorems that we have developed are found in Part I of Netto's Theory of Substitutions (American Edition). In some instances it seemed desirable to change the method of proof either because we had not yet developed the principles upon which Netto's proof is based, or because we desired to call attention to some special property. In a few instances our purposes required us to pursue the demonstration farther than is done by this author.

We did not enter into a special study of methods of operating with substitutions. Some of the more important ones have been incidentally explained. For further explanations we would refer to Senet's Algèbre Supérieure, Part IV, (this part is found in the second volume of this work), and to Part I of Netto.

In these works is also found considerable on the analysis of a substitution. The first 15 pages of the first volume of Gordan's Invariantentheorie contain considerable on this point. For the more advanced methods of operation we have to refer to the classical work on this subject, Jordan's Traité des Substitutions, and to the periodicals.

Before entering upon the development of more advanced methods of group construction we shall study some of the relations which exist between substitution groups and functions containing a finite number of letters. These relations will not only show how substitution groups may be utilized but they may also serve as a means of arriving at important properties of substitution groups.

Leipsig, Germany, September 20, 1895.

SIMULTANEOUS QUADRATIC EQUATIONS.

By I. H. BRYANT, M. A., Instructor of Mathematics, Waco High School, Waco, Texas.

[Continued from May Number.]

The discussion in this article is restricted to two unknown quantities. Cases 1, 2, 4, and 5 apply to two variables just as they are stated in the previous article. In Case 3, the restriction that each factor must occur twice is unnecessary when only two variables occur. It is sufficient for one factor to occur in each equation. This reduces Case 3 to Case 2. For two variables, Cases 6, 7, and 8 become one and the same, as no restrictions are necessary.

The following Cases are applicable to two variables only. Express the equations thus for Cases 9 and 10:

$$ax^2+by^2+cxy+dx+ey+f=0. \qquad 1.$$

$$a'x^2+b'y^2+c'xy+d'x+e'y+f'=0. \qquad 2.$$

CASE 9. When $a:a'::c:c'::d:d'$. If this is true, it is obvious that the terms containing x can be eliminated. This holds true when the terms of any one, or any two, of the three ratios are zero.

CASE 10. When $a:a'::b:b'::d^2:d'^2::e^2:e'^2$, and when $d:d'::e:e'$.

By alternation, $\dfrac{e}{d}=\dfrac{e'}{d'}$, $\dfrac{b}{a}=\dfrac{b'}{a'}$.

Let $\dfrac{e}{d}=r$. Then $\dfrac{e'}{d'}=r$, $\dfrac{b}{a}=r^2$, $\dfrac{b'}{a'}=r^2$.

$e=dr$, $e'=d'r$, $b=ar^2$, $b'=a'r^2$.

Divide equation 1 by ad, and equation 2 by $a'd'$.

$$\frac{\ +r^2y^2}{d}+\frac{x+ry}{a}+\frac{cxy+f}{ad}=0. \qquad \frac{x^2+r^2y^2}{d'}+\frac{x+ry}{a'}+\frac{c'xy+f'}{a'd'}=0.$$

Let $x=r(u+v)$, $y=u-v$, and substitute. Eliminate v^2 and solve the re- ing equation. The signs of d and e, and of d' and e' must be alike in both, unlike in both equations. They cannot be alike in one and unlike in the er. Symmetrical equations are a special form of this case.

CASE 11. When the two equations can be expressed as follows:

$$(mx+p)^2-(m'x+p')^2+r(ny+q+n'y+q')=0$$

$$(ny+q)^2-(n'y+q')^2+r'(mx+p+m'x+p')=0.$$

By factoring, one value, and only one, of x and y can be found. The arvard Catch" is a special form of this Case.

ARITHMETIC.

ducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

60. Proposed by ISAAC L. BEVERAGE, Monterey, Virginia.

A broker charges me 1½ per cent. brokerage for buying some uncurrent bank bills at r cent. discount. Of these bills, 4 of \$50. each become worthless, but the remainder I ose of at par, and make by the operation \$364. What was the face amount? [Which er is correct, \$3000, or \$3048 24/37?]

I. Solution by J. C. CORBIN, Pine Bluff, Arkansas; H. C. WILKES, Skull Run, West Virginia; and G. KERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Ar- a-Texas.

80% + 1½% = 81½%; 100% − 81½% = 18½%.

\$364 + (4 × \$50) = \$564, amount he would have made if he had disposed of at par.

\$564 ÷ 18½% = \$3048 24/37, the correct face amount.

The other result is obtained as follows:

1½% of 80% = 1⅕%.

$80\% + 1\frac{1}{5}\% = 81\frac{1}{5}\%$; $100\% - 81\frac{1}{5}\% = 18\frac{4}{5}\%$.

$\$564 \div 18\frac{4}{5}\% = \$3000$.

The latter is commission on money invested and not brokerage on bills bought.

II. Solution by F. M. McGAW, Bordentown, New Jersey.

Market Value+Brokerage equals whole cost, therefore gain % was $1.00-(.80+.015)=.185$.

The net gain in money was $364 to which we add the $200 lost, making a gross gain of $564. Then $18.5\% = \$564$, whence $\$564 \div .185 = \$3058\frac{24}{37}$, face.

To determine which answer is correct, assume the answer and work backwards..

I. Assume $\$3048\frac{24}{37}$ as face, then

$\$3048\frac{24}{37} \times .815 = \text{cost} = \$2484\frac{24}{37}$,

$\$3048\frac{24}{37} - \$200(\text{lost}) = \$2848\frac{24}{37}$

$\$2848\frac{24}{37} - \$2484\frac{24}{37} = \$364$, net gain. Answer.

II. Assume $3000 as face, then the same operations produce a gain of only $355.

Also solved by *A. P. REED*, *H. C. WHITTAKER*, *P. S. BERG*, and *J. SCHEFFER*.

We received solutions of problem 58, too late for credit in last issue, from *J. SCHEFFER*, *E. R. ROBBINS*, and *P. S. BERG*.

PROBLEMS.

62. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

A dealer buys milk at $m=5$ cents per quart, and sells it at $n=6$ cents per quart. How much water has he put with the milk, if his rate of profit is $p=60$ per cent.?

63. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics, Curry University, Pittsburg, Pennsylvania.

I owe A $100 due in 2 years, and $200 due in 4 years; when will the payment of $300 equitably discharge the debt, money being worth 6 per cent.?

64. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

If 27 men in 10 days of 7 hours each for $375 dig a ditch 70 rods long, 25 feet wide, and 4 feet deep, how long a ditch 40 feet wide and 3 feet deep will 15 men dig in 16 days of 9 hours each for $500?

[77 2-9 rods and 88 8-9 rods have been obtained. Which is correct?]

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

60. Proposed by ROBERT J. ALEY, A. M., Professor of Mathematics in Indiana University, Bloomington, Indiana.

Telegraph poles are a yards apart; for how many minutes must one count poles in order that the number of poles counted may be equal to the number of miles per hour that the train is running?

I. Solution by FREDERICK R. HONEY, A. B., New Haven, Connecticut.

The problem is independent of the number of poles counted, and of the number of miles per hour the train is running. We will call this number N.

$\therefore aN=$the number of yards the train runs while the poles are counted. Also, $1760N=$number of yards per hour the train runs. $\therefore aN/1760N=$the fraction of an hour during which the poles are counted.

$\therefore 60aN/1760N=3a/88=$number of *minutes* during which the poles are counted.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; and W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana.

Let $x=$the number of minutes, and let $r=$number of miles per hour the train is running. Also, $1760/a=$number of poles in a mile, and $rx/60=$number of miles the train runs in x minutes. Then, $rx/60\times1760/a=88rx/3a=$number of poles passed in x minutes, or while the train is running $rx/60$ miles.

$\therefore 88rx/3a=r$; whence $x=3a/88$.

The number of minutes depends upon the distance the poles are apart irrespective of the rate of the train.

Also solved by *O. W. ANTHONY, P. S. BERG, A. H. HOLMES, C. D. SCHMITT, H. C. WILKES, J. F. YANNEY*, and *G. B. M. ZERR*.

61. Proposed by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

Demonstrate the identity $2^{2n+1}\dfrac{d^n}{dx^n}\left(x^{n+\frac{1}{2}}\dfrac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}}\right)=e^{\sqrt{x}}$.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

It may be proved inductively that $\dfrac{d^n}{dx^n}e^{\sqrt{x}}=\dfrac{1}{4x}\,\dfrac{d^{n-2}}{dx^{n-2}}e^{\sqrt{x}}-(n-\tfrac{3}{2})\dfrac{d^{n-1}}{dx^{n-1}}e^{\sqrt{x}}$.

Change n to $n+3$; then

$$\frac{d^{n+3}}{dx^{n+3}}e^{\sqrt{x}}=\frac{1}{4x}\,\frac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}}-(n+\tfrac{3}{2})\frac{d^{n+2}}{dx^{n+2}}e^{\sqrt{x}}.$$

Clearing of fractions and transposing,

$$\frac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}}=4(n+\tfrac{3}{2})\frac{d^{n+2}}{dx^{n+2}}e^{\sqrt{x}}+4x\frac{d^{n+3}}{dx^{n+3}}e^{\sqrt{x}}.$$

Multiply through by $x^{n+\frac{1}{2}}$, and we have

$$x^{n+\frac{1}{2}}\frac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}}=4[(n+\tfrac{3}{2})x^{n+\frac{1}{2}}\frac{d^{n+2}}{dx^{n+2}}e^{\sqrt{x}}+x^{n+\frac{3}{2}}\frac{d^{n+3}}{dx^{n+3}}e^{\sqrt{x}}]$$

$$=4\frac{d}{dx}[x^{n+\frac{3}{2}}\frac{d^{n+2}}{dx^{n+2}}e^{\sqrt{x}}].$$

Multiply through by 2^{2n+1} and differentiate n times, and we have

$$2^{2n+1}\frac{d^n}{dx^n}(x^{n+\frac{1}{2}}\frac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}})=2^{2n+3}\frac{d^{n+1}}{dx^{n+1}}(x^{n+\frac{3}{2}}\frac{d^{n+2}}{dx^{n+2}}e^{\sqrt{x}}).$$

Hence, if the form given is true for n, it will be true for $n+1$. It may easily verified that it is true for $n=2$. Therefore it is *generally* true.

II. Solution by HENRY HEATON, M. S., Atlantic, Iowa, and G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

$$\frac{d}{dx}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}}{2\sqrt{x}},\ \frac{d^2}{dx^2}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}(\sqrt{x}-1)}{4x^{\frac{3}{2}}},\ \frac{d^3}{dx^3}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}(x-3\sqrt{x}+3)}{8x^{\frac{5}{2}}},$$

$$\frac{d^4}{dx^4}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}(x^{\frac{3}{2}}-6x+15\sqrt{x}-15)}{16x^{\frac{7}{2}}}.\quad\therefore x^{\frac{5}{2}}\frac{d^3}{dx^3}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}(x-3\sqrt{x}+3)}{8}.$$

$$\frac{d}{dx}\{x^{\frac{5}{2}}\frac{d^3}{dx^3}e^{\sqrt{x}}\}=\tfrac{1}{16}e^{\sqrt{x}}(\sqrt{x}-1),\ 2^5\frac{d^2}{dx^2}\{x^{\frac{5}{2}}\frac{d^3}{dx^3}e^{\sqrt{x}}\}=e^{\sqrt{x}}.$$

$$\text{Also } x^{\frac{7}{2}}\frac{d^4}{dx^4}e^{\sqrt{x}}=\tfrac{1}{16}e^{\sqrt{x}}(x^{\frac{3}{2}}-6x+15\sqrt{x}-15).$$

$$\frac{d}{dx}\{x^{\frac{7}{2}}\frac{d^4}{dx^4}e^{\sqrt{x}}\}=\tfrac{1}{32}e^{\sqrt{x}}(x-3\sqrt{x}+3),$$

$$\frac{d^2}{dx^2}\{x^{\frac{7}{2}}\frac{d^4}{dx^4}e^{\sqrt{x}}\}=\tfrac{1}{64}e^{\sqrt{x}}(\sqrt{x}-1),\ 2^7\frac{d^3}{dx^3}\{x^{\frac{7}{2}}\frac{d^4}{dx^4}e^{\sqrt{x}}\}=e^{\sqrt{x}}.$$

Hence generally $2^{2n+1}\frac{d^n}{dx^n}\{x^n+\frac{1}{2}\frac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}}\}=e^{\sqrt{x}}$.

Also solved by *B. F. YANNEY.*

PROBLEMS.

70. Proposed by J. A. CALDERHEAD, A. B., Professor of Mathematics, Curry University, Pittsburg, Pennsylvania.

Given $\sqrt[3]{a+x}+\sqrt[3]{a-x}=\sqrt[3]{c}$ to find x.

71. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

When $x=0$, find the the limit of the expression

$$U=\left(\frac{m+x}{m-x}\right)^{\frac{1}{x}}+\left(\frac{m-x}{m+x}\right)^{\frac{1}{x}}.$$

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

54. Proposed by I. J. SCHWATT, Ph. D., University of Pennsylvania, Philadelphia, Pennsylvania.

Prove geometrically:

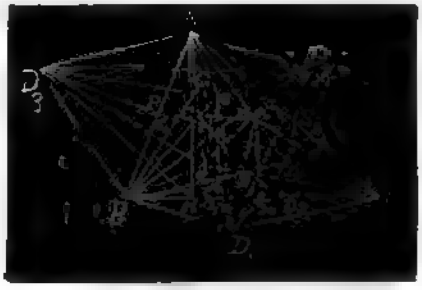
If through the center of perspective D of a given triangle ABC and its Brocard triangle $A'B'C'$ be drawn straight lines so as to pass through S_a, S_b and S_c (S_a, S_b, and S_c are the middle points of the sides BC, AC, and AB of the triangle ABC) and if S_aA_1 is made equal to DS_a, S_bD_2 equal to DS_b, and S_cD_3 equal to DS_c then are (1) the figures $D_1O'AO$, $D_2O'BO$ and $D_3O'CO$ parallelograms (O and O' are Brocard's points), (2) the triangles $D_1D_2D_3$ and ABC are equal, and (3) D_1A, D_2B, and D_3C intersect in S, (S is the middle point of OO').

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Since AC, DD_2 and BC, DD_1 bisect each other the quadrilaterals $ADCD_2$ and $BDCD_1$ are parallelograms, and AD_2, BD_1 both being equal and parallel to DC are equal and parallel to each other. Hence ABD_1D_2 is a parallelogram and AB is equal and parallel to D_1D_2. Similarly, AC is equal and parallel to D_1D_3, and BC is equal and parallel to D_2D_3.

$\therefore$ Triangle ABC is equal to triangle $D_1D_2D_3$. Also AD_1, BD_2, and

CD_3 intersect at the same point. For BD_2 and CD_3 bisect each other, also BD^2 and AD_1 bisect each other.



$\therefore BD_2, AD_1$, and CD_3 bisect one another in the same point. Since triangle BDC=triangle D_1AD_2, DD_a=the perpendiculer distance from A to D_2D_3.

Draw AH, OO_a, $O'O'_a$, DD_a perpendicular to BC; then the point of intersection of the three lines AD_1, BD_2, CD_3 is distant from BC, $\frac{1}{2}(AH-DD_a)$.

$$DD_a=\frac{2b^2c^2.\triangle}{a(a^2b^2+a^2c^2+b^2c^2)}.\quad \text{(Schwatt's Curves, p. 10).}$$

$$AH.a=2\triangle.\quad \therefore AH=\frac{2\cdot\triangle}{a}.$$

$$\frac{AH-DD_a}{2}=\frac{\triangle.a(b^2+c^2)}{(a^2b^2+a^2c^2+b^2c^2)}=\frac{OO_a+O'O'_a}{2}.\quad \text{(Schwatt's Curves, p. 9).}$$

$\therefore AD_1, BD_2, CD_3$ intersect at the mid-point of OO'.

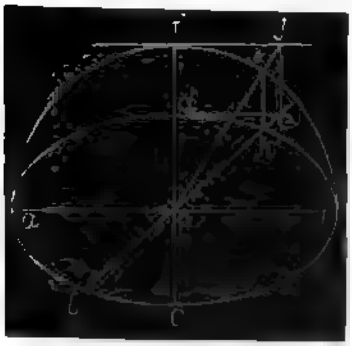
$\therefore$ Since AD_1, OO'; BD_2, OO'; CD_3, OO' all bisect one another, the quadrilaterals AOD_1O', BOD_2O', COD_3O' are parallelograms.

56. Proposed by FREDERICK R. HONEY, Ph. D., New Haven, Connecticut.

Let ab and cd be respectively the major and minor axes of an ellipse, and let α be the angle which a diameter lh forms with the major axis; it is required to find the length of this diameter.

I. Solution by the PROPOSER.

SOLUTION. Draw the semicircle afb on the diameter ab. Produce cd to f, and draw the tangents to the ellipse and the circle parallel to ab at the points d and f respectively. Produce lh to intersect de at e. Draw eg perpendicular to ab intersecting fg at g. Draw go intersecting the semicircle at k. Draw kh perpendicular to ab intersecting oe at h one extremity of the diameter lh.



ANALYSIS. The semiellipse adb may be considered as the projection on the plane of the paper of the semicircle afb, the latter being revolved about the diameter ab into a position when f is projected at d. The tangent fg which is parallel to ab is projected at de also parallel to ab. The points e and h are respectively the projections of g and k. Since the projection of every point on the micircle is found in a line drawn through it perpendicular to ab, the axis about hich the semicircle revolves, kh drawn perpendicular to ab intersects oe at and gives one point of the ellipse; and therefore one extremity of the diameter k.

II. Solution by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania; W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana; and J. O. MAHONEY, B. E., Assistant in Mathematics, Vanderbilt University, Nashville, Tennessee.

If we denote ab by a, cd by b and tangent α by m, we have the equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and } y=mx \text{ which intersect at}$$

$$\left(\frac{ab}{\sqrt{b^2+a^2m^2}}, \frac{abm}{\sqrt{b^2+a^2m^2}}\right) \text{ and } \left(-\frac{ab}{\sqrt{b^2+a^2m^2}}, -\frac{abm}{\sqrt{b^2+a^2m^2}}\right),$$

the distance between these points being equal to

$$2ab\sqrt{\frac{1+m^2}{b^2+a^2m^2}}.$$

Also solved by *G. B. M. ZERR*, *J. SCHEFFER*, and *WILLIAM HOOVER*.

PROBLEMS.

60. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Prove that the loci of the foci of the variable ellipses form a pair of circles passing through the extremities of the major axis of the fixed ellipse and having or diameters the semi-latusrectum of the fixed ellipse.

61. Proposed by WILLIAM E. HEAL, Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

Let the bisectors of the angles A, B, C of a triangle intersect in O nd meet the sides opposite A, B, C in A', B', C'. Prove that the perpendiculars form O on the sides of the triangle $A'B'C'$ are $p_1=\frac{rR}{d_1}$, $p_2=\frac{rR}{d_2}$, $p_3=\frac{rR}{d_3}$ where r, R are the radii of the inscribed and circumscribed circles of the triangle ABC and d_1, d_2, d_3 are the distances of the center of the circumscribed circle from the centers of the three escribed circles.

62. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Prove that two triangles are equal if they have two sides and *the median* *one of them* equal, each to each.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

49. Proposed by B. F. BURLESON, Oneida Castle, New York.

Find (1) in the leaf of the strophoid whose axis is a the axis of an inscribed leaf of the lemniscate, the node of the former coinciding with the crunode of the latter. Find (2) in a leaf of the lemniscate whose axis is b the axis a of an inscribed leaf of the strophoid, the node of the former also coinciding with the crunode of the latter.

Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

Solving the equatians, $r\cos\theta+a\cos2\theta=0$ (strophoid), and $r=e^2\cos2\theta$ (lemniscate), we find they coincide when $\sin\theta=\sqrt{\frac{1}{2}}$ (1), or $\sin\theta=\sqrt{\dfrac{a^2-e^2}{2a^2-e^2}}$ (2).

(1) shows that they coincide at the origin for all values of a and e. We have to find the relation between the axes a and e which will make the curves tangent at the points determined by (2), provided those points are on both the leaves. Let $\phi=\angle$ made by the tangent at any point, with the radius vector drawn to that point. Then by the formula $\tan\phi=r\dfrac{d\theta}{dr}$.

Now for the lemniscate $r=\pm e\sqrt{\cos2\theta}$. $\dfrac{dr}{d\theta}=\dfrac{\mp e\sin2\theta}{\sqrt{\cos2\theta}}$.

$$\tan\phi=\pm e\sqrt{\cos2\theta}\left(\frac{\sqrt{\cos2\theta}}{\mp e\sin2\theta}\right)=\frac{2\sin^2\theta-1}{2\sin\theta\sqrt{1-\sin^2\theta}}\ \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

For the strophoid $r=-a\cos2\theta/\cos\theta$.

$$dr/d\theta=-a(-2\cos\theta\sin2\theta+\cos2\theta\sin\theta)/\cos^2\theta.$$

$$\tan\phi=[a\cos^2\theta\cos2\theta]/[a\cos\theta(-2\cos\theta\sin2\theta+\cos2\theta\sin\theta)]$$

$$=[\sqrt{1-\sin^2\theta}(1-2\sin^2\theta)]/[\sin\theta(2\sin^2\theta-3)]\ \ldots\ldots\ldots\ldots(4).$$

Now equate (3) and (4) and substitute from (1),

$$[\sqrt{1-\sin^2\theta}(1-2\sin^2\theta)]/[\sin\theta(2\sin^2\theta-3)]=(2\sin^2\theta-1)/2\sin\theta\sqrt{1-\sin^2\theta}.$$

$$2\sin\theta(1-\sin^2\theta)(1-2\sin^2\theta)=\sin\theta(1-2\sin^2\theta)(3-2\sin^2\theta),$$

$$2\sqrt{\tfrac{1}{2}}(1-\tfrac{1}{2})(1-1)=\sqrt{\tfrac{1}{2}}(1-1)(3-1)\text{ or }0=0,$$

which shows that the curves are tangent at point ($\theta=\sin^{-1}\sqrt{\frac{1}{2}}$, $r=0$) for any value of a and e. Again substituting from (2),

$$2\sqrt{\frac{a^2-e^2}{2a^2-e^2}}\left(\frac{a^2}{2a^2-e^2}\right)\left(\frac{e^2}{2a^2-e^2}\right)=\sqrt{\frac{a^2-e^2}{2a^2-e^2}}\left(\frac{e^2}{2a^2-e^2}\right)\left(\frac{4a^2-e^2}{2a^2-e^2}\right).$$

This resolves into the three equations: $\sqrt{\frac{a^2-e^2}{2a^2-e^2}}=0$, whence $e=\pm a$..... (5);

$$\frac{e^2}{2a^2-e^2}=0, \text{ whence } e=0 \ldots\ldots (6);$$

$$\frac{2a^2}{2a^2-e^2}=\frac{4a^2-e^2}{2a^2-e^2}, \text{ whence } e=\pm a\sqrt{2} \ldots\ldots (7).$$

From (5) substituted in (2), $\sin\theta=0$. ∴ the curves are tangent at the extremity of the common axis, and the equations become,

$$r\cos\theta+a\cos2\theta=0 \ldots\ldots (8),$$

$$r^2=a^2\cos2\theta \ldots\ldots (9).$$

From (9) $r_1=\pm a\sqrt{\cos2\theta}$.

From (8) $r_2=\dfrac{-a\cos2\theta}{\cos\theta}=-a\sqrt{\cos2\theta}\sqrt{\dfrac{\cos2\theta}{1-\sin^2\theta}}=-a\sqrt{\cos2\theta}\sqrt{\dfrac{1-2\sin^2\theta}{1-\sin^2\theta}}$.

Since for any value of $\sin\theta$ numerically less than $\sqrt{\frac{1}{2}}$, $\sqrt{\dfrac{1-2\sin^2\theta}{1-\sin^2\theta}}$ is numerically less than 1, r_2 is then numerically less than r_1. But by tracing the curves the leaf of each is seen to be formed by values of θ determined by this limit. ∴ every point of the leaf of the strophoid lies within the lemniscate, and the former is in this case inscribed. From (6) equation of lemniscate becomes $r^2=0$, and the curve becomes a point. From (7) by substituting in (2) $\sin\theta=\sqrt{\dfrac{-a^2}{0}}$ an impossible value.

Accordingly the leaf of the strophoid can be inscribed in the leaf of the lemniscate when their axes are equal, and under no condition can the leaf of the lemniscate with an axis greater than 0 be inscribed in the leaf of the strophoid.

Also solved by *G. B. M. ZERR*, and the *PROPOSER*.

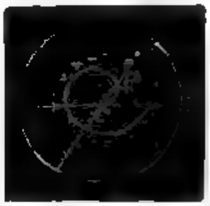
[It will be seen that Professor Black's result does not realise the intention of the problem as given by the Proposer. However, even for the Proposer's reading of the problem, his solution seems to us to be defective in several points. We may give Professor Zerr's solution later. EDITOR.]

58. Proposed by GEORGE LILLEY, Ph. D., LL. D., Principal of Park School, 394 Hall Street, Portland, Oregon.

A draw bridge, a feet in length, moves uniformly about a center axis. At the instant it began to open, a man stepped on the end; and, walking at a uniform rate in the straight line passing through its center, reached the opposite end just as it made n complete revolutions. Find the absolute path described by the man, and the ratio of his rate of motion in this path and the velocity of the end of the bridge. Apply the result to the case when $a=320$, and $n=2$.

Solution by E. L. SHERWOOD, A. M., Professor of Mathematics in Mississippi Normal College, Houston, Mississippi.

Let the man start at C and walk toward E, the table turning positively. He will traverse R, while the table turns $\frac{n}{2}.2\pi$. As velocities are uniform, we have,



$CP : PCE :: R : \pi n$, or $\rho : \theta :: R : \pi n$,

whence $\rho=\dfrac{R\theta}{\pi n}$ is the equation of the curve.

As $dl=[(\rho d\theta)^2+d\rho^2]^{\frac{1}{2}}$ we have,

$$dl=\frac{R}{\pi.n}(1+\theta^2)^{\frac{1}{2}}\,d\theta \text{ for this curve, and}$$

$$2l=\frac{2R}{\pi.n}\int_0^{\theta'}[1+\theta^2]^{\frac{1}{2}}\,d\theta, \text{ and}$$

$$L=\frac{2R}{\pi.n}\Big[\frac{\theta}{2}(1+\theta^2)^{\frac{1}{2}}+\tfrac{1}{2}\log(\theta+\sqrt{1+\theta^2})\Big]_0^{\pi.n}$$

$$L=\frac{R}{\pi.n}[\pi.n(1+\pi^2.n^2)^{\frac{1}{2}}+\log(\pi.n+\sqrt{1+\pi^2.n^2})].$$

If $a=100$ feet, and $n=2$,

$$L=\frac{50}{2\pi}[2\pi\sqrt{1+4\pi^2}+\log(2\pi+\sqrt{1+4\pi^2})]=338.3.$$

[No. 41, *Calculus.*]

If $a=36$ inches, and $n=1$, we have,

$$L=\frac{18}{\pi}[\pi\sqrt{1+\pi^2}+\log(\pi+\sqrt{1+\pi^2}]=69.6+ \text{ inches.}$$

[No. 45, *Calculus.*]

If $a=320$, and $n=2$, we have,

$$L=\frac{160}{2\pi}[2\pi\sqrt{1+4\pi^2}+\log(2\pi+\sqrt{1+4\pi^2})]=1082.56 \text{ feet.}$$

[No. 50, *Calculus.*]

The ratio of rates of extremity of the bridge and the man in his path is:

$$\frac{a}{2}\,d\theta+dl=\frac{\pi n}{\sqrt{1+\theta^2}}.$$

The ratio of rates of extremity of bridge and the man's *walking* is:

$$\frac{\pi an}{a}=\pi n.$$

Also solved by *G. B. M. ZERR* and *C. W. M. BLACK*.

PROBLEMS.

57. Proposed by F. M. McGAW, A. M., Mathematical Department, Bordentown Military Institute, Bordentown, New Jersey.

Solve the following equation: $(1+x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0.$

58. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

A line passes through a fixed point and rotates uniformly about this point. Another passes through a point which moves uniformly along the arc of a given curve and ro- uniformly about this point. Develop a method for finding the locus of intersection these two lines. Apply to case of circle and straight line.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

32. Proposed by OTTO CLAYTON, A. B., Fowler, Indiana.

The wheel of a wind pump has 60 fans, each turned at an angle of 45° to the direction the axis, and each having 150 square inches exposed to the wind. If the wind blows a velocity of V and the wheel rotates with velocity ω, what is the component of force pressure along the axis if it is turned at an angle α to the direction of the wind, assum- the resistance of the wheel in turning to be R?

No solution of this problem has been received.

38. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

At what angle with the axis of a stalk must a *sharp* wedge-shaped blade be struck, in order to *sever* the stalk with the *least* force?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let φ be the inclination of the axis of the stalk to the blade. A=area of of section made by blade, r=radius of stalk, and suppose resistance per unit of area to vary as $f(\varphi)$.

$\therefore$ R=resistance per unit of area$=mf(\varphi)$.

$\therefore$ $A=\pi r^2\operatorname{cosec}\varphi$.

$\therefore$ Work of cutting any section is $\pi r^2 mf(\varphi)\operatorname{cosec}\varphi$. This may be made a minimum when $f(\varphi)$ is known.

II. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Since the blade is *sharp* we may neglect the force required to cut through the fibres and only regard that required to produce longitudinal compression.

Call k the coefficient of longitudinal compression, θ the angle of the blade, and ϕ the angle which the lower surface of the blade makes with the horizontal. Then when the blade has just cut through the stalk the force upon each surface parallel to the axis of the stalk will be

$$dF=k[\tan(\phi+\theta)-\tan\phi]xydx.$$

Resolving these parallel to the surface of wedge and parallel to median line of wedge we have—

$$dF_1=k.\frac{\cos(\phi+\theta)+\cos\phi}{\sin\frac{\theta}{2}}(\tan(\phi+\theta)-\tan\phi)xydx,$$

where F_1 is the force perpendicular to base of wedge.

$$\text{Then } F_1=k.\frac{\cos(\phi+\theta)+\cos\phi)}{\sin\frac{\theta}{2}}[\tan(\phi+\theta)-\tan\phi]\int_0^{2a}xydx$$

$$=2k\cos\frac{\theta}{2}\Big[\sec(\phi+\theta)+\sec\phi\Big]\int_0^{2a}xydx.$$

$$\frac{dF_1}{d\phi}=0 \text{ for minimum.}$$

$\therefore$ $\sec(\phi+\theta)\tan(\phi+\theta)+\sec\phi\tan\phi=0$. By some obvious reductions

$$\sin(\phi+\frac{\theta}{2}).\{\cos^2(\phi+\frac{\theta}{2})+\sec^2\frac{\theta}{2}\}=0,$$

whence $\phi=-\frac{\theta}{2}$.

That is, the medial line is horizontal. The second factor gives imaginary results, except when $\theta=0$.

PROBLEMS.

39. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A person whose height is a and weight W stands in a swing whose length is l. Supposing the initial inclination of the swing to the vertical is α and that the person always crouches when in the highest position and stands up when in the lowest, his center of gravity moving through a distance b measured from lower part of swing upward, find how much the arc is increased after n complete vibrations.

40. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the law of the force, in order that the orbit may be a Cassinian Oval.

41. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

If the earth were a perfect sphere and had a frictionless surface, what would be the motion of a ball placed at a given latitude?

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

40. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

The sum of three positive integral *cubic* roots of an equation is a square. What is the equation?

I. Solution by E. L. SHERWOOD, A. M., Professor of Mathematics and Science in Mississippi Normal College, Houston, Mississippi.

Let a, b, and c be the roots of the equation.

We then have $a^3+b^3+c^3=\square$.

This condition is satisfied by the equation $v^4(v^2+8v^2+27v^2)=\square$, where

$a^3=v^6$, $b^3=8v^6$ and $c^3=27v^6$. Forming the equation from the roots, we have: $x^3-(a^3+b^3+c^3)x^2+(a^3b^3+a^3c^3+b^3c^3)x-a^3b^3c^3=0$.

Substituting values of a, b, c and reducing, we have:

$x^3-36v^6x^2+251v^{12}x-216v^{18}=0$, where "$v$" may be 1, 2, 3, etc., in succession.

II. Solution by A. H. HOLMES, Brunswick, Maine, and G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics in Texarkana College, Texarkana, Arkansas-Texas.

Let a, b, c be the roots of the cubic equation.

$\therefore x^3-(a+b+c)x^2+(ab+ac+bc)x=abc$, is the equation.

Let $a=5m^2$, $b=3m^2$, $c=m^2$. $\therefore 5m^2+3m^2+m=9m^2$.

$\therefore x^3-9m^2x^2+23m^4x=15m^6$ ....(1).

Let $a=m^2+mn$, $b=n^2-mn$, $c=2mn$, $m>n$.

$\therefore m^2+mn+n^2-mn+2mn=(m+n)^2$.

$\therefore x^3-(m+n)^2x^2+(3m^3n+3mn^3)x=2m^4n^2-2m^2n^4$............(2).

(1) and (2) both satisfy the conditions.

41. Proposed by H. C. WILKES, Skull Run, West Virginia.

Given $\dfrac{50(a+b)}{ab}=\dfrac{81(c+d)}{cd}$......(1); $\dfrac{56(a+c)}{ac}=\dfrac{75(b+d)}{bd}$......(2);

$\dfrac{65(b+c)}{bc}=\dfrac{66(a+d)}{ad}$......(3), to find the least integral values of a, b, c, d.

I. Solution by the PROPOSER.

The sum of equations (1), (2) and (3), after clearing of fractions, can be reduced to $20d(ab+ac+bc)=111abc$......(4).

Eliminating from (1) and (4), $6d=9c$.

Eliminating from (2) and (4), $5d=9b$.

Eliminating from (3) and (4), $4d=9a$.

$\therefore$ The numbers are in the ratio $a4$, $b5$, $c6$, $d9$, which will be the least integers that will satisfy the equation. [See problem No. 36.]

II. Solution by A. H. BELL, Hillsboro, Illinois.

Arranging,	$50acd+50bcd=81abc+81abd$.	(1).
	$75acd+56bcd=-75abc+56abd$.	(2).
	$65acd-66bcd=66abc-65abd$.	(3).
(1)×3	$150acd+150bcd=243abc+243abd$.	(4).
(2)×2	$150acd-112bcd=-150abc+112abd$.	(5).
(4)−(5)	$262bcd=393abc+131abd$.	(6).
(2)×13	$975acd-728bcd=-975abc+728abd$.	(7).
(3)×15	$975acd-990bcd=990abc-975abd$.	(8).
(7)−(8)	$262bcd=-1965abc+1703abd$.	(9).

(9)−(6), and reducing $3c=2d$. $\therefore c=2$, and $d=3$ (10).

These values in (1) and (2), etc., $a=4$ and $b=5$ (11).

To obtain the relative values between the two sets of values (10) and (11), take (6)×1703−(9)×131, results in $9a=4d$. ∴ $a=4$ and $d=9$, $b=5$ and $c=6$. These are prime to each other. ∴ are the least values.

III. Solution by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Nashville, Tennessee.

The equations can be written: $50\left(\frac{1}{a}+\frac{1}{b}\right)=81\left(\frac{1}{c}+\frac{1}{d}\right)$,

$56\left(\frac{1}{c}+\frac{1}{a}\right)=75\left(\frac{1}{b}+\frac{1}{d}\right)$, $65\left(\frac{1}{b}+\frac{1}{c}\right)=66\left(\frac{1}{a}+\frac{1}{d}\right)$.

Let $1/a=x$, $1/b=y$, $1/c=z$, and $1/d=u$, and the equations become $50x+50y-81z-81u=0$; $56x-75y+56z-75u=0$; $66x-65y-65z+66u=0$.

Thus we have three equations with four unknown quantities.

By determinants $x:y:z:u$ ::

$$\begin{vmatrix} 50, & -81, & -81 \\ -75, & 56, & -75 \\ -65, & -65, & 66 \end{vmatrix} : - \begin{vmatrix} 50, & -81, & -81 \\ 56, & 56, & -75 \\ 66, & -65, & 66 \end{vmatrix} : \begin{vmatrix} 50, & 50, & -81 \\ 56, & -75, & -75 \\ 66, & -65, & 66 \end{vmatrix} : - \begin{vmatrix} 50, & 50, & -81 \\ 56, & -75, & 56 \\ 66, & -65, & -65 \end{vmatrix}$$

Evaluating the determinants, we have,

$x:y:z:u$ :: $(131)^2 90 : (131)^2 72 : (131)^2 60 : (131)^2 40$,

or $x:y:z:u$ :: $90:72:60:40$.

Hence $1/a : 1/b : 1/c : 1/d$ :: $90:72:60:40$,

or $a:b$ :: $c:d$ :: $4:5:6:9$;

whence $a=4$, $b=5$, $c=6$, $d=9$ are the lowest values.

Also solved by *A. H. HOLMES.*

PROBLEMS.

47. Proposed by EDMUND FISH, Hillsboro Illinois.

A rectangular field, whose length and breadth in rods are in whole numders, is enclosed with a fence and subdivided by fences on both diagonals, the total length of the fences is 2204 rods; required the sides and area.

48. Proposed by SYLVESTER ROBBINS, North Branch Depot, New Jersey.

The edges of a rectangular parallelopiped are within 1 of the proportion 2 : 3 : 9, and they are $2x\pm1$, $3x$ and $9x$, $(2x\mp1)^2+(3x)^2+(9x)^2$=the diagonal squared$=94x^2\mp4x+1=\square$. To find four integral values for x.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

30. Proposed by F. P. MATZ, M. A., M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the average area of all the triangles which can be inscribed in a given circle.

I. Solution by the PROPOSER.

Let P_1OP_2 be any inscribed triangle; and through O draw any diameter OA. Two cases have now to be considered: (1), the triangle may lie *wholly* on one side of the diameter OA; (2), the triangle may lie *partly* on one side of the diameter OA.

I. Put $OA=2r$, $\angle AOP_1=\phi$, and $\angle AOP_2=\theta$; then $OP_1=2r\cos\phi$, $OP_2=2r\cos\theta$, and the area of the $\triangle$, P_1OP_2, $=A'$, $=2r^2\cos\phi\cos\theta\sin(\phi-\theta)$. Hence the average area of the triangles in this case, is

$$A_1=\int_0^{\frac{1}{2}\pi}\int_0^{\phi}A'd\phi d\theta\div\int_0^{\frac{1}{2}\pi}\int_0^{\phi}d\phi d\theta=\frac{8r^2}{\pi^2}\int_0^{\frac{1}{2}\pi}\phi\sin\phi\cos\phi d\phi$$

$$=\frac{r^2}{\pi^2}\Big[\sin2\phi-2\phi\cos2\phi\Big]_0^{\frac{1}{2}\pi}=\frac{r^2}{\pi}\text{............ (1).}$$

II. Put $\angle AOP_2=\psi$; then the area of the triangle P_1OP_2, $=A''$, $=2r^2\cos\theta\cos\psi\sin(\theta+\psi)$. Hence the average area of the triangles in this case, is

$$A_2=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}A''d\theta d\psi\div\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}d\theta d\psi=\frac{8r^2}{\pi^2}\Big[\frac{\pi}{4}\int_0^{\frac{1}{2}\pi}\sin\theta\cos\theta d\theta$$

$$+\tfrac{1}{2}\int_0^{\frac{1}{2}\pi}\cos^2\theta d\theta\Big]=\frac{8r^2}{\pi^2}\Big[\frac{\pi}{4}\times\tfrac{1}{2}+\tfrac{1}{2}\times\frac{\pi}{4}\Big]=\frac{2r^2}{\pi}\text{............ (2).}$$

Hence the *required* average area becomes

$$A=\tfrac{1}{2}(A_1+A_2)=3r^2/2\pi\text{...... ....(3).}$$

II. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

We readily get the area of triangle

$$=\frac{R^2}{2}\{\sin2A+\sin2B+\sin2C\},$$

which, by virtue of the relation $A+B+C=\pi$, reduces to

$$\frac{R^2}{2}\{\sin 2A+\sin 2B-\sin 2(A+B)\}.$$

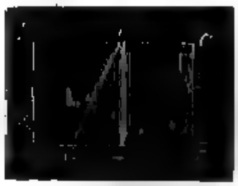
$$\therefore \text{ Average area}=\frac{R^2}{2}\,\frac{\int_0^\pi\int_0^{\pi-A}\{\sin 2A+\sin 2B-\sin 2(A+B)\}dAdB}{\int_0^\pi\int_0^{\pi-A}dAdB}=\frac{3R^2}{2\pi}.$$

31. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

Find the average length of a line drawn across the opposite sides of a rectangle, length l and breadth b.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas; and the PROPOSER.

Let $ABCD$ be the rectangle, FG the random line. Let $AB=l$, $BC=b$, [illegible], $AG=y$.



Then $FG=\{b^2+(x-y)^2\}^{\frac{1}{2}}$.

The limits of x are 0 and l; of y, 0 and x.

Hence the required average area is

$$\Delta=\frac{\int_0^l\int_0^x\{b^2+(x-y)^2\}^{\frac{1}{2}}dxdy}{\int_0^l\int_0^x dxdy}$$

$$=\frac{2}{l^2}\int_0^l\int_0^x\{b^2+(x-y)^2\}^{\frac{1}{2}}dxdy$$

$$=\frac{1}{l^2}\int_0^l\{x(b^2+x^2)^{\frac{1}{2}}+b^2\log[x+(b^2+x^2)^{\frac{1}{2}}]-b^2\log b\}dx$$

$$=\frac{1}{3l^2}(l^2+b^2)^{\frac{3}{2}}+\frac{b^2}{l}\log\{l+(l^2+b^2)^{\frac{1}{2}}\}-\frac{b^2}{l}\log b-\frac{1}{l^2}(l^2+b^2)^{\frac{1}{2}}-\frac{b^3}{3l^2}+\frac{b}{l^2}.$$

For the line KL, we get, by writing l for b and b for l,

$$\Delta_1=\frac{1}{3b^2}(l^2+b^2)^{\frac{3}{2}}+\frac{l^2}{b}\log\{b+(l^2+b^2)^{\frac{1}{2}}\}-\frac{l^2}{b}\log l-\frac{1}{b^2}(l^2+b^2)^{\frac{1}{2}}-\frac{l^3}{3b^2}+\frac{l}{b^2}.$$

Cor. I. If $l=b$, $\Delta=\frac{1}{3}(2l\sqrt{2})+l\log(1+\sqrt{2})-\frac{1}{l}\sqrt{2}-\frac{1}{3}l+\frac{1}{l}$.

Cor. II. If $l=b=1$, $\Delta=\frac{1}{3}(2-\sqrt{2})+\log(1+\sqrt{2})$, which is the same result as given in *Williamson's Integral Calculus*, page 409.

Also solved by *F. P. MATZ*.

PROBLEMS.

39. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

A man is at the center of a circular desert; he travels at a given rate but in a *perfectly* random manner. What is the probability that he will be off the desert in a given time?

40. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

If every point of an ellipse be joined with every other point, what is the average length of the chords thus drawn?

41. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

A line is drawn at random across the chord and *given* arc of a circular segment. Find the mean area of the *divisions*.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

35. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, California; P. O., Sebastopol, California.

To an observer whose latitude is 40 degrees north, what is the sidereal time when Fomalhout and Antares have the same altitude: taking the Right Ascension and Declination of the former to be 22 hours, 52 minutes, —30 degrees, 12 minutes; of the latter 16 hours, 23 minutes, —26 degrees, 12 minutes?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let λ=latitude of observer, α, δ, α_1, δ_1 the Right Ascension and Declination of Fomalhout and Antares, respectively, β=altitude, h, h_1 the hour angles.

$\therefore \sin\beta=\sin\lambda\sin\delta+\cos\lambda\cos\delta\cos h=\sin\lambda\sin\delta_1+\cos\lambda\cos\delta_1\cos h_1$.

Also $h+\alpha=h_1+\alpha_1$.

But $\lambda=40°$, $\alpha=343°$, $\alpha_1=245° \ 45'$, $\delta=-30° \ 12'$, $\delta_1=-26° \ 12'$.

$\therefore 66207\cos h-68734\cos h_1=3954 \ldots\ldots\ldots\ldots(1)$.

$h_1-h=\alpha-\alpha_1=97° \ 15'$, $\cos(h_1-h)=.12620 \ldots\ldots\ldots\ldots(2)$.

Let $\cos h=x$, $\cos h_1=y$.

$\therefore$ from (2) $y=.1262x\pm\sqrt{.98407-.98407x^2}$. This in (1) gives $57532.7692x \mp 68184.81534\sqrt{1-x^2}=3954$.

$\therefore x^2-.05716x=.58216$, $\therefore x=.79211$ or $-.73495$.

$\therefore h=37^\circ\ 37'$ or $137^\circ\ 18'\ 12''$. $h=2$ hours, 30 minutes, 28 seconds, or 9 ours, 9 minutes, 12.8 seconds.

$\therefore$ sidereal time=1 hour, 22 minutes, 28 seconds, or 8 hours, 1 minute, 3.8 seconds.

36. Proposed by J. K. ELLWOOD, A. M., Principal of the Colfax School, Pittsburg, Pennsylvania.

"What is the length of a chord cutting off one-fifth of the area of a circle whose iameter is 10 feet?"

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana ollege, Texarkana, Arkansas-Texas.

Let the chord subtend an angle$=2\theta$, a=radius of circle. Then the length f the chord$=2a\sin\theta$.

$\therefore a^2(\theta-\sin\theta\cos\theta)=\frac{1}{5}\pi a^2$.

$\therefore \theta-\sin\theta\cos\theta=\frac{1}{5}\pi$, $\therefore \theta=60^\circ 32'$ nearly.

$\therefore$ chord$=2a\sin\theta=10\sin\theta=8.7064$ feet.

II. Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, hio, and Prof. P. S. BERG, Larimore, North Dakota.

Let θ=the angle at the center, subtended by the required chord. Then $0\sin\theta$=the length of the required chord. Now $\frac{2\theta}{360}\pi 25$, the area of the sector, $5\sin\theta\sqrt{(25-25\sin^2\theta)}$, the area of the triangle, $=5\pi$, the given area of the segent. Whence, by reduction, $\frac{\theta}{180}\pi-\sin\theta\cos\theta=\frac{\pi}{5}$.

$\therefore \frac{\theta}{90}\pi-2\sin\theta\cos\theta=\frac{2}{5}\pi$. $\therefore .0349065\theta-\sin 2\theta=1.256637$.

From which we readily find, by supposition, the value of θ; and from his, the value of $10\sin\theta$ to be 8.706, the length of the chord required.

III. Solution by A. H. BELL, Hillsboro, Illinois.

By Reversion of Series. Let the given diameter $=10=D$ and $1/5$ of circle$=a\pi r^2/d$, radius$=r$. To obtain the greatest convergency in the series, let ACB, he angle at the center$=2\theta$ and take the sector $ACD=r^2\theta/2$ and $r^2\sin\theta\cos\theta/2=ACE$.

Then $r^2(\theta-\sin\theta\cos\theta)/2=a\pi r^2/2d$ or arc$\theta=a\pi/d$ $+\cos\theta\sqrt{(1-\cos^2\theta)}$ ....(1).

Make $\cos\theta=x$, and when expanded,

$$\theta=\frac{a\pi}{d}+x-\frac{x^3}{2}-\frac{x^5}{2.4}-\frac{3x^7}{2.4.6}-\frac{3.5x^9}{2.4.6.8},\ \text{etc.},\ \ldots\ldots\ldots\ (2).$$

By trigonometry or calculus, we have,

$$\text{arc}\theta=\frac{\pi}{2}-x-\frac{x^3}{2.3}-\frac{3x^5}{2.4.5}-\frac{3.5x^7}{2.4.6.7}-\frac{3.5.7x^9}{2.4.6.8.9},\ \text{etc.},\ \ldots\ldots\ (3).$$

(2)—(3) and ÷ by 2, etc.,

$$y=\frac{(d-2a)\pi}{4d}=x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{112}-\frac{5x^9}{1152}-\text{ etc.,}\ldots\ldots\ldots(4).$$

Assume $x=Ay+By^3+Cy^5+Dy^7+Ey^9+$ etc.,.....................(5).

The powers of x substituted in (4), $y=Ay+$

$$\left(B-\frac{A^3}{6}\right)y^3+\left(C-\frac{A^2B}{2}-\frac{A^5}{40}\right)y^5+\left(D-\frac{A^2C}{2}-\frac{AB^2}{2}-\frac{A^4B}{8}-\frac{A^7}{112}\right)y^7+\text{etc.}$$

$\therefore$ $A=1$, $B=1/6$, $C=13/120$, $D=493/5040$, $E=37369/362880$, etc., in (5).
$x=\cos\theta=y+y^3/6+13y^5/120+493y^7/5040+37369y^9/362880+$ etc,,....(A).

Substituting values, $y=3\pi/20=0.471239=$logarithm $\bar{1}.673241+$.

2nd$=0.017441$
3rd$=0.002517$
4th$=0.000505$
5th$=0.000118$
Estimated$=0.000025$

$\cos\theta=0.491845$

2nd term $y^3=\bar{1}.019724-$
6 0.778151.
$0.017441=\bar{2}.241573$

4th term $y^7=\bar{3}.712688$
493/5040....$\bar{2}.990416$
$0.000505-=\bar{4}.703104$

3rd term $y^5=\bar{2}.366206$
13/120$=\bar{1}.034762$
$0.002517+=\bar{3}.400968$

5th term $y^9=\bar{3}.059171$
37369/362880....$\bar{1}.012737$
$0.00018=\bar{4}.071908$

Chord $AB=10\sqrt{(1-\cos^2\theta)}=8.7068+$. $ACD=60^\circ 32' 17''$ nearly.

NOTE.—Formula (A) is also a general solution for the height of the circular segment (see problem 37, page 75, Vol. II). When the angle ACD is less than 50°, solve (1) for $\sin\theta$, and we have,

$$\theta<50^\circ=\sin\theta=\left(\frac{3a\pi}{2d}\right)^{\frac{1}{3}}-\frac{1}{10}\left(\frac{3a\pi}{2d}\right)^{\frac{3}{3}}-\frac{19}{1400}\left(\frac{3a\pi}{2d}\right)^{\frac{5}{3}}-\frac{71}{35000}\left(\frac{3a\pi}{2d}\right)^{\frac{7}{3}}-\text{, etc.,..}(B).$$

Chord$=D.\sin\theta$. It will be noticed that the convergency, in part, depends on the smallness of the value of y.

PROBLEMS.

42. Proposed by E. B. ESCOTT, 6123 Ellis Avenue, Chicago, Illinois.

To find a triangle whose sides and median lines are commensurable.

43. Proposed by H. C. WILKES, Skull Run, West Virginia.

To find, if possible, a right angled triangle, the bisectors of the acute angles of which, can be expressed by integral whole numbers.

44. Proposed by Prof. P. S. BERG, Larimore, North Dakota.

Two trees whose heights are 40 and 80 feet, respectively, stand on opposite sides of a stream 30 feet wide. What path does a squirrel take in leaping from the top of the higher to the top of the lower? What is the length of the path?

EDITORIALS.

The August-September number of the MONTHLY will be issued about the 20th of September.

The address of Editor Finkel, after July 1st, will be The University of Chicago, Chicago, Illinois.

This issue has been delayed on account of our engravers missending the plate for Mr. Miller's portrait.

The University of Pennsylvania has conferred the degree of Doctor of Philosophy on our valued contributor, H. C. Whitaker. We congratulate Prof. Whitaker on this merited recognition of his ability.

PERIODICALS.

Annual Recreation Number of the Outlook. The Outlook Publishing Co., 13 Astor Place, New York City.

The Outlook's seventh annual Recreation Number contains nearly a hundred pages and scores of illustrations. Nearly all of the special articles relate to outdoor life, sport, recreation, and vacation possibilities. Among the writers are Ian Maclaren, the Rev. Dr. Henry van Dyke, the Rev. Dr. Charles H. Parkhurst, Kirk Munroe, General A. W. Greely, Poultney Bigelow, and many others.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single number, 25 cents. Review of Reviews Co., New York City.

The June number of *The Review of Reviews* is, as usual, full of the history of the important events that are taking place in various parts of the world. Dr. Shaw, the editor, has given a close analysis of the political situation which is now being worked out at St. Louis.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hudson, New York.

The June number of *The Cosmopolitan* is keeping up its literary merit, but is each time improving in the artistic excellence which it embodies.

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APPLICATIONS OF SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Paris, France.

Lagrange seems to have been the first to give a clear statement and ıt least a partial proof* of the following fundamental

THEOREM I. *The number* (N) *of different formal values which are obtained* 'y *permuting the* n *elements of a given function in every possible manner is a divis-* r *of* $n!$.†

About thirty years later Ruffini proved that N cannot have the values 3 or when $n=5$, in his work, "*Teoria generale delle equazioni, in cui si dimostra im-* ossibile *la soluzione algebraica delle equazioni generali di grado superiore al* uarto," Bologna, 1799. He thus proved also that N cannot be equal to every ivisor of $n!$.

As it was known that the value of N is the quotient obtained by dividing ! by the order of the largest substitution group which transforms the function ıto itself it became an important problem to determine all the possible orders f the substitution groups of n elements, especially since it was believed that this ould throw light on the solution of the general equation of the n^{th} degree. This roblem has been solved only for small values of n.

The given theorem of Lagrange indicates the most direct application ' substitution groups and therefore naturally furnished the starting point for the ırly investigations in this subject. It may be readily proved in the following anner.‡

Let ψ, the given function, be unchanged only by the substitutions in the

*The proof given by Lagrange in his article, "*Reflexions sur la resolution algebrique des equations*," emoires de l'Academie de Berlin, 1770 and 1771, seems to have been generally considered as complete. . Mathieu, Comptes Rendus, 46, page 1047. Burkhardt, on the contrary, seems to regard it as incom-ete. Cf. Zeitschrift fur Mathematik, 1892, page 141.

†$n!=1.2.3......n$.

‡Cf. Netto's Theory of Substitutions (Cole's edition) §41.

first row of the following rectangle. (These form a group, for the product of any two leaves ψ unchanged and is therefore found in this row.)

$$\begin{array}{llll}
s_1=1 & s_2 & s_3 \ldots\ldots\ldots\ldots s_\alpha & \\
t_2 & s_2t_2 & s_3t_2 \ldots\ldots\ldots\ldots s_\alpha t_2 & \\
t_3 & s_2t_3 & s_3t_3 \ldots\ldots\ldots\ldots s_\alpha t_3 & \\
\vdots & \vdots & \vdots \quad \ldots\ldots\ldots\ldots \quad \vdots & \\
\vdots & \vdots & \vdots \quad \ldots\ldots\ldots\ldots \quad \vdots & \\
\vdots & \vdots & \vdots \quad \ldots\ldots\ldots\ldots \quad \vdots & \\
t_m & s_2t_m & s_3t_m \ldots\ldots\ldots\ldots s_\alpha t_m & m=\dfrac{n!}{\alpha}.
\end{array}$$

ψ will assume the same formal value if any one of the substitutions of a given row is applied to it, for the first factor leaves it unchanged and the second factor is the same throughout the row. If we assume that t_β $(\beta=2, 3, \ldots\ldots n)$ is not found in a preceding row the substitutions of the rectangle are all different, for from

$$t_{\beta_2}s_{\gamma_2}=t_{\beta_1}s_{\gamma_1} \qquad (\beta_2 \gtreqless \beta_1 \leqq m)$$

we would have $\qquad (\gamma_1, \gamma_2, \gamma_3 \leqq \alpha)$

$$t_{\beta_2}=t_{\beta_1}s_{\gamma_3}.$$

This is impossible unless $\beta_1=\beta_2$ and $s_{\gamma_3}=1$. In this case $t_{\beta_2}s_{\gamma_2}$ and $t_{\beta_1}s_{\gamma_1}$ occupy the same place in the given rectangle.

Since there are just $n!$ substitutions of n elements the given rectangle contains each substitution once and only once. If t_{β_1} and t_{β_2} would transform ψ into the same function (ψ_1) then would the products of all the substitutions in the rows containing t_{β_1} and t_{β_2} into a substitution* t_γ which transforms ψ_1 into ψ give 2α different substitutions that transform ψ into itself. This is contrary to the hypothesis. Therefore $N=m=$ a divisor of $n!$.

One of the best known functions to which these elementary principles of substitution groups are commonly applied is the anharmonic ratio of four points.† If the four points are represented by A, B, C, and D, their anharmonic or cross ratio may be represented by

$$\psi\equiv\frac{AB}{CB}\div\frac{AD}{CD} \text{ or } \frac{AB.CD}{AD.CB}.$$

It is required to find the number of formal values of ψ when the points are interchanged in every possible manner. We may do this by dividing $4!=24$ by the order of the largest group of degree four that transforms ψ into itself. Since ψ is unchanged by the substitution $AB.CD$ and also by the substitution

*Since the rectangle contains all the possible substitutions of n elements, it must contain the inverse of each of its substitutions. We shall always consider n to be a finite number.

†Cf. Harkness and Morley's.

).BC, it must be unchanged by the group generated by these substitutions, viz., $(AB.CD)_4$. We know that there are only three groups* of degree four which ;lude $(AB.CD)_4$ and that these contain either a substitution of the form AB or e of the form ABC. As no such substitution transforms ϕ into itself $B.CD)_4$ is the largest group that has this property. The number of different lues of ϕ is therefore $24\div4=6$.

To find these six values of ϕ we may arrange the substitutions of four ele- ›nts as follows:

1	$AB.CD$	$AC.BD$	$AD.BC$
AB	CD	$ACBD$	$ADBC$
AC	$ABCD$	BD	$ADCB$
AD	$ABDC$	$ACDB$	BC
ABC	ACD	BDC	ADB.
ACB	BCD	ABD	ADC

Since all the substitutions of a row transform ϕ into the same function we n find the six formal values of ϕ by applying to it the six substitutions of the st column in this rectangle.† We thus obtain the following, in order:

$$\frac{B.CD}{D.CB}=k;\ \frac{BA.CD}{BD.CA}=\frac{\frac{BA.CD}{AD.CB}}{\frac{AD.CB--AB.CD}{AD.CB}}=\frac{k}{k-1};\ \frac{CB.AD}{CD.AB}=\frac{1}{k}.$$

$$\frac{3.CA}{4.CB}=\frac{AD.CB-AB.CD}{AD.CB}=1-k;\ \frac{BC.AD}{BD.AC}=\frac{1}{1-k};\ \frac{CA.BD}{CD.BA}=\frac{k-1}{k}=1-\frac{1}{k}.$$

This example furnishes also a clear illustration of what we mean by "*dif- ent formal values.*" The six given values of ϕ are all different as to form but nay have such values that they are not all really different. E. g., if $k=-1$, ›y coincide in pairs. In this case the ratio is called *harmonic*. If $k=$an im- nary cube root of -1, they coincide in triplets and the ratio is called *ıianharmonic*.

It should be observed that each one of the four subgroups of $(ABCD)$all, ich are of the form (ABC)all, has one substitution in each row. Hence the lowing

THEOREM II. *The six different formal values of an anharmonic ratio 'our points may be obtained by transforming any three of its points symmetrically.*

*It is evident that the function is not symmetric. It would therefore only be necessary to examine ith respect to the other two groups.

†This is clearly only one of the 4096 different ways in which the six transforming substitutions may elected.

If G is the largest group which transforms a function (ψ) into itself we say that ψ *belongs to* G. The same relation is also expressed by saying that G *belongs to* ψ.* The former of these two expressions is to be preferred since only *one group belongs to any given* ψ *while an infinite number of functions belong to any given* G. This may be readily proved as follows :†

We first suppose that G is the symmetric group of n elements. Every symmetric function of these elements will then belong to G. That their number is infinite, follows directly from the fact that both a and b can have an infinite number of different values without impairing the symmetry of the following functions :‡

$$x_1^a+x_2^a+x_4^a+\ldots\ldots+x_n^a+bx_1x_2x_3\ldots\ldots x_n \quad\ldots\ldots\ldots\ldots A.$$

We may now suppose that G consists of a single substitution, viz., identity. In this case every function of the n elements which is changed in form by each substitution of these elements belongs to G. If we suppose that a_1, a_2, a_3, ..., a_n represent n different given numbers, the following function belongs to G:

$$a_1x_1+a_2x_2+a_3x_3+\ldots\ldots+a_nx_n \quad\ldots\ldots\ldots\ldots\ldots\ldots B.$$

We may now assign all the possible values of a_1 with the exception of the finite number of values represented by a_2, a_3, ..., a_n. In this way we obtain an infinite number of functions belonging to G.

We finally suppose that G represents any other group whose order is g. If we apply the substitutions of G to any one of the functions of B we obtain g different functions, ψ_1, ψ_2, ψ_3, ..., ψ_g. In any of the functions A we may suppose $n=g$ and the x's, in order, replaced by these ψ's. The resulting function belongs to this G. It is clear that we obtain an infinite number of such functions even by using a particular function of either A or B. We did not prove that all the functions belonging to G can be obtained in this way. In fact, this is not the case. As it follows from the definition that only one group belongs to a given function the proof is complete.

We have thus far only considered the relations between groups and functions when all the elements of the function which are permuted and no others are explicitly contained in the corresponding group. We have also only considered the number of values of a function when its elements are permuted according to the symmetric group. That the arguments which were employed apply to much more general cases may be illustrated by means of the following well-known Trigonometry formula

$$\text{Sin}\,\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

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*Cf. Netto, Theory of Substitutions (Cole's Edition) §28.

†Ibid, §30.

‡ If a and b are complex numbers, A represents ∞^4 different functions.

If we regard the first member of this equation as a function of the three angles A, B, C of a triangle it belongs to the group (BC) and is therefore a three-valued function of the angles. The second member belongs to the group (bc) and is therefore a three-valued function of the sides. Hence the formula says that *a given three-valued function of the angles is equal to a given three-valued function of the sides.*

As no special properties were imposed upon any the sides or angles in deriving the formula the three different values of the angles must correspond separately to the three different values of the sides. It remains only to find the substitutions which transform the given formula into the other two. To do this we may arrange the substitutions of the angles and the sides, in the usual manner, as follows:

1	BC	1	bc
AB	ABC	ab	abc
AC	ACB	ac	acb

Since the substitutions of a row transform the corresponding functions in the same way and the rows of the two rectangles evidently correspond in order, we may effect the required transformation by any two substitutions such that one belongs to the first and the other to the second of the following two rows:

$$AB.ab,\ AB.abc,\ ABC.ab,\ ABC.abc$$
$$AC.ac,\ AC.acb,\ ACB.ac,\ ACB.acb$$

If we use the last one of each of these rows we have the rule frequently given in the text-books, viz., "The corresponding formulas for the other two angles may be obtained from this by the cyclical interchange of the letters."

The given formula might also be studied by employing a single group in place of two. The most convenient group is the intransitive group of degree six and order 36 which is obtained by multiplying the symmetric group of the angles into the symmetric group of the sides.* Since the given formula is transformed into itself only by the following four substitutions of this group

$$1,\ BC,\ bc,\ BC.bc,$$

it is a nine-valued function with respect to this group.† Since substitutions of this group transform the first member of the given formula into its three values without affecting the second member, these nine values may be arranged into three triplets, each of which has the same second member. By very simple trials we can show that six of these relations are absurd. Since three must be true the remaining relations are the required formulas.

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*Cf. This journal, Vol. II, page 207.

†This may be proved in exactly the same way as Lagrange's theorem was proved.

Since similar remarks apply to a large number of the other Trigonometry formulas, it is clear that these formulas can be discussed in a more general and more definite manner by presupposing a thorough knowledge of the given elementary principles of substitution groups.

It is also easy to show that many problems of factoring can be discussed more completely by presupposing a knowledge of these groups. The following is a very simple illustration:

$$a^2-b^2-c^2+2bc=(a+b-c)(a-b+c) \ldots\ldots\ldots\ldots C.$$

The expression belongs to the group (bc) and is therefore a three-valued function; its factors belong to the groups (ab) and (ac) respectively and are therefore also three-valued functions. Hence C indicates an equality between a given three-valued function and the product of two other three-valued functions. These functions belong to three distinct groups. Arranging the substitutions of these groups in the usual manner, we have

1	*bc*	1	*ab*	1	*ac*
ab	*abc*	*ac*	*abc*	*ab*	*acb*
ac	*acb*	*bc*	*acb*	*bc*	*abc*

The three values of the given expression* may be obtained by applying to it one substitution from each of the three rows of the first rectangle, e. g., the first column. The factors of these transforms may evidently be found by applying the same substitutions to the given factors. Since ab and ac transform one of the factors into itself it follows that the three conjugate expressions contain only three distinct linear factors, viz., the three values of any one of them.

These observations indicate how we may readily determine the total number of substitutions by means of which the factors of all the conjugates of a given expression may be found from those of the given expression. They have brought us in contact with, at least, three important questions, viz.:

1. What relations exist between the factors of a system of conjugate expression?

2. What relations exist between the groups of the factors and the group of the expression?

3. To what extent may these relations be utilized in the process of factoring?

*The same idea is expressed by "the three conjugates of the given expression" or by "the three transforms of the given expression."

THE BINOMIAL THEOREM.

By G. B. M. ZERR, A. M., Ph. D., Texarkana, Texas.

I use the following rule for expanding all binomials, whether the exponent is integral or fractional, positive or negative.

The number of terms of a binomial expansion is one more than the exponent when the exponent is a positive integer, otherwise the number of terms is infinite. For the first term of the expansion, raise the first term of the binomial to the required power. For any other term of the expansion, multiply the preceding term by the second term of the binomial, and this product by the exponent of the power diminished by two less than the number of terms from the beginning, divide this product by the product of the first term of the binomial into one less than the number of terms from the beginning, always observing the proper algebraic signs of the binomial terms.

$$(ax+by)^m=(ax)^m+\frac{m(ax)^m by}{ax}+\frac{m(m-1)(ax)^m(by)^2}{1.2.(ax)^2}+\ldots\ldots$$

$$+\ldots\ldots+\frac{m(m-1)(m-2)\ldots\ldots(m-r+2)(ax)^m(by)^{r-1}}{1.2.3\ldots\ldots(r-1)(ax)^{r-1}}+\ldots\ldots(A).$$

(A) gives the expansion without reducing the terms.

(1). To expand $(3x\pm4y)^5$.

1st term $=(3x)^5=243x^5$; 2nd term $=\dfrac{243x^5\times(\pm4y)\times5}{3x}=\pm1620x^4y$;

3rd term $=\dfrac{\pm1620x^4y\times(\pm4y)\times4}{2.3x}=4320x^3y^2$;

4th term $=\dfrac{4320x^3y^2\times(\pm4y)\times3}{3.3x}=\pm5760x^2y^3$;

5th term $=\dfrac{\pm5760x^2y^3\times(\pm4y)\times2}{4.3x}=3840xy^4$;

6th term $=\dfrac{3840xy^4\times(\pm4y)\times1}{5.3x}=\pm1024y^5$.

$\therefore\ (3x\pm4y)^5=243x^5\pm1620x^4y+4320x^3y^2\pm5760x^2y^3+3840xy^4\pm1024y^5$.

(2). To expand $(a^2+2b)^7$.

$$\text{1st term} = (a^2)^7 = a^{14};\ \text{2nd term} = \frac{a^{14}.2b.7}{a^2} = 14a^{12}b;$$

$$\text{3rd term} = \frac{14a^{12}b.2b.6}{2.a^2} = 84a^{10}b^2;\ \text{4th term} = \frac{84a^{10}b^2.2b.5}{3.a^2} = 280a^8b^3;$$

$$\text{5th term} = \frac{280a^8b^3.2b.4}{4.a^2} = 560a^6b^4;\ \text{6th term} = \frac{560a^6b^4.2b.3}{5.a^2} = 672a^4b^5;$$

$$\text{7th term} = \frac{672a^4b^5.2b.2}{6.a^2} = 448a^2b^6;\ \text{8th term} = \frac{448a^2b^6.2b.1}{7.a^2} = 128b^7.$$

$$\therefore\ (a^2+2b)^7 = a^{14} + 14a^{12}b + 84a^{10}b^2 + 280a^8b^3 + 560a^6b^4 + 672a^4b^5 + 448a^2b^6 + 128b^7$$

(3). To expand $(2+x)^{-3}$.

$$\text{1st term} = (2)^{-3} = \tfrac{1}{8};\ \text{2nd term} = \tfrac{1}{8} \times \frac{x\times(-3)}{2} = -\frac{3x}{16};$$

$$\text{3rd term} = -\frac{3x}{16} \times \frac{x\times(-4)}{2.2} = \frac{3x^2}{16};\ \text{4th term} = \frac{3x^2}{16} \times \frac{x\times(-5)}{3.2} = -\frac{5x^3}{32};$$

$$\text{5th term} = -\frac{5x^3}{32} \times \frac{x\times(-6)}{4.2} = \frac{15x^4}{128}.$$

$$\therefore\ (2+x)^{-3} = \tfrac{1}{8} - \frac{3x}{16} + \frac{3x^2}{16} - \frac{5x^3}{32} + \frac{15x^4}{128} - \ldots\ldots$$

(4). To expand $(1+\frac{2x}{3})^{\frac{3}{2}}$.

$$\text{1st term} = (1)^{\frac{3}{2}} = 1;\ \text{2nd term} = \frac{1.\frac{2x}{3}.\frac{3}{2}}{1} = x;$$

$$\text{3rd term} = \frac{x.\frac{2x}{3}.\frac{1}{2}}{2.1} = \tfrac{1}{6}x^2;\ \text{4th term} = \frac{\frac{1}{6}x^2.\frac{2x}{3}.(-\frac{1}{2})}{3.1} = -\tfrac{1}{54}x^3;$$

$$\text{5th term} = \frac{-\frac{1}{54}x^3.\frac{2x}{3}.(-\frac{3}{2})}{4.1} = \tfrac{1}{216}x^4.$$

$\therefore\ (1+\frac{2x}{3})^{\frac{3}{2}}=1+x+\frac{1}{6}x^2-\frac{1}{54}x^3+\frac{1}{216}x^4-\ \ldots\ldots$

(5). To expand $(8+12a)^{\frac{2}{3}}$.

$$\text{1st term}=(8)^{\frac{2}{3}}=4;\ \text{2nd term}=\frac{4.12a.\frac{2}{3}}{8}=4a;$$

$$\text{3rd term}=\frac{4a.12a.(-\frac{1}{3})}{2.8}=-a^2;\ \text{4th term}=\frac{-a^2.12a.(-\frac{4}{3})}{3.8}=\frac{2a^3}{3};$$

$$\text{5th term}=\frac{\frac{2a^3}{3}.12a.(-\frac{7}{3})}{4.8}=-\frac{7a^4}{12}.$$

$\therefore\ (8+12a)^{\frac{2}{3}}=4+4a-a^2+\frac{2}{3}a^3-\frac{7}{12}a^4+\ldots\ldots$

(6). To expand $(4a-8x)^{-\frac{1}{2}}$.

$$\text{1st term}=(4a)^{-\frac{1}{2}}=\frac{1}{2a^{\frac{1}{2}}};\ \text{2nd term}=\frac{1}{2a^{\frac{1}{2}}}.\frac{(-8x)(-\frac{1}{2})}{4a}=\frac{x}{2a^{\frac{3}{2}}};$$

$$\text{term}=\frac{x}{2a^{\frac{3}{2}}}.\frac{(-8x)(-\frac{3}{2})}{2.4a}=\frac{3x^2}{4a^{\frac{5}{2}}};\ \text{4th term}=\frac{3x^2}{4a^{\frac{5}{2}}}.\frac{(-8x)(-\frac{5}{2})}{3.4a}=\frac{5x^3}{4a^{\frac{7}{2}}};$$

$$\text{5th term}=\frac{5x^3}{4a^{\frac{7}{2}}}.\frac{(-8x)(-\frac{7}{2})}{4.4a}=\frac{35x^4}{16a^{\frac{9}{2}}}.$$

$$\therefore\ (4a-8x)^{-\frac{1}{2}}=\frac{1}{2a^{\frac{1}{2}}}+\frac{x}{2a^{\frac{3}{2}}}+\frac{3x^2}{4a^{\frac{5}{2}}}+\frac{5x^3}{4a^{\frac{7}{2}}}+\frac{35x^4}{16a^{\frac{9}{2}}}+\ldots\ldots$$

$$=\frac{1}{2a^{\frac{1}{2}}}\left(1+\frac{x}{a}+\frac{3x^2}{2a^2}+\frac{5x^3}{4a^3}+\frac{35x^4}{8a^4}+\ldots\ldots\right).$$

The r^{th} term in (A) is $\frac{m(m-1)(m-2)\ldots\ldots(m-r+2)(ax)^m(by)^{r-1}}{1.2.3\ldots\ldots(r-1)(ax)^{r-1}}$.

(7). Find the 4th term of $(\frac{a}{3}+9b)^{10}$.

$$m=10,\ r=4.\quad \therefore\ \text{4th term}=\frac{10\times9\times8\times(\frac{a}{3})^{10}(9b)^3}{1.2.3.(\frac{a}{3})^3}=40a^7b^3.$$

(8). Find the 28th term of $(5x+8y)^{30}$.
$m=30$, $r=28$.

$$\therefore\ 28\text{th term}=\frac{30.29.28\ldots\ldots6.5.4.(5x)^{30}(8y)^{27}}{1.2.3\ldots\ldots26.27.(5x)^{27}}=\frac{\lfloor 30}{\lfloor 27\ \lfloor 3}(5x)^{3}(8y)^{27}.$$

(9). Find the 8th term of $(1+2x)^{-\frac{1}{2}}$.
$m=-\frac{1}{2}$, $r=8$.

$$\therefore\ 8\text{th term}=\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})(-\frac{9}{2})(-\frac{11}{2})(-\frac{13}{2})(1)^{-\frac{1}{2}}(2x)^{7}}{1.2.3.4.5.6.7.(1)^{7}}=-\frac{429x^{7}}{16}.$$

(10). Find the 10th term of $(1+3a^{2})^{\frac{16}{3}}$.
$m=\frac{16}{3}$, $r=10$.

$$\therefore\ 10\text{th term}=\frac{\frac{16}{3}.\frac{13}{3}.\frac{10}{3}.\frac{7}{3}.\frac{4}{3}.\frac{1}{3}(-\frac{2}{3}).(-\frac{5}{3}).(-\frac{8}{3})(1)^{\frac{16}{3}}(3a^{2})^{9}}{1.2.3.4.5.6.7.8.9.(1)^{9}}=-\frac{1040a^{18}}{81}.$$

(11). Find the 5th term of $(3a-2b)^{-1}$.
$m=-1$, $r=5$.

$$\therefore\ 5\text{th term}=\frac{(-1)(-2)(-3)(-4)(3a)^{-1}(2b)^{4}}{1.2.3.4.(3a)^{4}}=\frac{16b^{4}}{243a^{5}}.$$

These are enough examples to illustrate both the rule and the general term.

I have used this method with my classes for several years and find it easier and better than any other method I have ever used. I have never seen this method in this form. If any of the readers of the MONTHLY have ever seen it, I would be pleased to know where to find it.

A GEOMETRICAL PROOF THAT $0\times\infty$ IS INDETERMINATE.

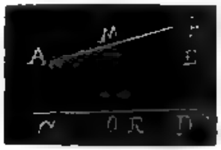
y B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

The Proof that I shall offer is not new perhaps, but I have never seen it print, and for that reason I shall give it in the MONTHLY.

The proof follows as a corollary of the following

PROPOSITION: *The area of the surface generated by a straight line revolving ut an axis in its plane is equal to the product of the projection of the line on the by the circumference whose radius is a perpendicular erected at the middle ut of the line and terminated by the axis.*

Let AB be the straight line revolved about CD as an axis. When AB is parallel to the axis CX, the surface generated by AB is the convex surface of frustum of a cone.



$\therefore$ Area generated by $AB=AB\times2\pi MO$.

But $MO:AE::MR:AB$, or

$$AB\times MO=AE\times MR=CD\times MR.$$

$\therefore$ Area generated by $AB=CD\times2\pi MR$. Now if AB is made to approach pendicularity, MR will approach parallelism to CX, and, in consequence, CD approach 0 as its limit and MR will approach ∞ as its limit. Hence, in the it, we have

$$\text{area generated by } AB=CD\times2\pi MR=0\times2\pi\times\infty.$$

But area generated by AB when AB is perpendicular to CX is $C^2-AC^2)$. Hence, $\pi(BC^2-AC^2)=0\times2\pi\times\infty$, or $BC^2-AC^2=2\times0\times\infty$ $\times\infty$. When $AC=0$, we have $0\times\infty=BC^2$. Now BC is entirely arbitrary. ice, $0\times\infty$ is indeterminate. But when BC is a definite quantity, as for nple 3, then $0\times\infty$ has the definite value 9.

The fundamental type of symbols of indetermination is $\frac{0}{0}$, and to this type ∞ may be reduced. Thus, $0\times\infty=0\times\frac{1}{\frac{1}{\infty}}=\frac{0\times1}{0}=\frac{0}{0}$. The indeterminate $\frac{\infty}{\infty}$, $=\frac{\frac{1}{\infty}}{\frac{1}{\infty}}=\frac{0}{0}$. Also $\infty-\infty=\frac{1}{\frac{1}{\infty}}-\frac{1}{\frac{1}{\infty}}=\frac{1}{0}-\frac{1}{0}=\frac{0}{0}$; $0^0=0^0\div0^0=\frac{0^0}{0^0}$ $\frac{0}{0}$; $\infty^0=\infty^0\div\infty^0=\frac{\infty}{\infty}=\frac{\frac{1}{\infty}}{\frac{1}{\infty}}=\frac{0}{0}$.

When these forms occur as the answers of problems, they have, in general, perfectly definite values, and these definite values must be found. But when these forms stand apart from the consideration of problems, they are perfectly meaningless,

Drury College, September 14, 1896.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

60. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

A pipe 1 foot long and 27-32 inch in diameter has a half-inch orifice and weighs $1\frac{3}{4}$ pounds. What is the diameter of a pipe the same length and orifice, but weighing 41 ounces?

I. Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, New Jersey.

Let V_1 = volume of "solid" pipe.

Let V_2 = volume of bore.

Then $V_1 - V_2$ = volume of metal $=\pi(\frac{1419}{1024})$ cubic inches.

Since weights are proportional to volumes, $\pi(\frac{1419}{1024}) : V_3 = 28 : 41$, where V_3 = volume of required size of pipe.

Add to this volume of bore $=V_2$, and we have,

$$V_3 + V_2 = V_4 = \text{new "solid" pipe} = \pi(\tfrac{79683}{28672}) \text{ cubic inches.}$$

$$\text{Hence } R = \{\pi(\tfrac{79683}{28672}) \div \pi.12\}^{\frac{1}{2}} = .481\tfrac{1}{4} \text{ and } D = .9625 \text{ inches.}$$

II. Solution by EDWARD R. ROBBINS, Master in Mathematics and Physics, Lawrenceville School, Lawrenceville, New Jersey.

The volumes of the two pipes will have the same ratio as their weights.

$$\text{Hence, } \frac{\pi l\left[\left(\frac{D}{2}\right)^2 - \left(\frac{d}{2}\right)^2\right]}{\pi l\left[\left(\frac{D'}{2}\right)^2 - \left(\frac{d}{2}\right)^2\right]} = \frac{w}{w'}; \text{ or } \frac{D^2 - d^2}{D'^2 - d^2} = \frac{w}{w'},$$

where the D's represent the diameters of the pipes, and d the common diameter of their *orifices*. From this

$$D'=\sqrt{\frac{w'D^2-w'd^2+wd^2}{w}}=\sqrt{\frac{w'}{w}(D^2-d^2)+d^2}.$$

$$=\sqrt{[\text{illegible}]\left[([\text{illegible}])^2-([\text{illegible}])^2\right]+[\text{illegible}]}=\sqrt{[\text{illegible}]}=.96248+\text{ inches.}$$

Also solved by *G. B. M. ZERR*, *H. C. WILKES*, and *J. SCHEFFER*.

61. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Insured my store for a/bth$=\frac{3}{4}$th of its value, at $r=1\frac{1}{4}\%$; but soon afterward the store was burned down, and my loss over the insurance was $\$L=\$4150$. What was the value of my store?

I. Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

Construing the terms of this question as they are used in legal and insurance circles the solution is $\$4,150\times4=\$16,600$.

But the proposer evidently intends to reckon the premium paid as a part of the "loss."

Then for every \$4.00 of value \$3.00 was insured at a cost of 3.75 cents, leaving \$1.0375 of loss.

Hence 1.0375 : 4 :: 4150 : 16,000.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

The value of the policy is $\frac{a}{b}\cdot\frac{r}{100}x$, x representing the value of the store.

We have, therefore, obviously,

$$x\left[1-\frac{a}{b}\left(1-\frac{r}{100}\right)\right]=L,\ \therefore x=L\div\left[1-\frac{a}{b}\left(1-\frac{r}{100}\right)\right].$$

Substituting numerical value, we find $x=\$16,000$.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas; P. S. BERG, Larimore, North Dakota; and A. P. READ, A. M., Clarence, Missouri.

Let s=value of store.

$$\text{Then } s-\frac{as}{b}=\left(\frac{b-a}{b}\right)s;\ \frac{r}{100}\times\frac{as}{b}=\frac{ars}{100b}.\ \therefore\left(\frac{b-a}{b}+\frac{ar}{100b}\right)s=L.$$

$$\therefore s=\frac{100bL}{100b-100a+ar}=\frac{400(4150)}{100+\frac{15}{4}}=\$16,000.$$

Also solved by *EDWARD R. ROBBINS*, and *F. M. McGAW*.

PROBLEMS.

65. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Bought April 4, 1894, 250 yards of broadcloth at \$5.37½ per yard, less 12½ and 10% discount for cash payment. Sold September 5, 1894, at 15, 10, and 5% on *quoted price*, the cloth; and in settlement received a 90-day note which I had discounted at 5½%, October 19, 1894, by the First National Bank of Baltimore, Maryland. Reckoning 6% interest on the *money invested* in the cloth, what is the profit made?

66. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Brown adds $m=10\%$ of water to the pure wine he buys, and then sells the mixture at a price $n=10\%$ greater than the cost price of the pure wine. What is his rate per cent. of profit?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

62. Proposed by Professor C. E. WHITE, A. M., Trafalgar, Indiana.

Prove that every algebraic equation can be transformed into another equation of the same degree, but which wants its n^{th} term.

I. Solution by HENRY HEATON, M. Sc., County Surveyor, Atlantic, Iowa.

To illustrate, let $x^4+ax^3+bx^2+cx+d=0$ be any equation of the fourth degree. Put $x=y+p$; then the equation becomes

$$y^4+(4p+a)y^3+(6p^2+3ap+b)y^2+(4p^3+3ap^2+2bp+c)y+p^4+ap^3+bp^2+cp+d=0.$$

Since we are at liberty to give p any value, we may give it the value that will make $4p+a=0$ or $-a/4$; then will the coefficient of y^3 disappear. It is also evident that we may give p such a value that any desired coefficient will disappear. It is also evident that to find the desired value of p by this method requires for the second term, the solution of an equation of the first degree; for the third term, the solution of an equation of the second degree, etc. It is further evident that this is true without regard to the degree of the original equation.

II. Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

If not already so, any equation of the n^{th} degree may be reduced to the form $x^n+Ax^{n-1}+Bx^{n-2}+\ldots\ldots+L=0$. Now, by putting for x, $x+a$, we obtain a new equation whose roots differ from the corresponding roots of the given equation by a, (and whose degree, therefore, is still the n^{th}) viz.:

$$x^n+(na+A)x^{n-1}+\left(\frac{n(n-1)}{\underline{|2}}a^2+(n-1)Aa+B\right)x^{n-2}$$

$$+\ldots\ldots+(a^n+Aa^{n-1}+Ba^{n-2}+\ldots\ldots+L=0.$$

As a is an arbitrary constant, it may be selected so that $(na+A)=0$, or

$$\left(\frac{n(n-1)}{\underline{|2}}a^2+(n-1)Aa+B\right)=0,$$

or any coefficient, except the first, $=C$. Hence, any term, except the first, may thus be removed.

III. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Every algebraic equation may be written

$$X^k-\Sigma\alpha.X^{k-1}+\Sigma\alpha\beta.X^{k-2}\ldots\ldots=0.$$

The coefficient of the n^{th} term will be $\Sigma\alpha\beta\gamma\ldots\ldots$to $n-1$ factors. Now in place of X write $X+h$; then α, β, γ, etc., will be changed into $\alpha+h$, $\beta+h$, $\gamma+h$, etc. The coefficient of n^{th} term will then be $\pm\Sigma(\alpha+h)(\beta+h)(\gamma+h)\ldots\ldots$to $n-1$ terms. If we equate this to zero, we may consider it an equation of degree $n-1$ in h. This will give $n-1$ values of h. Therefore there are $n-1$ transformations which will make the n^{th} term vanish. Consider the first term, $n-1$; there are in that case no transformations.

Also solved by *PROF. E. W. MORRELL.*

63. Proposed by J. A. CALDERHEAD, A. B., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

Given $x^2+x\sqrt{xy}=10$, and $y^2+y\sqrt{xy}=20$ to find x and y by quadratics.

I. Solution by E. L. BROWN, A. M., Professor of Mathematics, Capital University, Columbus, Ohio; HENRY HEATON, M. Sc., Atlantic, Iowa; and G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Factoring, we have $x^{\frac{3}{2}}(x^{\frac{1}{2}}+y^{\frac{1}{2}})=10$, $y^{\frac{3}{2}}(x^{\frac{1}{2}}+y^{\frac{1}{2}})=20$.

$$\therefore y^{\frac{3}{2}}/x^{\frac{3}{2}}=2,\ y^{\frac{3}{2}}=2x^{\frac{3}{2}}.\quad \therefore y=\sqrt[3]{4}x.$$

$\therefore y^{\frac{1}{2}}=\pm x^{\frac{1}{2}}\sqrt[3]{2}$, this in either equation gives

$$x^2(1\pm\sqrt[3]{2})=10, \quad \therefore x=\pm\sqrt{\frac{10}{1\pm\sqrt[3]{2}}},\ y=\pm\sqrt[3]{4}\sqrt{\frac{10}{1\pm\sqrt[3]{2}}}.$$

II. Solution by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania; COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee; and M. A. GRUBER, A. M., War Department, Washington, D. C.

Factoring the given equations, we obtain

$$x\sqrt{x}(\sqrt{x}+\sqrt{y})=10=a, \ldots\ldots\ldots\ldots(1), \quad y\sqrt{y}(\sqrt{x}+\sqrt{y})=20=b, \ldots\ldots\ldots\ldots(2).$$

$$(1)\div(2) \text{ gives } \frac{x\sqrt{x}}{y\sqrt{y}}=\frac{a}{b}. \text{ Squaring and reducing, we get}$$

$$y=\frac{x\sqrt[3]{b^2}}{\sqrt[3]{a^2}}, \text{ and } \sqrt{xy}=\frac{x\sqrt[3]{b}}{\sqrt[3]{a}}.$$

Substituting in first given equation, we have $x^2+\dfrac{x^2\sqrt[3]{b}}{\sqrt[3]{a}}=a$;

$$\text{whence } x=\pm\left(\frac{a\sqrt[3]{a}}{\sqrt[3]{a}+\sqrt[3]{b}}\right)^{\frac{1}{2}}=\pm\left(\frac{10}{1+\sqrt[3]{2}}\right)^{\frac{1}{2}},$$

$$\text{and } y=\pm\left(\frac{b\sqrt[3]{b}}{\sqrt[3]{a}+\sqrt[3]{b}}\right)^{\frac{1}{2}}=\pm\left(\frac{20\sqrt[3]{2}}{1+\sqrt[3]{2}}\right)^{\frac{1}{2}}.$$

III. Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio; and Prof. E. W. MORRELL, Montpelier Seminary, Montpelier, Vermont.

The given equations may be written,

$$x\sqrt{xy}=10-x^2 \ldots\ldots\ldots\ldots(1), \quad y\sqrt{xy}=20-y^2 \ldots\ldots\ldots\ldots(2).$$

$$(1)\times(2),\ x^2y^2=200-20x^2-10y^2+x^2y^2. \quad \therefore 2x^2+y^2=20 \ldots\ldots\ldots\ldots(3).$$

$$\text{From (2) and (3), } 2x^2=y\sqrt{xy}. \quad \therefore y=x\sqrt[3]{4} \ldots\ldots\ldots\ldots(4).$$

$$(4) \text{ in } (1),\ x=\pm\sqrt{\frac{10}{1+\sqrt[3]{2}}}. \quad \therefore y=\pm\sqrt{\frac{20\sqrt[3]{2}}{1+\sqrt[3]{2}}}.$$

IV. Solution by J. H. DRUMMOND, LL. D., Portland, Maine; A. H. HOLMES, Brunswick, Maine; and O. W. ANTHONY, M. Sc., New Windsor College, New Windsor, Maryland.

Let $y=v^2x$, then $x^2(1+v)=a=10$, and $v^3x^2(1+v)=b=20$.

$$\therefore v=\sqrt[3]{\frac{b}{a}}, \text{ and } x=\pm\frac{a^{\frac{2}{3}}}{\sqrt{a^{\frac{1}{3}}+b^{\frac{1}{3}}}}, =\pm\left(\frac{10}{1+\sqrt[3]{2}}\right)^{\frac{1}{2}}.$$

$$y=\pm\frac{b^{\frac{3}{4}}}{\sqrt{a^{\frac{1}{2}}+b^{\frac{1}{2}}}},\quad =\pm\left(\frac{20\sqrt[3]{2}}{1+\sqrt[3]{2}}\right)^{\frac{3}{4}}.$$

V. Solution by CHAS. A. HOBBS, A. M., Master of Mathematics in the Belmont School, Belmont, Massachusetts.

$$x^{\frac{4}{3}}+x^{\frac{3}{3}}y^{\frac{1}{3}}=10,\quad y^{\frac{4}{3}}+x^{\frac{1}{3}}y^{\frac{3}{3}}=20.\quad \text{Let } y=vx.$$

$$\text{Then } x^{\frac{4}{3}}+v^{\frac{1}{3}}x^{\frac{4}{3}}=10,\quad v^{\frac{3}{3}}x^{\frac{4}{3}}+v^{\frac{4}{3}}x^{\frac{4}{3}}=20.$$

$$\therefore x^{\frac{4}{3}}=\frac{10}{1+v^{\frac{1}{3}}},\ \text{and } x^{\frac{4}{3}}=\frac{20}{v^{\frac{3}{3}}+v^{\frac{4}{3}}}.\quad \therefore \frac{10}{1+v^{\frac{1}{3}}}=\frac{20}{v^{\frac{3}{3}}+v^{\frac{4}{3}}}.$$

Dividing by 10, and clearing of fractions, $v^{\frac{3}{3}}=2$, $v=2^{\frac{3}{3}}$.

$$\therefore x^{\frac{4}{3}}=\frac{10}{1+2^{\frac{1}{3}}},\quad x=\sqrt{\frac{10}{1+\sqrt[3]{2}}}.\quad y=2^{\frac{3}{4}}\sqrt{\frac{10}{1+\sqrt[3]{2}}}=\sqrt{\frac{20\sqrt[3]{2}}{1+\sqrt[3]{2}}}.$$

VI. Solution by J. W. WATSON, Middle Creek, Ohio; and H. C. WILKES, Skull Run, West Virginia.

Put $x=m^3$, $y=n^3$. Then, the given equations become, after factoring,

$m^3(m+n)=10$ ....(1), and $n^3(m+n)=20$.........(2). Whence $n=m\sqrt[3]{2}$.

Then in (1) $m^3(m+m\sqrt[3]{2})=10$, or $m^4(1+\sqrt[3]{2})=10$.

$$\therefore m^4=\frac{10}{1+\sqrt[3]{2}},\ \text{and } m^2=\pm\sqrt{\frac{10}{1+\sqrt[3]{2}}},\ =x.$$

Also, n^2, $=y$, $=\pm\sqrt{\dfrac{20\sqrt[3]{2}}{1+\sqrt[3]{2}}}$.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

56. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The locus of the centers of the isogonal transformations of all the diameters of the circumcircle of any triangle is the nine-points circle. *Brocard.*

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Let O and H be the circum and ortho-centers respectively of the triangle ABC. Draw the diameter DE, connect E and H, and from F the mid-point of EH draw FG parallel to OE.

Now H and O are inverse points.

G is the mid-point of HO and $GF=\frac{1}{2}OE=$a constant.

$\therefore$ G is the center and GF the radius of the nine-point circle.

$\therefore$ The locus of F is the nine-point circle.

II. Solution by the PROPOSER.

Let $l\alpha+m\beta+n\gamma=0$..........(1) be any diameter. The isogonal transformation of (1) is

$$\frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}=0\text{......................(2).}$$

Now (1), passing through the center of the circumcircle, the coordinates of which are proportional to $\cos A$, $\cos B$, $\cos C$, gives the relation

$$l\cos A+m\cos B+n\cos C=0\text{.....................(3).}$$

Also, the center of (2), which is an equilateral hyperbola, with condition (3), is given by

$$\frac{l}{n}=\frac{-a\alpha^2+b\alpha\beta+c\alpha\gamma}{b\beta\gamma-c\gamma^2+a\alpha\gamma},\quad \frac{m}{n}=\frac{a\alpha\beta-b\beta^2+c\beta\gamma}{b\beta\gamma-c\gamma^2+a\alpha\gamma}\text{.... ....(4).}$$

Dividing (3) by n, and substituting equations (4), and reducing,

$$a\beta\gamma+b\alpha\gamma+c\alpha\beta-a\alpha^2\cos A-b\beta^2\cos B-c\gamma^2\cos C=0\text{.... .....(5),}$$

the nine-points circle.

67. Proposed by J. OWEN MAHONEY, B. E., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tennessee.

Show that pairs of points, on a straight line, may be so related harmonically that a pair of real points will be harmonic with regard to a pair of imaginary points, and by this means prove that there are an indefinite number of conjugate pairs of imaginary points on a real line.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

If the four points be A, B, C, D, and the axis of x coincide with the given straight line, A, B may be supposed given by

$$\alpha x^2+2\beta x+\gamma=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\text{or } x=\frac{-\beta\pm\sqrt{\beta^2-\gamma^2}}{\alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and C, D, by $\alpha' x^2+2\beta' x+\gamma'=0$ ....(3).

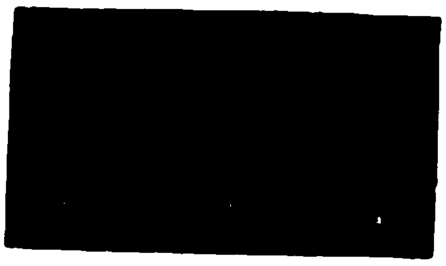
Now as long as γ exceeds β, (2) gives imaginary values for x, and so for a like pair of values for (3), which does not violate the condition

$$\alpha\gamma'+\alpha'\gamma=2\beta\beta' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

any number of values of β, γ in (2) always being consistent with (4).

II. Solution by JOHN B. FAUGHT, A. M., Instructor in Mathematics, Indiana University, Bloomington, Indiana.

Using trilinear coordinates, take B and C for the two real points on the real line $\alpha=0$, i. e., $b\beta+c\gamma=2\triangle$. $B^2+K^2\gamma^2=0$, is the equation of two lines through A; that is $\beta+Ki\gamma=0$, and $\beta-Ki\gamma=0$. These lines form with $\beta=0$ (AC) and $\gamma=0$ (AB) a harmonic pencil, and hence intersect BC in two points forming with B and C a harmonic range.



Moreover these lines are imaginary for all real values of K and hence must intersect BC in imaginary points, otherwise they would contain two real points, which is impossible.

The coordinates of the points of intersections of these imaginary lines may be found by solving with $b\beta+c\gamma=2\triangle$. Thus $\beta=-Ki\gamma$ gives

$$(c-bKi)\gamma=2\triangle \text{ and } \gamma=\frac{2\triangle c}{c^2+b^2K^2}+\frac{2\triangle Kb}{c^2+b^2K^2}i$$

$$\text{and } \beta=\frac{2\triangle K^2b}{c^2+b^2K^2}-\frac{2\triangle Kc}{c^2+b^2K^2}i, \text{ and } \beta=Ki\gamma, \text{ gives}$$

$$\gamma=\frac{2\triangle c}{c^2+b^2K^2}-\frac{2\triangle Kb}{c^2+b^2K^2}i, \text{ and } \beta=\frac{2\triangle K^2b}{c^2+b^2K^2}+\frac{2\triangle Kc}{c^2+b^2K^2}i.$$

If P and Q denote the imaginary points of intersection, we see that their coordinates are conjugates. These points are called "conjugués harmoniques" with respect to B and C, by M. Chasles.

It is evident that by giving different values to K an infinite number of such points can be found.

III. **Solution by the PROPOSER.**

The roots of $ax^2+2bx+c=0$ and $a'x^2+2b'x+c'=0$ will be harmonic if $ac'+a'c-2bb'=0$ (see Scott's Geometry, page 45).

Let $x^2=p^2$ give the points A and B. Let $x=OM=K<(OB=p)$ be midway between the other points, P and Q. The equation giving P and Q is

$$a'x^2+2b'x+c'=0, \text{ with the conditions } \frac{b'}{a}=-K, \text{ and } c'-p^2a'=0,$$

$$\text{or } x^2-2Kx+p^2=0.$$ [redacted]

But since $K<p$, $K^2-p^2<0$, the roots of this equation are imaginary, and since there are an indefinite number of values for $K<p$, there will be an indefinite number of pairs of imaginary points on the line harmonic with the given real pair. (Scott's Geometry, page 45.)

Solved in a similar manner by *G. B. M. ZERR*.

PROBLEMS.

63. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O. University, Mississippi.

Prove, analytically :—A rectangular hyperbola cannot be cut from a right circular cone unless the angle at its vertex is greater than a right angle.

64. Proposed by WILLIAM E. HEAL, Member of the London Mathematical Society and Treasurer of Grant County, Marion, Indiana.

Let the bisectors of the angles A, B, C of a triangle meet the sides opposite A, B, C in A', B', C'. Let AA', BB', CC' meet the sides of the triangle $A'B'C'$ in A'', B'', C''. Let this process continue indefinitely. Express the sides and angles of the triangle $A^{(m)}B^{(m)}C^{(m)}$ in terms of the sides and angles of the original triangle ABC.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

34. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

A particle is placed within a thin cylindrical shell without ends. Find the resultant attraction, the cylinder being composed of matter attracting according to the laws of nature.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Let $r=2a\sin\theta$ be the equation to the cylinder, so that the origin is in the surface of the cylinder at one end, then $y=r\sin\theta=2a\sin^2\theta$, $z=r\cos\theta=2a\sin\theta\cos\theta$, l=length of cylinder, (x, y, z) coordinates of any point in the shell, ρ=density, k=thickness of shell. It is always possible to take the axes of coordinates so that the particle will lie in the plane of the axis of y; let (m, n, o) be the coordinates of the particle, mass unity.

$$ds=2adxd\theta,\ p=\sqrt{(x-m)^2+(y-n)^2+z^2}=\sqrt{(x-m)^2+n^2-4a(n-a)\sin^2\theta},\ n>a.$$

This will give attraction for all possible positions of the particle. For $n<a$,

$$p=\sqrt{(x-m)^2+(2a-n)^2-4a(a-n)\sin^2(\tfrac{1}{2}\pi-\theta)},$$

and the solution would be the same as for $n>a$.

$$\text{Let } \frac{4a(n-a)}{m^2+n^2}=b^2,\quad \frac{4a(n-a)}{(l-m)^2+n^2}=c^2,\quad \frac{4a(n-a)}{n^2}=d.$$

Resolving the attractions parallel to the axes, we easily get

$$X=2a\rho k\int_0^{\pi}\int_0^{l}\frac{(x-m)d\theta dx}{\{(x-m)^2+n^2-4a(n-a)\sin^2\theta\}^{\frac{3}{2}}}$$

$$=2a\rho k\int_0^{\pi}\left\{\frac{1}{\sqrt{m^2+n^2-4a(n-a)\sin^2\theta}}-\frac{1}{\sqrt{(l-m)^2+n^2-4a(n-a)\sin^2\theta}}\right\}d\theta$$

$$=\frac{2a\rho k}{\sqrt{m^2+n^2}}E_0^{\pi}(b,\theta)-\frac{2a\rho k}{\sqrt{(l-m)^2+n^2}}E_0^{\pi}(c,\theta).$$

$$Y=2a\rho k\int_0^\pi\int_0^l\frac{(2a\sin^2\theta-n)d\theta dx}{\{(x-m)^2+n^2-4a(n-a)\sin^2\theta\}^{\frac{3}{2}}},$$

$$=2a\rho k\int_0^\pi\left\{\frac{l-m}{\sqrt{(l-m)^2+n^2-4a(n-a)\sin^2\theta}}\right.$$

$$\left.+\frac{m}{\sqrt{m^2+n^2-4a(n-a)\sin^2\theta}}\right\}\frac{(2a\sin^2\theta-n)d\theta}{n^2-4a(n-a)\sin^2\theta}$$

$$=\frac{a\rho kn(2a-n)}{n-a}\int_0^\pi\left\{\frac{l-m}{\sqrt{(l-m)^2+n^2-4a(n-a)\sin^2\theta}}\right.$$

$$\left.+\frac{m}{\sqrt{m^2+n^2-4a(n-a)\sin^2\theta}}\right\}\frac{d\theta}{n^2-4a(n-a)\sin^2\theta}.$$

$$-\frac{a\rho k}{n-a}\int_0^\pi\left\{\frac{l-m}{\sqrt{(l-m)^2+n^2-4a(n-a)\sin^2\theta}}+\frac{m}{\sqrt{m^2+n^2-4a(n-a)\sin^2\theta}}\right\}d\theta,$$

$$\therefore\ Y=\frac{a\rho kn(2a-n)}{n^2(n-a)}\left[\frac{l-m}{\sqrt{(l-m)^2+n^2}}\Pi_0^\pi(-d,c,\theta)\right.$$

$$\left.+\frac{m}{\sqrt{m^2+n^2}}\Pi_0^\pi(-d,b,\theta)\right]$$

$$-\frac{a\rho k(l-m)}{(n-a)\sqrt{(l-m)^2+n^2}}E_0^\pi(c,\theta)-\frac{a\rho km}{(n-a)\sqrt{m^2+n^2}}E_0^\pi(b,\theta).$$

$$Z=2a\rho k\int_0^l\int_0^\pi\frac{2a\sin\theta\cos\theta dx d\theta}{\{(x-m)^2+n^2-4a(n-a)\sin^2\theta\}^{\frac{3}{2}}}=0.$$

F=resultant attraction $=\sqrt{X^2+Y^2+Z^2}$.

When $n=a$, the particle is on the axis of the cylinder, then

$$F=X=2\pi a\rho k\left\{\frac{1}{\sqrt{m^2+a^2}}-\frac{1}{\sqrt{(l-m)^2+a^2}}\right\}.$$

When $m=\frac{1}{2}l$, the particle is at the center of the cylinder, and $F=0$.

When $m=l$, $n=a$, $F=2\pi a\rho k\left\{\frac{1}{\sqrt{l^2+a^2}}-\frac{1}{a}\right\}$.

When $m=0$, $n=a$, $F=-2\pi a\rho k\left\{\frac{1}{\sqrt{l^2+a^2}}-\frac{1}{a}\right\}$.

When $n=2a$ the particle is on the surface of the cylinder,

$$\text{then } b^2=\frac{4a^2}{m^2+4a^2},\quad c^2=\frac{4a^2}{(l-m)^2+4a^2},\quad d=1.$$

$\therefore$ The elliptic function of the third order in Y disappears.

PROBLEMS.

42. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Find the time of vibration of a particle *slightly* displaced from the center of a solid cylinder in direction of the axis, the matter of the cylinder attracting according to the laws of nature.

43. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Two weights P and Q rest on the concave side of a parabola whose axis is horizontal, and are connected by a string, length l, which passes over a smooth peg at the focus, F. [*Bowser's Analytic Mechanics, page 54.*]

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

42. Proposed by W. B. ESCOTT, 6123 Ellis Avenue, Chicago, Illinois.

In a parallelogram, sides a and b, diagonals c and d, $2a^2+2b^2=c^2+d^2$. Find all the parallelograms, not rectangles, whose sides and diagonals are rational.

Examples:	a	b	c	d
	4	7	9	7
	16	7	21	13
	8	9	13	11
	8	11	17	9

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

By means of the sides and diagonals we can form, in each parallelogram, two different triangles, the sides of one being a, b, and c, and of the other, a, b, and d.

Take the triangle, sides a, b, and c and put $a=n$, $b=n+p$, and $c=2n\pm q$. From the relations of the sum and the difference of any two sides to the third side, we have the following conditions: $p\mp q>0$ and $p\mp q<2n$. For $p-q$, $c=2n+q$; and for $p+q$, $c=2n-q$.

The *median* upon c is $1/2\sqrt{2(a^2+b^2)-c^2}$. But as the diagonals of a parallelogram bisect each other, this median equals $\frac{1}{2}d$. Whence $d^2=2(a^2+b^2)-c^2=4n(p\mp q)+2p^2-q^2$.

Then $n=\dfrac{d^2-2p^2+q}{4(p\mp q)}$. But we have found that $2n>p\mp q$. Therefore

$\dfrac{d^2-2p^2+q^2}{2(p\mp q)}>p\mp q$. Whence $d>2p\mp q$.

Put $d=2p\mp q+t$. Then $a=n=\dfrac{(2p\mp q+t)^2-2p^2+q^2}{4(p\mp q)}$;

$$b=n+p=\frac{(2p\mp q+t)^2+(2p\mp q)^2-2p^2}{4(p\mp q)};$$

$$\text{and } c=2n\pm q=\frac{(2p\mp q+t)^2-(p\mp q)^2-p^2}{2(p\mp q)},$$

in which p, q, and t are any integers. p and q may also be *zero*, but only one of them in the same operation. When $p=q$ and when $q>p$, we use q only as *positive*, $[+q]$; but when $p>q$, we can use q as both *positive* and *negative*.

When numerical values, assigned to p, q, and t, render a and b or a, b, and c fractional, integral results are obtained by multiplying a, b, c, and d by the least common denominator of the fractions.

Examples:—(1). Put $p=2$, $q=1$, and $t=2$. Then, for $p+q$, $a=7/2$, $b=11/2$, $c=6$, and $d=7$; or in integers, 7, 11, 12, and 14.

(2). Put $p=3$, $q=1$, and $t=2$. Then $a=4$, $b=7$, $c=7$, and $d=9$. Also $a=4$, $b=7$, $c=9$, and $d=7$.

For $p-q$, $a=9/2$, $b=13/2$, $c=10$, $d=5$; or in integers, 9, 13, 20 and 10.

When $q=0$, or when $c=2a$, we have $a=[(2p+t)^2-2p^2]/4p$, $b=[(2p+t)^2+2p^2]/4p$, $c=[(2p+t)^2-2p^2]/2p$, and $d=2p+t$.

Examples:—(1). Put $p=1$, and $t=2$. Then $a=7/2$, $b=9/2$, $c=7$, and $d=4$; or in integers, 7, 9, 14, and 8.

(2). Put $p=t=2$. Then $a=7/2$, $b=11/2$, $c=7$, and $d=6$; or in integers, 7, 11, 14, and 12.

When $p=0$, or when $a=b$, we have $a=[(t\mp q)^2+q^2]/4q=b$,

$c=[(t\mp q)^2-q^2]/2q$, and $d=t\mp q$; or, in integral form, $a=b=(t\mp q)^2+q^2$, $c=2t(t\mp 2q)$, and $d=4q(t\mp q)$.

Examples :—(1). Put $t=q=1$. Then $a=b=5$, $c=6$, and $d=8$.

(2). Put $t=3$ and $q=1$. Then $a=b=17$, $c=30$, and $d=16$. Also $a=b=5$, $c=6$, and $d=8$.

When $q=p$, we have $a=[(3p+t)^2-p^2]/8p$, $b=[(3p+t)^2+7p^2]/8p$, $c=[(3p+t)^2-5p^2]/4p$, and $d=3p+t$.

When $t=q=p$, we have, in integral form, $a=15p$, $b=23p$, $c=22p$, and $d=32p$.

Thus we continue making general values for a, b, c, and d, under a number of other conditions; as, $t=q$; $t=p$; $t=2q=2p$, etc.

43. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the series of integral numbers in which the sum of any two consecutive terms is the square of their difference.

I. Solution by J. H. DRUMMOND, LL. D., Portland, Maine, and the PROPOSER.

Let x and $x+m$ be two consecutive numbers. Then we have $2x+m=m^2$, and $x=m(m-1)/2$, and $x+m=m(m+1)/2$. But $m(m+1)/2$ is the sum of the terms in the series $1+2+3+4$ .... $+m$. Hence the m^{th} term of the series required is the sum of m terms of this series, and we have 1, 3, 6, 10, 15, ... $m(m-1)/2$.

II. Solution by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee; O. W. ANTHONY, M. Sc., Professor of Mathematics, New Windsor College, New Windsor, Maryland; and BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio.

By the conditions we must have $x+y=(x-y)^2$, x and y representing two consecutive terms in the series. Solving as a quadratic in x, we have $x=(2y+1)/2\pm\sqrt{(8y+1)}/4$. Hence $8y+1$ must be a square.

When $y=1$, $8y+1=3^2$, $x=3$;

$y=3$, $8y+1=5^2$, $x=6$;

$y=6$, $8y+1=7^2$, $x=10$;

and the series is, 1, 3, 6, 10, 15, 21, 28, 36, 45, etc., or the system of *triangular* numbers as set forth in Pascal's Triangle.

Also solved by *A. H. HOLMES*, *E. W. MORRELL*, *H. C. WILKES*, and *G. B. M. ZERR*.

44. Proposed by A. H. HOLMES, Box 963, Brunswick, Maine.

The hypotenuse of a right-angled triangle ABC, right-angled at A, is extended equally at both extremities so that $BE=CD$. Draw AD and AE. Find integral values for all the lines in the figure thus made.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Construct the figure as indicated by the problem. Then draw BF equal and parallel to AC, and draw CF, AF, EF, and DF. Then will $ABFC$ be a rectangle; and the diagonals BC and AF are equal.

It is also evident that $AE=DF$ and $AD=EF$. Whence $AEFD$ is an oblique-angled parallelogram, or rhomboid, of which AE and AD are the sides, and ED and AF the diagonals.

Let $x=BE=CD$, and put $a=AD$, $b=AE$, $c=ED$, and $d=AF=BC$, taking $c>d$. Then $2x+d=c$, and $x=(c-d)/2$. If $d>c$, AE and AD fall inside of AB and AC, and the hypotenuse BC would be *contracted* instead of *extended*.

We now find integral values for a, b, c, and d. This has been done in the solution of No. 42, in this issue, and need not be reproduced here.

By this process we find integral values for all the lines except the two legs, AB and AC, of the right-angled triangle. By means of the median and the perpendicular upon BC, we readily find

$$\overline{AB}^2=d[4b^2-(c-d)^2]/4c \text{ and } \overline{AC}^2=d[4a^2-(c-d)^2]/4c.$$

Now, if these expressions can be rendered squares, without destroying the relations of a, b, c, and d, AB and AC will also be rational and integral. But I have not yet succeeded in accomplishing this. We shall now illustrate by means of a few examples.

From Diophantine problem No. 42, take the set of values, $a=4$, $b=7$, $c=9$, and $d=7$. Then $2x+7=9$; whence $x=1$. $\therefore$ $AD=4$, $AE=7$, $ED=9$, $BC=AF=7$, $BE=DC=1$, $\overline{AB}^2=112/3$, and $\overline{AC}^2=35/3$.

Take the set of values, $a=8$, $b=11$, $c=17$, and $d=9$. Then $2x+9=17$; and $x=4$. Also $\overline{AB}^2=945/17$, and $\overline{AC}^2=432/17$.

Partial solutions also received from *J. H. DRUMMOND*, *A. H. BELL*, and the *PROPOSER*.

PROBLEMS.

51. Proposed by H. C. WILKES, Skull Run, West Virginia.

The difference between the roots of two successive triangular square numbers equals the sum of two successive integral numbers, the sum of whose squares will be a square number. Demonstrate.

52. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Prove that a "magic square" of nine integral elements, whose rows, columns, and diagonals have a constant sum, is only possible when this sum is a multiple of three.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

REPLY TO THE REPLIES TO MY "NOTE ON AVERAGE AND PROBABILITY."

BY ARTEMAS MARTIN, LL. D., U. S. COAST AND GEODETIC SURVEY OFFICE, WASHINGTON, D. C.

I WISH to say first that I reaffirm all that I stated on pages 370 and 371, Vol. II., No. 12, and then peoceed to consider the replies of the Repliers.

I. Professor Zerr starts out with the statement that "The problem that gives the result $\frac{1}{4}a^2$ is different from the problem that gives the result $\frac{a^2}{2\pi}$." This is superfluous information; I had clearly set forth that fact in my "Note." But the truth of the next sentence, "In the former the right angle remains fixed and does not lie on a circle as Dr. Martin states," I do not admit, and will proceed to prove its falsity.

Let $AB=a$, the given hypotenuse, which shall remain *fixed*.

Draw A_1B, A_2B, A_3B, A_4B, and so on, the sides AA_1, AA_2, AA_3, AA_4, etc., increasing uniformly from 0 towards a, the consecutive differences, AA_2-AA_1, AA_3-AA_2, AA_4-AA_3, etc., being all equal to each other, and each difference less than any assignable quantity. Thus will be had *all possible* right-angled triangles having the hypotenuse a, and, as I stated on page 371, the right angles *are* all situated on the semi-circumference whose diameter is the given hypotenuse a; but they (the right angles) are not uniformly distributed on this semicircumference because the chords AA_1, AA_2, AA_3, AA_4, etc., increase (or vary) uniformly and therefore their arcs can not increase (or vary) uniformly.

Professor Zerr continues: "The problem [the one that gives the result $\frac{1}{4}a^2$] is as follows: 'Find the average area of all triangles formed by a straight line of constant length a sliding so that its extremities constantly touch two fixed straight lines at right angles to one another'." With all due deference to Professor Zerr, I beg leave to say that I have *not* conceived the triangles to be generated in any such way, as I have clearly shown by the diagram above.

The remainder of Professor Zerr's "Note" does not require considering as it has nothing to do with the matter in hand.

II. I discard the "tail" in italics Professor Matz has appended to the problem; it is not needed to "fly the kite."

I will take up his third and fourth paragraphs. In his third paragraph he says that I, by making the number of possible right-angled triangles "proportional to the given hypotenuse," *ignore* an infinitude of right-angled triangles. Now if Professor Matz can *prove* that there *are any* right-angled triangles having the hypotenuse a besides those obtained by varying one leg uniformly from 0 to a, I—would like to see the proof. How *can* there *be* any other triangles, if we have a leg for *every possible* value from 0 to a?

III. I will pass over the first and second paragraphs of the Editor's "Reply." In regard to the third paragraph I deny that any triangles *can* be interpolated, and demand proof. If one leg takes *all possible* values from 0 to a, every triangle has been included and there *can not* be any other.

IV. My solution, which I desire to reproduce here, is as follows:

Let x denote one leg of any one of the triangles, then $\sqrt{(a^2-x^2)}$ will denote the other leg. The area of this triangle is $\frac{1}{2}x\sqrt{(a^2-x^2)}$, and the true average of this is

$$\int_0^a \tfrac{1}{2}xdx\sqrt{(a^2-x^2)} \div \int_0^a dx, =\tfrac{1}{6}a^2.$$

V. I think I have considered and fully refuted every objection that has been raised against my solution.

Correction.—Vol. II., page 371, for "p. 82" read p. 282.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

35. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, California; P. O., Sebastopol, California.

To an observer whose latitude is 40 degrees north, what is the sidereal time when Fomalhout and Antares have the same altitude; taking the Right Ascension and Declination of the former to be 22 hours, 52 minutes, —30 degrees, 12 minutes; of the latter 16 hours, 23 minutes, —26 degrees, 12 minutes?

II. Corrected solution by JOHN M. ARNOLD, Crompton, Rhode Island; and Prof. G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let λ=latitude of observer, α, δ, α_1, δ_1 the Right Ascension and Declination of Fomalhout and Antares, respectively, β=altitude, h, h_1 the hour angles

This event can happen only when Antares is west and Fomalhout east of the meridian.

$$\therefore \left.\begin{aligned}\sin\beta&=\sin\lambda\sin\delta+\cos\lambda\cos\lambda\cos h\\&=\sin\lambda\sin\delta_1+\cos\lambda\cos\delta_1\cos h_1\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\alpha-h=\alpha_1+h_1, \text{ or } h+h_1=\alpha-\alpha_1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

But $\lambda=40°$, $\alpha=343°$, $\alpha_1=245° \ 45'$. $\delta=-30° \ 12'$, $\delta_1=-26° \ 12'$.

$\therefore$ $662065\cos h-687337\cos h_1=39538 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$.

$$\cos(h+h_1)=\cos 97° \ 15'=-.12620 \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Let $\cos h=x$, $\cos h_1=y$. From (4) $y=-.12620x\pm.992005\sqrt{1-x^2}$. This
3) gives, $748806.9294x\mp681841.7407\sqrt{1-x^2}=39538$.

$\therefore$ $x^2-.057736x=.461771$. $\therefore$ $x=.701626$ or $-.643890$.

$\therefore$ $h=45° \ 26' \ 31''$ or $130° \ 4' \ 57''$. The first value of h gives h_1 positive.

$\therefore$ $h=3$ hours, 1 minute, 46 seconds.

$\therefore$ sidereal time$=\alpha-h=19$ hours, 50 minutes, 14 seconds.

37. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A gentleman owned and lived in the center, R, of a rectangular tract of land whose onal, D, was 350 rods, dividing the tract into two equal right-angled triangles, in each hich is inscribed the largest square field, F and F_1, possible; the north and south dary lines of the two square fields being extended and joined formed a little rectan-r lot, R, in the center around the residence. The difference in the area of the *entire ngular* tract and the *sum* of the areas of the two square fields, F, F_1, is 187½ acres. the dimensions and area of the entire tract, and one of the square fields, F or F_1.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana p, Texarkana, Arkansas-Texas.

Let $AB=a$, $AD=b$, $AH=x$. $\therefore$ $a^2+b^2=122500 \ldots\ldots\ldots\ldots\ldots\ldots(1)$.

$ab-2x^2=187\frac{1}{2}$ acres$=30000$ square rods...(2).

$ax+bx=ab \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$,

n triangles BAD and BEK.

From (3) $x^2(a^2+2ab+b^2)=a^2b^2 \ldots\ldots\ldots(4)$.

(1) and (2) in (4) gives $62500x^2=900000000$.

$\therefore$ $x^2=14400$ square rods$=90$ acres.

$\therefore$ $x=120$ rods.

$\therefore$ $ab=58800$ square rods$=367$ acres.

$\therefore$ $a+b=490$ rods. $a-b=70$ rods. $\therefore$ $a=280$, $b=210$.

II. Solution by ISAAC L. BEVERAGE, Monterey, Virginia.

If $a=AB$ and $b=AD$, then $ab=$area of entire farm. Now $ab/(a+b)=$, since it is the side of an inscribed square of a triangle.

$\therefore$ $[ab/(a+b)]^2=$the area of F or F_1. Hence, we readily obtain,

$$ab-2[ab/(a+b)]^2=187\tfrac{1}{2}\times160 \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\text{and } \sqrt{a^2+b^2}=350 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Whence $a=280$ rods, and $b=210$ rods; also $ab=58800$ square rods$=367\frac{1}{2}$ acres. $\therefore ab/(a+b)=120$ rods, and $[ab/(a+b)]^2=14400$ square rods$=90$ acres.

III. Solution by A. H. BELL, Box 184, Hillsboro, Illinois.

Let 350 rods$=87\frac{1}{2}$ chains$=2a$, $DK=a-y$ and $BK=a+y$. Also, $187\frac{1}{2}$ acres$=1875$ square chains$=b^2$, and side of square$=x$; then $DG=EB=\sqrt{(a+y)^2-x^2}$, $DH=BF=\sqrt{(a-y)^2-x^2}$,

$$(\sqrt{(a+y)^2-x^2}+x)^2+(\sqrt{(a-y)^2-x^2}+x)^2=4a^2 \ldots\ldots\ldots\ldots(1).$$

Plainly, $$x\sqrt{(a+y)^2-x^2}+x\sqrt{(a-y)^2-x^2}=b^2 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

$(2)\times 2$, and subtracted from (1), when expanded, $y^2=a^2-b^2 \ldots\ldots\ldots(3).$

$a+y : a-y :: \sqrt{(a+y)^2-x^2} : x. \quad \therefore x^2=(a^2-y^2)^2/2(a^2+y^2) \ldots\ldots(4).$

Substituting values, $y=6.25$ chains, $x^2=90$ acres; $x=30$ chains, $EB=DG=40$ chains, $BF=DH=22.5$ chains, $AB=DC=70$ chains, $AD=BC=52\frac{1}{2}$ chains, $DC\times AD=367\frac{1}{2}$ acres, in the rectangle.

Also solved by *P. S. BERG, A. H. HOLMES*, and *B. F. YANNEY*.

PROBLEMS.

45. Proposed by EDWARD R. ROBBINS, Master in Mathematics and Physics, Lawrenceville School Lawrenceville, New Jersey.

Required several numbers each of which, divided by 10 leaves a remainder 9; by 9 leaves 8; by 8 leaves 7; by 7 leaves 6, and so on. Also find the least such number which, when divided by 28 leaves 27; by 27 leaves 26; by 26 leaves 25; by 25 leaves 24, *et cetera ad unum*.

46. Proposed by A. H. HOLMES, Box 963, Brunswick, Maine.

The base BC of the triangle ABC is $2c$, the sum of the two sides, AB and BC, is $2a$. BP is always perpendicular to AB and cuts AC in P. What is the locus of the point P?

47. Proposed by S. HART WRIGHT, A. M., Ph. D., Penn Yan, New York.

In longitude 75 degrees west of Greenwich, latitude 43 degrees, 30 minutes north on January 1, 1895, at 8 o'clock A. M., local time. What points of the ecliptic were then rising, setting and on the meridian? Any other necessary data may be taken from an ephemeris.

48. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

In case of *mischance*, with what force would the cow, weighing $w=700$ pounds, jumping over the moon, have struck Her Lunar Majesty in the face?

EDITORIALS.

Our valued contributor, Sylvester Robbins, who is visiting in Southern ιo, made Prof. William Hoover a pleasant call a few days ago.

Professors W. W. Beman and D. E. Smith are preparing a translation of ɔin's Vorträge über ausgewählte Fragen der Elementargeometrie. It will be ıed during the winter by Ginn & Co.

We are grieved to record the death of our valued contributor and subscriber, ɔf. H. A. Newton, of Yale University, whose death occured August 12th. In ıture number of the MONTHLY will appear a biographical sketch of his life, by colleague, Prof. A. W. Phillips.

The friends of Drury College will be pained to learn of the death a member of its Faculty, Prof. William J. Whitney, of the Department History, whose death, caused by typhoid fever, occurred on September 26th, the home of his father, near Findley's Lake, New York. His broad scholar-p, his accurate judgment, and the fine qualities of his character made him a at favorite among the Faculty and students of the College. Professor ıitney was a most intimate and helpful friend of Editor Finkel, and in his ıth we sustain a great loss.

BOOKS.

Elements of Plane and Spherical Trigonometry, A Text-book for Colleges l Schools. By Edwin S. Crawley, Ph. D., Assistant Professor of Mathematics he University of Pennsylvania. Second edition, revised and enlarged. 8vo. th, 178 pages. Price, $1.00. Published by the Author, Philadelphia, Penn.

This book contains all that is needed on the subject of Trigonometry in our best col-s. The author has omitted nothing that is necessary in studying the branches of hematics following Trigonometry. Such important subjects as De Moivre's Theorem, ɹerbolic Functions, Theorems relating to the escribed circles and Brocard's points are ɔisely treated: The book is very beautifully printed, and substantially bound in cloth. do not hesitate to recommend this book to teachers and students desiring a good text he subject treated. B. F. F.

Higher Mathematics. A Text-book for Classical and Engineering Colleges. ted by Mansfield Merriman, Professor of Civil Engineering in Lehigh Uni-ıity, and Robert S. Woodward, Professor of Mechanics in Columbia Univer-. Large 8vo., 576 pages. Price, $5.00. New York: John Wiley & Sons.

This volume is designed especially for the use of Junior and Senior Classes in Col-s and Technical Schools, but it is equally well adapted to the use of advanced students and readers of Mathematics generally. The editors have called to their assistance the best mathematicians in the country, and thus given the book weight of authority never before given an American Mathematical Text-book. The book contains a concise treatment of the following subjects, not commonly found in text-books but upon which lectures are now given in our best classical and technical institutions:

Chapter I. The Solution of Equations, by Mansfield Merriman, Professor of Civil Engineering in Lehigh University; Chapter II. Determinants, by Laenas Gifford Weld, Professor of Mathematics in State University of Iowa; Chapter III. Projective Geometry, by George Bruce Halsted, Professor of Mathematics in the University of Texas; Chapter IV. Hyperbolic Functions, by James McMahon, Associate Professor of Mathematics in Cornell University; Chapter V. Harmonic Functions, by Professor William E. Byerly, Professor of Mathematics in Harvard University; Chapter VI. Functions of a Complex Variable, by Thomas S. Fiske, Adjunct Professor of Mathematics in Columbia University; Chapter VII. Differential Equations, by W. Woolsey Johnson, Professor of Mathematics in the U. S. Naval Academy; Chapter VIII. Grassmann's Space Analysis, by Edward W. Hyde, Professor of Mathematics in the University of Cincinnati; Chapter IX. Vector Analysis and Quaternions, by Alexander Macfarlane, Lecturer in Civil Engineering in Lehigh University; Chapter X. Probabilities and Theory of Errors, by Robert S. Woodward, Professor of Mechanics in Columbia University; Chapter XI. History of Modern Mathematics, by David Eugene Smith, Professor of Mathematics in Michigan State Normal School.

It is to be hoped that all classical colleges and other institutions of learning that have no provision for mathematical study in the Junior and Senior years will so arrange the course of study that the Higher Mathematics as here presented may be pursued during the last two years of college work, so that the student, during these years, may not be deprived of the rigid discipline of mind and the culture derived from its study. B. F. F.

Elementary Algebra. By H. S. Hall, M. A., and S. R. Knight, B. A. Revised and Enlarged for the use of American Schools by F. L. Sevenoak, A. M., Assistant Principal of the Academic Department, Stevens Institute of Technology. 8vo. Cloth and Leather Back. 416 pages. Price, $1.10. New York: Macmillan & Co.

Only words of commendation can be said of this book. The complete and accurate treatment of each subject, the abundant illustrations, the scientific arrangement of the subjects, go to make up all that could be desired in a good text-book. This book together with the author's Higher Algebra, makes a very exhaustive course in Algebra. B. F. F.

Euclidian Geometry. By J. A. Gillet, Professor in New York Normal College. 8vo. Cloth and Leather Back. 436 pages. New York: Henry Holt & Co.

This book, as its name implies, reverts to the purely geometric methods of Euclid. The author maintains sharply throughout the work, the distinction between the processes of pure geometry on the one hand and those of arithmetic and algebra on the other. The author says, "Euclidian Geometry bears to modern geometry very much the same relation that arithmetic bears to algebra. Its theorems are less general and it admits of positive magnitude only. For this reason its simple and rigorously logical methods can never be replaced by those of synthetic geometry, either as a factor in general education or as a foundation for advanced study." We can not agree with the Author in his last statement. It has been our experience in teaching geometry that the boy or girl, who studies geometry for the mental discipline it gives him and not merely for grades, feels better satisfied when he has demonstrated a proposition in its entirety, than he does when he has demonstrated one which he feels must be enlarged, as he advances in the study of Mathematics, to satisfy all cases. However, there is much in the book to commend it favorably to teachers. B. F. F.

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MATHEMATICAL INFINITY AND THE DIFFERENTIAL.

By FRANKLIN A. BECHER, Milwaukee, Wisconsin.

Mathematics, as defined by the great mathematician, Benjamin Pierce, is the science which draws necessary conclusions. In its broadest sense, it deals with conceptions from which necessary conclusions are drawn. A mathematical conception is any conception which, by means of a finite number of specified elements, is precisely and completely defined and determined. To denote the dependence of a mathematical conception on its elements, the word "manifoldness," introduced by Riemann, has been recently adopted. Manifoldness may be looked upon as the genus, and function, as the species. This conception reaches down to the very foundation of mathematical concepts and principles. It is the central idea from which the whole field and range of the mathematical sciences may be surveyed. Time, space, and numbers are included in the notion, manifoldness.

Manifoldness may be defined according to Dr. Cantor as being in general every *muchness* or complexity which may be conceived as a unit, or a number of objects, conceptions, or elements which are united in one law or system.

Manifoldness may be devided into discrete and continuous. Proceeding with the conception of whole numbers as it is obtained by counting and extending the same by means of the divisibility of numbers so as to include the conception of the rational system of numbers, we have one of the elements which enter into the conception of a discrete manifoldness. The irrational system of numbers is included in the conception of continuous manifoldness. This must not be considered as an inherent division, for it is well to note here that in the higher analysis, in one instance and for one purpose, a conception may be considered as a discrete manifoldness and for another purpose as a continuous manifoldness.

The three laws of operation, i. e., the law of commutation, of association, and of distribution, hold good in all forms of calculation, whether discrets or continuous manifoldness. From these laws, the four processes, addition, subtraction, multiplication, and division are derived.

Number, in all its forms, whether finite or infinite, rational or irrational, constant or variable, continuous or discontinuous, is included as one of the elements of manifoldness.

We will now consider number with special reference to its limits, infinity and zero, by the introduction of the conception of variability, of continuity, and of the differential.

By means of an unlimited continuous series of rational numbers, $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots\ldots \alpha_n$, whose terms have the property that there be given to every number δ, however small, a place n, from which the difference of all succeeding numbers remains smaller than δ, we define a definite number which is called a limit of this series. The creation of this conception admits of a comparison of rational numbers with respect to their magnitude. If all the numbers of the series differ after the place α_n by less than the number δ, then the limit is a number which lies between $\alpha_n - \delta$ and $\alpha_n + \delta$, which, because δ may be chosen as small as we please, can be expressed by a rational number as near as we please.

The totality of all numbers of an interval, for example, 0 to 1, consists not only of all numbers between 0 and 1, but of the totality of all numbers which may be interpolated between the limiting values of the defined series of numbers. This totality we dominate the aggregate or inclusive of the continuous series of numbers.

It is apparent that the conception of a limit of variability and of continuity have their root in irrationality. The two conceptions attached to a limit are in their nature entirely different. In the first instance, a limit may be defined as a limit of a variable, a limitless increasing or decreasing; in the second instance a limit means that which exceeds all limits of measurable number, either because it possesses no magnitude or because the amount or extent would not be exhausted by means of all the series of all numbers though they were being perfected. In the first case, we deal with variable numbers ; in the second case with the conception of the absolute value of the numbers derived from the formation of zero and the conception of infinity.

Zero and infinity are the limits of the natural series of numbers. They are derived in the same manner as the rational series of numbers. Infinity is the result of unlimited addition of unity or other positive numbers, the unlimited multiplication of whole numbers except unity. Zero is derived from the subtraction of two equal numbers. These are the fundamental conceptions of zero and infinity as derived in the lower analysis. It is evident from the different ways in which each of these symbols are derived that they have different meanings attached to them. We may note here that every problem carries inherently with it its solution. The meaning of every symbol depends upon its origin, derivion and relation. In different problems they may have different meanings. ymbols of quantity, like words, have different definitions, and these are to be etermined according to the nature of the problem and their relation to other ymbols.

In the higher analysis, the conceptions of infinity and zero present themelves more systematically in the developement of infinite series, infinite products, infinite continued fractions, etc. An infinite number is defined as a variale number, whose absolute value is conceived as being in an unlimited state of icreasing or decreasing. In the first instance it is called infinitely large, in the econd, infinitely small. The addition of a number of infinitely large or infinitey small numbers will produce an infinitely large or small number. The differnce between two infinitely large or infinitely small numbers, where either or oth are equal, is zero. However, if they are not equal, the difference can never e a finite number, but must always be an infinite number; otherwise an infinite umber would be increased or decreased by a finite number, which is without ieaning.

The addition and subtraction of infinite numbers can never produce anyhing else than infinite numbers or, in a particular case, zero. Again the multilication or division of an infinite number by a finite number or by infinity will roduce like results, i. e., it may be merely an indicated operation, not a comleted operation. For instance,

$$2\times\infty=2\,\infty\,;\ n\times\infty=n\,\infty\,;\ \infty\times\infty=\infty^2,\ \&c.$$

t is apparent that the unlimited number of changes which may be thought of nder the conception of infinity as defined here are extraordinarily manifold.

If we conceive an infinite number to grow so that it is continuously twice s large as any other infinite number, then the first is derived from the second y multiplying by two or the second by dividing by two. Multiplying an infiite number by another gives us infinity of a higher power or dividing gives us nfinity of a lower power. Every change in value of a variable suggests an inrement.

There are two kinds of conceptions associated with increments: the one is hat the absolute value of the increment is capable of divisibility. The condiions, however, of which are such that it cannot be conceived smaller. The othr is that the absolute value is incapable of divisibility. In the first instance he increments are of such a nature that the variables must stand in a certain reation to one another and if this takes place they are known in higher mathematcs as differentials; those of the second kind are of that nature that they do not tand in any relation to one another; these may be called absolute elements of uantity.

Thus, if we pass from one interval of value of a variable to another, there ies between the two a difference which must be considered as possessing quanity, but does not possess the capability of divisibility and this difference in it crements is an element of quantity. Increments and differentials are not identical. The former are vested with quantity, while the latter are vested with quality, i. e., they are formal in their nature.

Milwaukee, Wisconsin, September, 1896.

A PROBLEM IN ASTRONOMY.

By G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

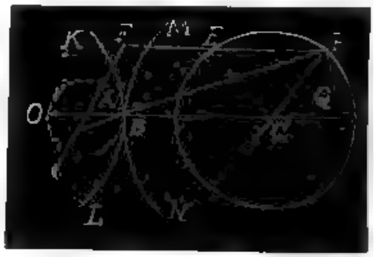
To find the Distance from the Earth to the Sun knowing the distance from an Inferior planet to the Sun supposing the planets to describe circles around the Sun.

Let P be a point on the epicyclic curve PQ, OC the radius of the deferent, CP the radius of the epicycle. Let $CO:CB=n:1$. $\therefore CB=\frac{CO}{n}$.

Then $BO=CO-CB=CO-\frac{CO}{n}=CO(1-\frac{1}{n})$.

Now the angular velocity$=\frac{\text{transverse velocity}}{\text{radius vector}}$.

The transverse velocity of E is represented by EA in magnitude, and in direction by BA. Let the linear velocity of the mean point (C) be V, the linear velocity of the moving point in the epicycle is



$$nV.\frac{PC}{CO}.$$

$\therefore$ The transverse velocity$=$

$$\frac{nV}{CO}.EB\cos BEO;$$

$$\text{but } \cos BEO=\frac{(BE)^2+(EO)^2-(BO)^2}{2BE.EO}.$$

$\therefore$ Transverse velocity $=\frac{nV}{CO}.\frac{BE^2+EO^2-BO^2}{2EO}$. Also radius vector$=EO$.

$$\therefore \text{Angular velocity}=\frac{nV}{CO}.\frac{BE^2+EO^2-BO^2}{2EO^2}\ \ldots\ldots\ldots\ldots (A).$$

Let the deferential angle $=\theta$, then angle $ECD=(n-1)\theta$.

$\therefore\ BE^2=CE^2+CB^2+2CE.CB\cos(n-1)\theta$

$$=CE^2+\frac{CO^2}{n^2}+2CE.\frac{CO}{n}\cos(n-1)\theta$$

$$EO^2=CO^2+CE^2+2CO.CE\cos(n-1)\theta$$

$$BO^2=CO^2\left(1-\frac{1}{n}\right)^2.$$

Substituting these values in (A) we get

$$\text{Angular velocity}=\frac{V}{CO}\cdot\frac{CO^2+nCE^2+(n+1)CO.CE\cos(n-1)\theta}{CO^2+CE^2+2CO.CE\cos(n-1)\theta}.$$

Now in inferior conjunction, if the moving planet is inferior, $(n-1)\theta=180^\circ$.

$$\therefore\ \text{Angular velocity}=\frac{V}{CO}\cdot\frac{CO^2+nCE^2-(n+1)CO.CE}{CO^2+CE^2-2CO.CE}.$$

Let $CO=R$, $CE=r$.

$$\text{Then Angular velocity}=\frac{V}{R}\cdot\frac{R^2+nr^2-(n+1)R.r}{(R-r)^2}.$$

Now $n=\left(\frac{R}{r}\right)^{\frac{3}{2}}$, also putting $\frac{V}{R}=\omega$.

$$\therefore\ \text{Angular velocity}=\omega.\frac{R^2+\left(\frac{R}{r}\right)^{\frac{3}{2}}r^2-\left\{\left(\frac{R}{r}\right)^{\frac{3}{2}}+1\right\}R.r}{(R-r)^2}$$

$$=\omega\frac{1+\left(\frac{R}{r}\right)^{\frac{3}{2}}\frac{r^2}{R^2}-\frac{R^{\frac{3}{2}}+r^{\frac{3}{2}}}{r^{\frac{3}{2}}}\cdot\frac{r}{R}}{\left(1-\frac{r}{R}\right)^2}\ \ldots\ldots\ldots\ldots\ldots(B)$$

$$=\omega\frac{\frac{R^2}{r^2}+\left(\frac{R}{r}\right)^{\frac{3}{2}}-\left\{\left(\frac{R}{r}\right)^{\frac{3}{2}}+1\right\}\frac{R}{r}}{\left(\frac{R}{r}-1\right)^2}\ \ldots\ldots\ldots\ldots(C).$$

Let the distance from the earth to the sun be known to find the distance from the planet to the sun.

Let $\frac{r}{R}=\rho$, then (B) becomes

$$\text{Angular velocity}=\omega.\frac{1+\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}}-\rho}{(1-\rho)^2}=\omega.\frac{(1-\rho)-\rho^{-\frac{1}{2}}(1-\rho)}{(1-\rho)^2}$$

$$=\omega.\frac{1-\rho^{-\frac{1}{2}}}{1-\rho}=-\frac{\omega}{\sqrt{\rho}}.\frac{1-\rho^{\frac{1}{2}}}{1-\rho}=-\frac{\omega}{\sqrt{\rho}+\rho}\ \ldots\ldots\ldots\ldots(D).$$

Let the distance from the planet to the sun be known to find the distance from the earth to the sun.

Let $\frac{R}{r}=\rho'$, then (C) becomes

$$\text{Angular velocity}=\omega.\frac{\rho'^2+\rho'^{\frac{1}{2}}-\rho'^{\frac{3}{2}}-\rho'}{(\rho'-1)^2}=\omega\frac{\rho'(\rho'-1)-\rho'^{\frac{1}{2}}(\rho'-1)}{(\rho'-1)^2}$$

$$=\omega.\frac{\rho'-\rho'^{\frac{1}{2}}}{\rho'-1}=-\frac{\omega\rho'}{1+\sqrt{\rho'}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(E).$$

Case I. A planet transits the sun's disc at such a rate that the sun's diameter S would be traversed in time t. Find the planet's distance from the sun.

Let ρ=planet's distance, unity being the earth's distance, and let ω be the earth's angular velocity around the sun=sun's angular velocity around the earth, and let t' be the time in which the sun in his annual course moves through a distance equal to his own apparent diameter; then $\omega t'=S$. From (D) the planet's angular velocity about the earth$=-\frac{\omega}{\sqrt{\rho}+\rho}$.

∴ That is the planet's retrograde gain on the sun is

$$\frac{\omega}{\sqrt{\rho}+\rho}+\omega=\frac{S}{t}=\frac{\omega t'}{t}.$$

$$\therefore\ \rho+\rho^{\frac{1}{2}}=\frac{t}{t'-t},\quad \therefore\ \rho^{\frac{1}{2}}=\tfrac{1}{2}\Big(\pm\frac{\sqrt{3t+t'}}{\sqrt{t'-t}}-1\Big),$$

$$\therefore\ \rho=\tfrac{1}{2}\left(\frac{t'+t}{t'-t}-\sqrt{\frac{3t+t'}{t'-t}}\right)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Case II. If we wish to find the earth's distance knowing the planet's distance, then let the planet's distance be unity and the earth's distance$=\rho'$.

Proceeding the same as before using (E) we get

$$\frac{\omega\,\rho'}{1+\sqrt{\rho'}}+\omega=\frac{S}{t}=\frac{\omega t'}{t}.\quad \therefore\ \rho'-\frac{t'-t}{t}\sqrt{\rho'}=\frac{t'-t}{t};$$

$$\therefore\ \sqrt{\rho'}=\frac{1}{2t}\left\{(t'-t)+\sqrt{t'^2-3t^2+2tt'}\right\}.$$

$$\therefore\ \rho'=\frac{t'-t}{2t^2}\left\{(t'+t)+\sqrt{t'^2-3t^2+2tt'}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Suppose Venus transits the sun's disc at such a rate that the sun's apparent diameter would be traversed in 7⅓ hours, and at the same time the sun in his annual course moves through a distance equal to his own apparent diameter in 12 hours. Required (1) the distance from Venus to the sun, the earth's distance being unity, and (2) the distance from the earth to the sun, Venus's distance being unity.

Now $t=7\frac{1}{3}$, $t'=12$; hence for first case substituting in (1)

$$\rho=\tfrac{1}{2}\left(\tfrac{29}{7}-\sqrt{\tfrac{51}{7}}\right)=.721824.$$

For the second case substitute in (2)

$$\rho'=\tfrac{7}{484}\{58+\sqrt{1428}\}=1.38538.$$

(The above is suggested in Proctor's Geometry of the Cycloid.)

A PROPOSITION IN DETERMINANTS.

By ALFRED HUME, C. E., D. Sc., Professor of Mathematics in the University of Mississippi.

THEOREM.—The product of two numbers, each the sum of four squares, is the sum of eight squares.

$$\begin{vmatrix} a+b\sqrt{-1} & -c+d\sqrt{-1} \\ c+d\sqrt{-1} & a-b\sqrt{-1} \end{vmatrix} \times \begin{vmatrix} \alpha+\beta\sqrt{-1} & -\gamma+\delta\sqrt{-1} \\ \gamma+\delta\sqrt{-1} & \alpha-\beta\sqrt{-1} \end{vmatrix}$$

$$=\begin{vmatrix} a+b\sqrt{-1} & -c+d\sqrt{-1} & 0 \\ c+d\sqrt{-1} & a-b\sqrt{-1} & 0 \\ 0 & 0 & 1 \end{vmatrix} \times(-1) \begin{vmatrix} \alpha+\beta\sqrt{-1} & 0 & -\gamma+\delta\sqrt{-1} \\ \gamma+\delta\sqrt{-1} & 0 & \alpha-\beta\sqrt{-1} \\ 0 & 1 & 0 \end{vmatrix}$$

$$=(-1)\begin{vmatrix} a\alpha-b\beta+(a\beta+b\alpha)\sqrt{-1} & a\gamma-b\delta+(a\delta+b\gamma)\sqrt{-1} & -c+d\sqrt{-1} \\ c\alpha-d\beta+(c\beta+d\alpha)\sqrt{-1} & c\gamma-d\delta+(c\delta+d\gamma)\sqrt{-1} & a-b\sqrt{-1} \\ -\gamma+\delta\sqrt{-1} & \alpha-\beta\sqrt{-1} & 0 \end{vmatrix}$$

$$= \begin{vmatrix} c\alpha-d\beta+(c\beta+d\alpha)\sqrt{-1} & c\gamma-d\delta+(c\delta+d\gamma)\sqrt{-1} \\ -c\gamma+d\delta+(c\delta+d\gamma)\sqrt{-1} & c\alpha-d\beta-(c\beta+d\alpha)\sqrt{-1} \end{vmatrix}$$

$$+ \begin{vmatrix} a\alpha-b\beta+(a\beta+b\alpha)\sqrt{-1} & a\gamma-b\delta+(a\delta+b\gamma)\sqrt{-1} \\ -a\gamma+b\delta+(a\delta+b\gamma)\sqrt{-1} & a\alpha-b\beta-(a\beta+b\alpha)\sqrt{-1} \end{vmatrix}$$

or $(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2)=(c\alpha-d\beta)^2+(c\beta+d\alpha)^2$
$+(c\gamma-d\delta)^2+(c\delta+d\gamma)^2+(a\alpha-b\beta)^2+(a\beta+b\alpha)^2+(a\gamma-b\delta)^2+(a\delta+b\gamma)^2$.

Euler's Theorem is an easy corollary of this, and *vice-versa*.

University of Mississippi, March, 1896.

A METHOD OF SOLVING QUADRATIC EQUATIONS.

By Prof. HENRY HEATON, M. Sc., Atlantic, Iowa.

Let it be required to solve the equation

$ax^2+bx+c=0$ .... (1).

Transposing the middle term we have

$ax^2+c=-bx$ .... (2).

Squaring, $a^2x^4+2acx^2+c^2=b^2x^2$ .... (3).

Subtracting $4acx^2$, $a^2x^4-2acx^2=(b^2-4ac)x^2$ .... (4).

Extracting the square root, $ax^2-c=\pm(\sqrt{b^2-4ac})x$ .... (5).

Adding equation (2), $2a^2x^2=(-b\pm\sqrt{b^2-4ac})x$ .... (6).

Whence $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Let it be required to solve the equation $3x^2-2x=21$.

Transposing $2x$ to the second member and 21 to the first, the equation becomes

$3x^2-21=2x$ .... (7).

Squaring, $9x^4-126x^2+441=4x^2$ ....(8).

Adding twice $126x^2$, $9x^4+126x^2+441=256x^2$ ....(9).

Extracting the square root, $3x^2+21=\pm16x$ ....(10).

Adding equation (7), $6x^2=18x$ or $-14x$.

$\therefore x=3$ or $-2\frac{1}{3}$.
Is this new?

[NOTE.—We do not remember of ever having seen this method. If any of our readers have seen it 'here, please let us know. Editor.]

ON THE DOCTRINE OF PARALLELS.

By Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

I desire to enter my protest against any assumption that parallel lines, exled to an infinite distance, do, or do not, intersect. The human mind cannot prehend the infinite and, therefore, we cannot determine the question. We ' use modes of reasoning involving infinite quantities, but we can rely upon results *only so far as human experience shows that they are correct.* It is true, a mode of reasoning in such cases. which leads to a result found by human ›rience to be correct in a particular case, may generally be assumed to be ect in all cases. Without human experience, the proposition that if two ob› are moving in the same line in the same direction at different velocities, the in advance will move over an appreciable space while the other is moving ' the space between them and, therefore, that the one can never overtake the ›r, could never have been successfully denied. I hold that this doctrine ap› to much of the discussion of the present day, and some of the propositions ›ve been able to deny, and old propositions denied I have been able to affirm, ›use I knew that *human experience had settled the matter.*

Whether Euclid's reasoning was, or was not correct, I have never seen a in which the result which he reached has not been found to be absolutely ect by human experience.

The quotation which Professor Lyle makes from Lotze (Vol. II. › 375) involves the arrogant assumption that the human mind is infinite in scope of its reasoning power. Mathematicians, of all men, should not claim a proposition involving the infinite, cannot be true, because we cannot *comend the possibility* of its being true.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

62. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

A dealer buys milk at $m=5$ cents per quart, and sells it at $n=6$ cents per quart. How much water has he put with the milk, if his rate of profit is $p=60\%$?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas, and J. F. YOTHERS, Westerville, Ohio.

$m(1+p)$=price at which a quart of pure milk would sell at a profit of $p\%$.

$\frac{m(1+p)}{n}$=number of quarts at n cents sold for $m(1+p)$ cents.

$\therefore \frac{m(1+p)}{n}-1=\frac{m(1+p)-n}{n}$=amount of water added to each quart of milk. Let $m=5$, $n=6$, $p=.60$.

$\therefore \frac{m(1+p)-n}{n}=\frac{1}{3}$. $\therefore$ He adds one quart of water to 3 quarts of milk.

Also solved by *E. R. ROBBINS, P. S. BERG, F. R. HONEY.*

63. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

I owe A $100 due in 2 years, and $200 due in 4 years; when will the payment of $300 equitably discharge the debt, money being worth 6%?

I. Solution by P. S. BERG, Larimore, North Dakota; EDWARD R. ROBBINS, Lawrenceville, New Jersey; FREDERICK R. HONEY, Ph. B., New Haven, Connecticut.

Present worth of $100 for 2 years at 6%=$89.28.
Present worth of $200 for 4 years at 6%=$161.29.
$250.57=$89.28+$161.29=sum of present worths,
The time required for $250.57 at 6% to amount to $300 is the time sought.
Interest of $250.57 for one year at 6%=$15.0342.
$300−$250.57=$49.43, interest for the whole term.
Hence time equals 49.43÷15.0342=3.2878 years.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas; J. F. YOTHERS, Westerville, Ohio.

The interest on $100 for 2 years at 6%=$12.
The interest on $200 for 4 years at 6%=$48.
The interest on $300 for 1 year at 6%=$18.
($12+$48)÷$18=$60÷$18=3 years, 4 months.

Or, \$100 for 2 years = \$ 200 for 1 year.
\$200 for 4 years = \$ 800 for 1 year.

\$1000 for 1 years.

\$1000 ÷ \$300 = 3 years, 4 months.

PROBLEMS.

67. **Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.**

A agreed to work a year for \$300 and a suit of clothes. At the end of five months he left, receiving for his wages \$60 and the clothes. What was the suit worth?

68. **Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.**

The population of a city is annually increasing $m=2\frac{1}{2}\%$. If the population now is $P=68921$, what was it $n=3$ years ago? At this rate of increase, what will the population be $n=3$ years hence?

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

58. **Proposed by I. J. SCHWATT, Ph. D., Instructor in Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania.**

1. The point of intersection K_a' of the tangent drawn to the circumcircle about the triangle ABC at A and the side BC is harmonic conjugate to K_a with respect to BC. (K_a is the point where the symmedian line through A of the triangle ABC meets the side BC.)

2. The point K_a' is the center of the Apollonius circle passing through A of the triangle ABC.

3. Grebes point is on the line joining the middle point of any side of a triangle with the middle point of the altitude to this side.

I. **Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

1. In trilinears, the equation to the circumcircle of the triangle of reference is

$$a\beta\gamma+b\alpha\gamma+c\alpha\beta=0\text{.......................}(1).$$

The tangent to this circle at A is

$$b\gamma+c\beta=0 \quad\text{.........................(2).}$$

The equation to the symmedian through A is

$$b\gamma-c\beta=0 \quad\text{.........................(3).}$$

(2) and (3) are conjugates to $\beta=0$, $\gamma=0$.

2. The circle of Apollonius passes through A and the points of intersection of the internal and external bisectors of the angle A of the triangle of reference with the side BC. The coordinates of the center of this circle are plainly the half sum of those of the intersections of the bisectors with BC, or $\left(0, \dfrac{ab^2\sin C}{b^2-c^2}, \dfrac{ac^2\sin B}{b^2-c^2}\right)$. This is the point of intersection of (2) and $\alpha=0$.

3. The coordinates of Grebe's point are proportional to a, b, c.

The mid-point of a is $(0, \frac{a}{2}\sin C, \frac{a}{2}\sin B)$, and of the altitude on a, $\left(\frac{\Delta}{a}, \frac{b}{4}\sin 2C, \frac{c}{4}\sin 2B\right)$.

The line through these points is $\alpha\sin(B-C)+\gamma\sin C-\beta\sin B=0$, which is satisfied by $\alpha=a$, $\beta=b$, $\gamma=c$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Using trilinear coordinates we get,

(1), equation to AK_a' is, $\beta\sin C+\gamma\sin B=0$, or $\beta c+\gamma b=0$; to AB, $\gamma=0$; to AK_a, $\beta c-\gamma b=0$; to AC, $\beta=0$.

$\therefore$ AK_a', AB, AK_a, AC form a harmonic pencil.

$\therefore$ K_a' and K_a are harmonic conjugate.

(2). Draw AK_a''' bisecting $\angle DAB$.

Then $\angle AK_a'B=\angle K_a'K_a'''A+\angle K_a'AK_a'''=B-C$, and

$\angle K_a'AK_a'''=\angle K_a'''AB-\angle K_a'AB=90^\circ-\frac{1}{2}A-C=\frac{1}{2}(B-C)$.

$\therefore$ $\angle K_a'K_a'''A=\angle K_a'AK_a'''$; $\therefore$ $K_a'K_a'''=AK_a'=AK_a''$.

$\therefore$ K_a' is the center of the required circle.

(3). The equation to the straight line through Grebe's point and the mid-point of BC is the same as the equation of the straight line through mid-point of BC and the midpoint of the perpendicular from A on BC, both being $\sin(B-C)\alpha-\sin B\beta+\sin C\gamma=0$.

III. Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, New Jersey, and J. C. CORBIN, Pine Bluff, Arkansas.

(1). Let ABC be $\triangle$; AK_a', the tangent at A; and K_a point where symmedian meets the side BC. Take $AN=AC$, and $AM=AB$. Then in $\triangle ANM$, AH, the median, is the symmedian of $\triangle ABC$. Lines BC, and MN are antiparallel. Also, since $\angle ABC=\angle BAD$, each being measured by ½arc AC, the lines BC and AK_a' are antiparallel. Wherefore MN is parallel to the tangent line AK_a'.

Now we have a pencil of four rays AB, AH, AC, AK_a' in which one ray AH bisects a line parallel to its conjugate, and included between the other pair of conjugate rays; hence the pencil is harmonic, and any line, as BK_a', drawn across the pencil will cut out an harmonic range $\{BC, K_aK_a'\}$. Q. E. D.

(3). Draw ST perpendicular to AC at its middle point S; draw BT, and it is a symmedian line (Halsted: Syn. Geom. §648.), hence it passes through Grebe's point (or Lemoine's Point) K. Now as $A\{BC, K_aK_a'\}$ is an harmonic pencil, $\{BR, KT\}$ is an harmonic range; whence $S\{BR, KT\}$ is an harmonic pencil. Draw altitude BP, and it is ‖ to ray ST, and is therefore bisected by the ray SK, the conjugate of ST. Therefore the line joining the middle point, S, of a side, and the middle point of the altitude to that side passes through Grebe's point.

Q. E. D.

60. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Show that the tangent plane at any point of the surface $a^2x^2+b^2y^2+c^2z^2=2bcyz+2acxz+2abxy$ intersects the surface $ayz+bzx+cxy=0$ in two straight lines at right angles to one another.

Solution by the PROPOSER.

The tangent plane at (x', y', z') to $F=0$..........................(1) is

$$(x-x')\frac{dF}{dx'}+(y-y')\frac{dF}{dy'}+(z-z')\frac{dF}{dz'}=0 \quad \text{..........(2).}$$

Here $F=a^2x^2+b^2y^2+c^2z^2-2bcyz-2acxz-2abxy$..................(3).

$$\frac{dF}{dx'}=2a(ax'-by'-cz'),\ \frac{dF}{dy'}=2b(-ax'+by'-cz'),$$

$$\frac{dF}{dz'}=2c(-ax'-by'+cz') \quad \text{..........(4).}$$

Then (2) becomes by aid of (3),

$$a(ax'-by'-cz')x+b(-ax'+by'-cz')y+c(-ax'-by'+cz')z=0 \quad \text{..........(5).}$$

It may be shown that the condition that

$lx+my+nz=0$ ....(6) cuts $ayz+bxz+cxy=0$........... .....(7)

in two straight lines including a right angle is $amn+bnl+clm=0$...........(8).

Comparing (5) and (6), $l=a(ax'-by'-cz')$, $m=b(-ax'+by'-cz')$, $n=c(-ax'-by'+cz')$, and (8) becomes

$abc\{a^2x'^2-(by'-cz')^2+b^2y'^2-(cz'-ax')^2+c^2z'^2-(ax'-by')^2\}=0$ ....(9),

an identity by aid of (3).

Also solved by *HENRY HEATON* and *J. SCHEFFER*.

PROBLEMS.

66. Proposed by I. J. SCHWATT, Ph. D., Professor of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania.

Prove in a pure geometrical way the following:

The axes of the ellipse isogonal to Lemoine's line with respect to a triangle (Steiner's ellipse) are parallel to Simson's lines belonging to the extremities of Brocard's Diameter.

66. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The locus of points whose polars with respect to a given parabola touch the circle of curvature at the vertex is an equilateral hyperbola.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

32. Proposed by OTTO CLAYTON, A. B., Fowler, Indiana.

The wheel of a wind pump has 60 fans, each turned at an angle of 45° to the direction of the axis, and each having 150 square inches exposed to the wind. If the wind blows with a velocity of V and the wheel rotates with velocity ω, what is the component of force or pressure along the axis if it is turned at an angte α to the direction of the wind assuming the resistance of the wheel in turning to be R?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let A=projecting area of fans exposed to the wind, in square feet,
V=velocity of wind in feet per second,
H=horse hower of pump,
R=extreme radius of fans in feet,

r=inner radius of fans in feet,

$l=\sqrt{\frac{R^2+r^2}{2}}$, =radius of center of percussion, in feet,

n=number of revolutions of fans per minute,

β=mean angle of fans to the plane of motion.

By Nystrom's Mechanics we get $H=\frac{Aln\sin\beta\cos\beta}{1.540000}\left(V-\frac{2ln\sin\beta}{19}\right)^2$.

Let AB, CD, FE be the direction of the axis, fans, and wind, respectively.

$\angle HKG=\alpha$, $\angle KGH=\frac{\pi}{4}$. $\therefore \angle GKH=\left(\frac{3\pi}{4}-\alpha\right)$.

Then $A=60\times150\times\sin\left(\frac{3\pi}{4}-\alpha\right)$

$+144=125\sin\left(\frac{3\pi}{4}-\alpha\right)$.

$n=\omega$, $\beta=\frac{\pi}{4}$. $\therefore H=\frac{125l\omega\sin\left(\frac{3\pi}{4}-\alpha\right)\left(19V-l\omega\sqrt{2}\right)^2}{117040000}$

$H'=H-R/33000$=effective horse power.

36. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

A man weighs 150 pounds; his balloon with all its attachments weighs 500 pounds. What volume of pure hydrogen must be made and put into the balloon so that it will be on the point of ascending with the man? How many kilograms of zinc and of hydrogen sulphate will be used generating the hydrogen? Give volume of hydrogen in cubic feet, given that one litre of hydrogen weighs .0896 grams.

Solution by P. S. BERG, Larimore, North Dakota, and the PROPOSER.

1 grain=.0022046 pounds. 1 cubic foot=28.315 litres.

Let temperature and pressure be normal.

$\therefore$ 1 cubic foot of hydrogen weighs $28.315\times.0896\times.0022046=.005593123$ pounds.

1 cubic foot of air weighs $28.315\times1.293\times.0022046=.080713261$ pounds.

$\therefore$ The lifting power of 1 cubic foot of hydrogen is .080713261 pounds $-$.005593123 pounds=.075120138 pounds.

500 pounds+150 pounds=650 pounds.

$650\div.075120138=8652.806$ cubic feet of hydrogen.

$8652.806\times.005593123\div2.2046=21.9524$ kilograms of hydrogen used.

$Zn+H_2SO_4=ZnSO_4+H_2$. $\therefore H_2SO_4 : H_2=x : 21.9524$.

$98 : 2=x : 21.9524$. $\therefore x=1075.6676$ kilograms of hydrogen sulphate.

$Zn : H_2=x : 21.9524$. $65 : 2=x : 21.9524$. $\therefore x=713.453$ kilograms of zinc.

Also solved by *A. P. REED*.

PROBLEMS.

44. **Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.**

There is a triangle whose sides repulse a center of force within the triangle with an intensity that varies inversely as the distance of the center of force from each point of the sides of the triangle. What is the position of equilibrium of the center?

45. **Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.**

A fifty-pound cannon-ball is projected vertically upward with a velocity of 800 feet per second. Find the height to which it will rise and the time of flight, assuming the initial resistance of the air on the ball to be 10 pounds and the resistance to vary as the square of the velocity.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

64. **Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.**

A man raises 1 chicken the first year; 6, the second; 35, the third; 180, the fourth; 921, the fifth; 4626, the sixth; 23215, the seventh; 116160, the eighth; and so on. How many does he raise the 20th year, and how many in the twenty years?

I. Solution by A. H. HOLMES, Box 963, Brunswick, Maine.

We easily find by inspection $U_{x+1}-5U_x=\dfrac{4^{\frac{x+1}{2}}-1}{3}$, or $\dfrac{4^{\frac{x+2}{2}}-1}{3}$, according as x is odd or even. Integrating and reducing, we have

$$U_x=\tfrac{1}{4}\Big[5^x+4\times5^{x-2}+4^2\times5^{x-4}+\text{etc.}-\frac{4^{\frac{x+1}{2}}-1}{3}\text{ or }\frac{4^{\frac{x+2}{2}}-1}{3}\Big].$$

$$\text{Summing, } S_x=\tfrac{1}{16}\Big[5^{x+1}+4\times5^{x-1}+4^2\times5^{x-3}+\text{etc.}-\frac{23\times4^{\frac{x+2}{2}}-12x-47}{9},$$

$$\text{or }\frac{11\times4^{\frac{x+2}{2}}-12x-47}{9}\Big].$$

Putting $x=20$, and performing operations indicated, we have,
$U_{20}=28,383,163,779,300$, and $S_{20}=35,478,954,491,110$.

II. **Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.**

The numbers in the problem may be represented under the following form :

1	6	35	180	921	4626
$5\times0+1$,	$5\times1+1$,	$5\times6+5$,	$5\times35+5$,	$5\times180+21$,	$5\times921+21$,

23215	116160	
$5\times4626+85$,	$5\times23215+85$,	etc.

The general term of the numbers 1, 5, 21, 82, etc., is $\frac{1}{3}(4^x-1)$, as can be easily found by Finite Differences. Expressing the $(2x-1)$th term of the above series by $F(2x-1)$, we have, by Finite Differences, $F(2x-1)=C.5^{2x-1}+C_1.4^x+C_2$. Substituting for x successively 1, 2, 3, we have the three equations: $5C+4C_1+C_2=1$, $125C+16C_1+C_2=35$, $3125C+64C_1+C_2=921$, whence $C=25/84$, $C_1=-1/7$, $C_2=1/12$.

$$\therefore F(2x-1)=\frac{5^{2x+1}}{84}-\frac{4^x}{7}+\tfrac{1}{12} \quad\ldots\ldots\ldots\ldots\text{(I)}.$$

To find $F(2x)$, multiply $F(2x-1)$ by 5 and add $\frac{1}{3}(4^x-1)$, thus,

$$F(2x)=\frac{5^{2x+2}}{84}-\tfrac{8}{21}.4x+\tfrac{1}{12} \quad\ldots\ldots\ldots\ldots\text{(II)}.$$

By summing the geometrical series $5^3+5^5+\ldots\ldots 5^{2x-1}$, $5^4+5^6+\ldots\ldots 5^{2x+2}$, $4+4^2+4^3+\ldots\ldots 4^x$, we find

$$\sum F(2x-1)=\frac{5^{2x+3}}{2016}-\frac{4^{x+1}}{21}+\tfrac{1}{12}x+\tfrac{259}{2016}, \text{ and}$$

$$\sum F(2x)=\frac{5^{2x+4}}{2016}-\tfrac{8}{63}.4^{x+1}+\tfrac{1}{12}x+\tfrac{399}{2016}.$$

Consequently $$\sum_{x=1}^{x=2n-1}(x)=\frac{5^{2n+2}}{336}-\tfrac{5}{63}.4^{n+1}+\tfrac{1}{6}n+\tfrac{35}{144} \quad\ldots\ldots\ldots\ldots\text{(III)};$$

$$\sum_{x=1}^{x=2n}(x)=\frac{5^{2n+3}}{336}-\tfrac{11}{63}.4^{n+1}+\tfrac{1}{6}n+\tfrac{47}{144} \quad\ldots\ldots\ldots\ldots\text{(IV)}.$$

The formulae I and III are to be employed for an odd number of terms, and II and IV for an even one. Thus, $F(20)=\frac{5^{22}}{84}-\frac{8}{21}.4^{10}+\frac{1}{12}=28383163779300$;

$$\sum F(20)=\frac{5^{23}}{336}-\tfrac{11}{63}.4^{11}+\tfrac{10}{6}+\tfrac{47}{144}=35478954491110.$$

III. Solution by A. M. HUGHLETT, A. M., Associate Principal and Professor of Mathematics in Randolph-Macon Academy, Bedford City, Virginia.

1. Write out to "n" terms the series: $1, 5, 25, 125, 625, \ldots\ldots 5^{n+1}$.
2. Begin at 3rd term and write the series: $4, 20, 100, \ldots\ldots 4.5^{n-3}$.
3. Begin at 5th term and write the series: $16, 4^2.5, 4^2.5^{n-5}$.

....

4. Begin at $(n-1)$th term and write the series: $4^{\frac{n-1}{2}}, 4^{\frac{n-2}{2}}.5$, n being even.

5. Begin at nth term and write the series: $4^{\frac{n-1}{2}}$, n being odd.

The nth term in the required series is the sum of all the numbers in the nth term of the above arrangement plus all that precede it. Denote the sum by s, then, if n is even,

$$s=\left\{\begin{array}{l}1+5+25\ldots\ldots\ldots+5^{n-1}\\ +4+4.5\ldots\ldots+4.5^{n-3}\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ +\qquad 4^{\frac{n-2}{2}}+4^{\frac{n-1}{2}}.5\end{array}\right\}=\frac{5^n-1}{4}+\frac{4(5^{n-2}-1)}{4}\ldots\ldots\ldots+\frac{4^{\frac{n-2}{2}}(5^2-1)}{4} \quad (1).$$

If n is odd,

$$s=\left\{\begin{array}{l}1+5+35\ldots\ldots\ldots.5^{n-1}\\ 4+4.5\ldots\ldots\ldots.4.5^{n-3}\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ +\qquad 4^{\frac{n-1}{2}}\end{array}\right\}=\frac{5^n-1}{4}+\frac{4(5^{n-2}-1)}{4}\ldots\ldots\ldots 4^{\frac{n-1}{2}}\frac{(5-1)}{4} \quad (2).$$

(1) finally becomes: $\frac{1}{24}\{5^{n+2}+7-8.4^{\frac{n+2}{2}}\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$,

and (2) becomes $\frac{1}{24}\{5^{n+2}+7-12.4^{\frac{n+1}{2}}\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$.

(3) gives the even terms; (4) gives the odd terms. To sum the series:—
1st term by $(4)=\frac{1}{24}(5^3+7-12.4)$
2nd term by $(3)=\frac{1}{24}(5^4+7-8.4^2)$
3rd term by $(4)=\frac{1}{24}(5^5+7-12.4^2)$
4th term by $(3)=\frac{1}{24}(5^6+7-8.4^3)$
5th term by $(4)=\frac{1}{24}(5^7+7-12.4^3)$
....

nth term by $(3)=\frac{1}{24}(5^{n+2}+7-8.4^{\frac{n+2}{2}})$, n being even,

nth term by $(4)=\frac{1}{24}(5^{n+2}+7-12.4^{\frac{n+1}{2}})$, n being odd.

Denote the sum by S, then, *n being even*,

$$S = \frac{1}{84}\left\{\frac{5^{n+3}-125}{4}+7n-11\frac{(4^{\frac{n+4}{2}}-16)}{3}\right\}\ \ldots\ldots\ldots\ldots(5).$$

Similarly, n being odd,

$$S = \frac{1}{84}\left\{\frac{5^{n+3}-125}{4}+7n-\frac{20.4^{\frac{n+3}{2}}-176}{3}\right\}\ \ldots\ldots\ldots\ldots(6).$$

By (3), the 20th term is: $\frac{1}{84}\{5^{22}+7-8.4^{11}\}=28,383,163,779,300.$

(5), the twenty terms are: $\frac{1}{84}\{\frac{5^{23}-125}{4}+140-11\ \frac{(4^{12}-16)}{3}\}$

$=35,478,954,491,110.$

IV. **Solution by the PROPOSER.**

Let it be required to sum to n terms and find the nth term of the series:

$1+6x+35x^2+180x^3+921x^4+4626x^5+23215x^6+116160x^7+\ldots\ldots$

Let the scale of relation be denoted by m, n, p, q.

$\therefore 921x^4=180qx^3+35px^2+6nx+m$ ....(1).

$4626x^5=921qx^4\times 180px^3+35nx^2+6mx$ ....(2).

$23215x^6=4626qx^5+921px^4+180nx^3+35mx^2$ ....(3).

$116160x^7=23215qx^6+4626px^5+921nx^4+180mx^3$ ....(4).

$\therefore m=20x^4,\ n=-24x^3,\ p=-x^2,\ q=6x.$

Since the series has a quadruple scale of relation it must be composed of e sum of four geometrical series. The ratios of these series will be the roots the biquadratic equation

$r^4=6xr^3-x^2r^2-24x^3r+20x^4$ ....(5).

$\therefore r_1=2x,\ r_2=-2x,\ r_3=5x,\ r_4=x.$

Let a_1, a_2, a_3, a_4 be the first terms of these sets of series; then

$a_1+a_2+a_3+a_4=1$ ....(6).

$a_1r_1+a_2r_2+a_3r_3+a_4r_4=2a_1-2a_2+5a_3+a_4=6$ ....(7).

$a_1r_1^2+a_2r_2^2+a_3r_3^2+a_4r_4^2=4a_1+4a_2+25a_3+a_4=35$ ....(8).

$a_1r_1^3+a_2r_2^3+a_3r_3^3+a_4r_4^3=8a_1-8a_2+125a_3+a_4=180$ ....(9).

$\therefore a_1=-2/3,\ a_2=2/21,\ a_3=125/84,\ a_4=1/12.$

Hence the series are:

$-2/3-4x/3-8x^2/3-16x^3/3-32x^4/3-64x^5/3-$ ....(10).

$2/21-4x/21+8x^2/21-16x^3/21+32x^4/21-64x^5/21+$ ....(11).

$5/84+625x/84+3125x^2/84+15625x^3/84$

$+78125x^4/84+390625x^5/84+$ ....(12).

$1/12+x/12+x^2/12+x^3/12+x^4/12+x^5/12+$ ....(13).

Let A_n^1, A_n^2, A_n^3, A_n^4, S_n^1, S_n^2, S_n^3, S_n^4 represent the nth terms, and the sum of n terms of the series (10), (11), (12), (13). Then,

$$A_n^1=-\tfrac{8}{3}(2x)^{n-1},\ A_n^2=\tfrac{8}{21}(\pm 2x)^{n-1},\ A_n^3=\tfrac{125}{21}(5x)^{n-1},\ A_n^4=\tfrac{1}{3}x^{n-1},$$

$$S_n^1=-\tfrac{8}{3}\left(\frac{2^nx^n-1}{2x-1}\right),\ S_n^2=\tfrac{8}{21}\left(\frac{\pm 2^nx^n-1}{-2x-1}\right),\ S_n^3=\tfrac{125}{21}\left(\frac{5^nx^n-1}{5x-1}\right),$$

$$S_n^4=\tfrac{1}{3}\left(\frac{x^n-1}{x-1}\right).$$

Let A_n, S_n, be the nth term and the sum of n terms of the original series.

$$\therefore\ A_n=\tfrac{1}{21}\{5^{n+2}+7(1-2^{n+2})\mp 2^{n+2}\}x^{n-1}.$$

$$S_n=\tfrac{1}{21}\left\{\frac{125(5^nx^n-1)}{5x-1}-\frac{56(2^nx^n-1)}{2x-1}+\frac{8(\pm 2^nx^n-1}{-2x-1}+\frac{7(x^n-1)}{x-1}\right\}.$$

The upper sign to be used when n is even. Now let $x=1$, $n=20$, and we will get the required results for the problem. $A_{20}=28383163779800$, the number the twentieth year; $S_{20}=35478954491110$, the number in twenty years.

Also solved by *EDWARD R. ROBBINS.*

65. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A, B, and C bought unequal shares in 200 acres of land at the same price per acre, which they sold for \$286.90. A gained as much per cent. on his part as he had acres, B gained 5-8 as much per cent. on his part as A did, and C lost \$9.10 on the cost of his part; the total net gain was 43 9-20 per cent. How much land did each buy, and what did each receive per acre at the sale?

I. Solution by W. H. CARTER, Professor of Mathematics in Centenary College of Louisiana, Jackson, Louisiana.

Let x, y, and z be the number of acres bought by A, B, and C, respectively. $\therefore x+y+z=200$ ....(1).

Since the selling price is \$286.90 and the gain per cent. is 43.45, the cost is \$200. Let m=cost per acre; then mx, my, and mz represent the cost of the shares of A, B, and C, respectively. $\therefore m(x+y+z)=200$. $\therefore m=1$. $\therefore$ the cost of the share of each=number of acres he bought.

x=A's gain per cent., and $5x/8$=B's gain per cent.

$\therefore x+x^2/100+y+5xy/800+z-\$9.10=\$286.90$.

$\therefore x^2/100+5xy/800=\$96$. $\therefore 8x^2+5xy=76800$.

$$\therefore\ y=\frac{76800-8x^2}{5x}=\frac{15360}{x}-\frac{8x}{5}\ \ ....(2).$$

If the number of acres bought by each is to be integral, then (1) and (2) are to be solved for *positive integral* values of x, y, and z. Since y is to be integral, x must be a factor 15360 and must be divisible by 5. $15360=5\times3\times2^{10}$. $\therefore$ the factors of 15360 which are divisible by 5, are 5, 10, 15, 20, 30, 40, 60, 80, 120, 160, etc. If x has any of these values less than 80, z will be negative; if x has any values greater than 80, y is negative. If $x=80$, $y=64$, and $z=56$. $\therefore$ 80, 64, and 56 are the shares of A, B, and C.

The amounts each received per acre at the sale are easily found to be \$1.80, \$1.50, and \$0.83¾.

II. Solution by EDWARD R. ROBBINS, Master in Mathematics and Physics in the Lawrenceville School, Lawrenceville, New Jersey.

Let x, y, and $200-x-y$ represent the number of acres which A, B, and C bought, respectively. Then by the problem,

$$x+x^2/100+y+5xy/800+200-x-y-9.10=286.90.$$

This gives $8x^2+5xy=76,800$; or $y=(76800-8x^2)/5x$. Solving for positive integers in x, we have, when

$$x=\ 75,\quad 80,\quad 85,\quad 90,$$
$$y=107\tfrac{4}{5},\quad 64,\quad 44\tfrac{12}{17},\quad 26\tfrac{2}{3},$$

Accepting the integral values we obtain:
A's purchase consisted of 80 acres and sold for \$144;
B's purchase consisted of 64 acres and sold for \$96;
C's purchase consisted of 56 acres and sold for \$46.90.
Hence A received \$1⅘ per acre; B, \$1½; and C, \$$\frac{67}{80}$.

III. Solution by H. C. WILKES, Skull Run, West Virginia.

Since by the terms of the problem the price paid for the land was \$1 per acre, let $8x$, y, z be the number of acres bought, and the number of dollars paid, by A, B, and C, respectively.

Then $8x+y+z=200\ldots(1)$. $8x+64x^2/100+y+5xy/100+z=296\ldots(2)$.

Subtracting (1) from (2), and clearing, $64x^2+5xy=9600$. Factoring, $x(64x+5y)=10(960)$. Let $x=10$; then $5y=320$, and $y=64$.

$\therefore$ 80, 64, 56 are numbers satisfying the conditions. See solution of a similar problem on page 76 of Vol. II.

IV. Solution by A. M. HUGHLETT, A. M., Associate Principal and Professor of Mathematics in Randolph-Macon Academy, Bedford City, Virginia.

Let x, y, and z represent the shares of A, B, and C, respectively. $x+y+z=200\ \ldots\ldots(1)$. Since C lost \$9.10, he must have bought at least 9.10 acres. Therefore 190.90 is the maximum limit of $x+y$.

$$x+x^2/100+y+xy/160+z=296\ldots\ldots(2).$$

(1) in (2) gives $x^2/100+xy/160=96$. $\therefore y=(76800-8x^2)/5x\ldots\ldots(3)$.

$\therefore$ 190.90$>x+(76800-8x^2)/5x$. $\therefore$ 190.90$>(76800-3x^2)/5x$.
As x decreases, $(76800-3x^2)/5x$ increases.

$\therefore$ the equation $(76800-3x^2)/5x=190.90$..............................(4)

gives the minimum limit of x.

$\therefore$ 66.48+ is the minimum limit of x......... ....(5).

From (3), $y=(76800-8x^2)/5x$, we get, since y must have some value, $76800>8x^2$; hence $8x^2=76800$ gives maximum limit of x. $\therefore$ 97.97+ is the maximum limit of x. Hence, any values of x between 66.48+ and 97.97+ will satisfy the conditions of the problem. *Example:* Let $x=77\frac{1}{4}$. Then from (3) $y=75\frac{43}{65}$; $\therefore z=47\frac{11}{65}$.

$\therefore$ A received \$136.65$\frac{136}{441}$; B received \$112.17$\frac{103}{441}$. $\therefore$ C received \$38.07$\frac{303}{441}$; but he paid \$47.17$\frac{303}{441}$. $\therefore$ C lost \$9.10.

Also solved by *A. H. HOLMES*, *J. SCHEFFER*, and *G. B. M. ZERR*.

PROBLEMS.

72. Proposed by CHAS. C. CROSS, Laytonsville, Maryland.

Prove that $\dfrac{2\sqrt{2+\sqrt{3}}}{4\times\sqrt{6-\sqrt{2}}}=\sqrt{6}-\sqrt{2}+\sqrt{3}-2$, when reduced to its lowest terms.

73. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Find the worth of each of five persons, A, B, C, D, and E, knowing, 1st, that when A's worth is added to a times what B, C, D, and E are worth, it is equal to m; 2nd, when B's worth is added to b times what A, C, D, and E are worth, it is equal to n; 3rd, when C's worth is added to c times what A, B, D, and E are worth, it is equal to p; 4th, when D's worth is added to d times what A, B, C, and E are worth, it is equal to q; 5th, when E's worth is added to e times what A, B, C, and D are worth, it is equal to r.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

51. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the maximum ellipsoid that can be cut out of a given right conic frustum.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

A complete solution of this problem without any assumptions would be a task greater than I care to undertake at present. We will, therefore, assume the cone to be one of revolution. Let $2h$=height of frustum, R, r radii of the lower and upper bases, respectively, l, m, p the coordinates of the vertex.

$x^2/a^2+y^2/b^2+z^2/c^2=1$, the equation to the ellipsoid.

$\therefore (p-z)^2+(n-y)^2=[(R-r)/2h]^2(m-x)^2$ is the equation to the cone.

We will further assume that this cone is the tangent cone to the maximum ellipsoid, then the equation to the cone is

$$(m^2/a^2+n^2/b^2+p^2/c^2-1)(x^2/a^2+y^2/b^2+z^2/c^2-1)$$
$$=(mx/a^2+ny/b^2+pz/c^2-1)^2.$$

From these two equations to the cone we get $n=p=0$.

$[(R-r)/2h]^2=Rr/(m^2-h^2)$ or $m=[(R+r)/(R-r)]h$.

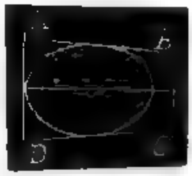
$\therefore$ The center of the frustum and the center of the ellipsoid coincide, and the ellipsoid is one of revolution.

$\therefore x^2/a^2+(y^2+z^2)/b^2=1$ is its equation. $\therefore a=h$, $b=\sqrt{Rr}$.

$V=\frac{4}{3}\pi hRr$=volume of maximum ellipsoid.

II. Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

The figure shows vertical section of frustum and inscribed ellipsoid, with axis of x coinciding with axis of cone, and axis of y in base. Let d and c be radii of bases, and h the altitude. Then $(0, d)$ and (h, c) represent points A and B, respectively.



$(x-a)^2/a^2+y^2/b^2=1$ is equation to inscribed ellipse. $\therefore$ equation to AB, as tangent to ellipse, is

$$a^2yy_1+b^2(x-a)(x_1-a)=a^2b^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

x_1 and y_1 being coordinates of point of contact. Substituting coordinates of A and B for x and y in (1),

$$\left.\begin{array}{l} a^2dy_1+b^2(-x)(x_1-a)=a^2b^2 \\ a^2cy_1+b^2(h-a)(x_1-a)=a^2b^2 \end{array}\right\} \quad\ldots\ldots\ldots\ldots\ldots(2).$$

Solving (2) for x_1 and y_1,

$$\left.\begin{array}{l} x_1=ahd/(bd-ad+ac) \\ y_1=b^2h/(bd-ad+ac) \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Substituting from (3) for x and y in equation of ellipse and solving we obtain $b^2=[(hd^2+2ad(c-d)]/h$.

Now volume of ellipsoid $V=4/3(\pi ab^2)=4\pi/3h[ad^2h+2a^2d(c-d)]$.

$$dV/da=4\pi/3h[d^2h+4ad(c-d)]\ldots\ldots\ldots\ldots(4).$$

Equating (4) to 0, we find $a=dh/4(d-c)$..........................(5),

$$\text{and } b^2=d^2/2\ldots\ldots\ldots\ldots(6).$$

Also, $d^2V/da^2=16\pi d(c-d)/3h$, which is negative since $d>c$. Now the ellipsoid will be entirely within the frustum if $2a$ is not greater than h, which from (5) gives, $dh/2(d-c)$ is not greater than h or c is not greater than $\frac{1}{2}d$. So volume of maximum ellipsoid=

$\frac{4\pi}{3}\cdot\frac{dh}{4(d-c)}\cdot\frac{d^2}{2}=\frac{\pi}{6}\frac{d^3h}{d-c}$, if c is not greater than $\frac{1}{2}d$, or $\frac{4\pi}{3}hb^2$, if $c>\frac{1}{2}d$, the latter result being true, since (4) shows but one maximum, and V is a continuous function of A.

III. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Take the base of the frustum as the plane xz, and the axis of the frustum as the axis of y. We may, without loss of generality, take one axis parallel to the axis of z. The equation of the ellipsoid may then be written:

$$Ax^2+By^2+Cxy+Dx+Ey+Hz^2+F=0\ldots\ldots\ldots\ldots(1).$$

We find the axes of the ellipsoid to be:

$$a=\sqrt{R/P},\ b=\sqrt{R/Q},\ c=\sqrt{R/H},$$

where $R=F(c^2-4AB)+AE^2+BD^2-CD^2)/(4AB-C^2)$.

$P=1/2[A+B\pm\sqrt{(A-B)^2+C^2}]$,

$Q=1/2[A+B\mp\sqrt{(A-B)^2+C^2}]$,

Volume of ellipsoid$=4/3(\pi abc)$

$$=\tfrac{4}{3}\pi\frac{[F(C^2-4AB)+AE^2+BD^2-CDE]^{\frac{3}{2}}}{[4AB-C^2]^2}\cdot\frac{1}{\sqrt{H}}\ldots\ldots\ldots\ldots(2).$$

A little consideration will show that the ellipsoid to be a maximum *must touch* the larger base of the frustum and also the conical surface. The condition that it touch the lower base is $D^2-4AF=0$..............................(3).

The condition that it shall not cut the upper base is

$$(Ch+D)^2-4A(Bh^2+Eh+F(<0\ldots\ldots\ldots\ldots(4),$$

where h is the altitude of the frustum.

To find the condition that the ellipsoid shall be tangent to conical surface, we assume the equation of complete cone to be:

$$m^2(x^2+z^2)=(y-k)^2 \qquad\qquad (5).$$

For intersection of (1) and (5),

$$(A-H)m^2x^2+(Bm^2+H)y^2+Cm^2xy+Dm^2x+(Em^2-2k)+(Hk^2+Fm^2)=0.$$

If this ellipse have no axes,

$$(Hk^2+Fm^2)[c^2m^2-4(A-H)(Bm^2+H)]+(A-H)(Em^2-2k)^2+(Bm^2+H)D^2m^2$$
$$-CD(Em^2-2k)m^2=0.$$

Solving this for B we obtain,

$$B=\frac{CD(Em^2-2k)m^2-(A-H)(Em^2-2k)^2-C^2m^2(Hk^2+Fm^2)}{m^2[D^2m^2-4(A-H)(Hk^2+Fm^2)]}-\frac{H}{m^2}.$$

Substitute the value of A given in (3),

$$B=\frac{4FCD(Em^2-2k)m^2-(D^2-4FH)(Em^2-2k)^2-4FC^2m^2(Hk^2+Fm^2)}{m^2[FD^2m^2-(D^2-4FH)(Hk^2+Fm^2)]}-\frac{H}{m^2}.$$

If we were then to substitute these values of A and B in equation (2), we should obtain a value of V which contains the variables C, D, E, F, and H, independent, except as to the condition given in (4). By the ordinary methods of maximum and minimum, five equations can be formed and the maximum critical values of the five letters determined. But life is too short to do this.

If we assume that two of the axes are parallel to the bases of the frustum, we obtain $V=\frac{4}{3}\pi\tan^2\phi(h^2b-2hb^2)$, where h=altitude of complete cone, ϕ=semi-angle of cone, and b=semi-vertical axis of ellipsoid. From this for maximum, $b=p/4$.

52. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

There are two lights of intensities m and n. Where must a target, whose surface is parallel to the line joining the two lights, be set up in order that it shall receive the maximum illumination per unit of area?

I. Solution by the PROPOSER.

If we take the point where the light with intensity l is situated as the origin of coordinates, we have readily from the principles of Optics,
$I=ly/(x^2+y^2)^{\frac{3}{2}}+my/[(a-x)^2+y^2]^{\frac{3}{2}}$, x and y being the coordinates of the bull's-eye.

$$\frac{dI}{dx}=\frac{-3lxy}{(x^2+y^2)^{\frac{5}{2}}}+\frac{3m(a-x)y}{[(a-x)^2+y^2]^{\frac{5}{2}}}=0\ldots\ldots\ \ldots\ldots\ldots(1).$$

$$\frac{dI}{dy}=\frac{l(x^2-2y^2)}{(x^2+y^2)^{\frac{5}{2}}}-\frac{m[(a-x)^2-2y^2]}{[(a-x)^2+y^2]^{\frac{5}{2}}}=0\ \ldots\ldots\ldots\ldots\ldots(2).$$

From (1) $y=0$ ....(a),

$$\text{or } \frac{[(a-x)^2+y^2]^{\frac{5}{2}}}{[x^2+y^2]^{\frac{5}{2}}}=\frac{m}{l}\cdot\frac{a-x}{x}\ \ldots\ldots\ldots\ldots\ldots\ldots(b).$$

From (2) $\frac{[(a-y)^2+y^2]^{\frac{5}{2}}}{[x^2+y^2]^{\frac{5}{2}}}=-\frac{m}{l}\cdot\frac{(a-x)^2-2y^2}{x^2-2y^2}$ ....(c).

From (b) and (c) $y^2=x(a-x)/2$ ....(d).

From (b) $y=\pm x^{\frac{1}{5}}(a-x)^{\frac{1}{5}}\sqrt{\frac{m^{\frac{2}{5}}x^{\frac{8}{5}}-l^{\frac{2}{5}}(a-x)^{\frac{8}{5}}}{l^{\frac{2}{5}}x^{\frac{2}{5}}-m^{\frac{2}{5}}(a-x)^{\frac{2}{5}}}}$ ....(e).

By (a) and (d); $x=0$; $x=a$; that is, the lights themselves must be used, as bull's-eye. By (a) and (e) we obtain the additional condition $x=al^{\frac{1}{4}}/(l^{\frac{1}{4}}+m^{\frac{1}{4}})$, which is the point of minimum illumination on the line joining the two lights. Other critical points will be obtained by solving (d) and (e) simultaneously,—a task which seems to be almost impossible.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let A, B, be the lights, intensities m, n; E the center of the target, radius $ED=r$, $AB=a$, $AF=x$, $EF=s$, $\angle DAB=\theta$, $\angle CBA=\phi$.

$\therefore m\sin\theta/AD^2+n\sin\phi/BC^2=I$.

$\sin\theta=s/AD=s/\sqrt{s^2+(x+r)^2}$,

$\sin\phi=s/\sqrt{s^2+(a-x+r)^2}$.

$$\therefore \frac{ms}{\{s^2+(x+r)^2\}^{\frac{3}{2}}}+\frac{ns}{\{s^2+(a-x+r)^2\}^{\frac{3}{2}}}=I.$$

Differentiating with reference to s,

$$\frac{m(x+r)^2-2ms^2}{\{s^2+(x+r)^2\}^{\frac{5}{2}}}+\frac{n(a-x+r)^2-2ns^2}{\{s^2+(a-x+r)^2\}^{\frac{5}{2}}}=0\ \ldots\ldots\ldots\ldots\ (1).$$

Differentiating with respect to x,

$$\frac{m(x+r)}{\{s^2+(x+r)^2\}^{\frac{5}{2}}}=\frac{n(a-x+r)}{\{s^2+(a-x+r)^2\}^{\frac{5}{2}}}\ \ldots\ldots\ldots\ldots\ldots\ (2).$$

From (1) and (2), $z=\sqrt{\frac{1}{2}(x+r)(a-x+r)}$. This value of z in (2) gives

$$\frac{m(x+r)}{\{(x+r)(a+x+3r)\}^{\frac{3}{2}}}=\frac{n(a-x+r)}{\{(a-x+r)(2a+3r-x)\}^{\frac{3}{2}}},$$

n equation of the eighth degree to find x.

If $m=n$, $x=\frac{1}{2}a$, $z=(\frac{1}{2}a+r)\sqrt{\frac{1}{2}}=\frac{1}{2}(\frac{1}{2}a+r)\sqrt{2}$.

If $n=0$, $x=0$, $z=\frac{1}{2}r\sqrt{2}$.

III. Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

Let A be any position of target, $AD(=y)$ be perpendicular from A to BC, e line connecting the positions of the two lights. Let x equal part of $BC(=a)$ off by AD. By laws of light, intensity of light eived from B at A

$$\frac{m\sin\theta}{AB^2}=\frac{m}{x^2+y^2}\times\frac{y}{\sqrt{x^2+y^2}}=\frac{my}{(x^2+y^2)^{\frac{3}{2}}}.$$

Similarly, that received from $C=\dfrac{ny}{[(a-x)^2+y^2]^{\frac{3}{2}}}$.

Then total intensity at A or $u=my(x^2+y^2)^{-\frac{3}{2}}+ny[(a-x)^2+y^2]^{-\frac{3}{2}}$. ..(1).

$$du/dx=-3mxy(x^2+y^2)^{-\frac{5}{2}}+3n(a-x)y[(a-x)^2+y^2]^{-\frac{5}{2}}\ldots\ldots\ldots\ldots(2).$$

$$/dy=m(x^2+y^2)^{-\frac{3}{2}}-3my^2(x^2+y^2)^{-\frac{5}{2}}+n[(a-x)^2+y^2]^{-\frac{3}{2}}-3ny^2[(a-x)^2+y^2]^{-\frac{5}{2}}\ldots\ldots\ldots\ldots(3).$$

Equating (3) to 0, we have

$$y=0\ldots\ldots\ldots\ldots(4).$$

$$\frac{[(a-x)^2+y^2]^{\frac{5}{2}}}{(x^2+y^2)^{\frac{5}{2}}}=\frac{n(a-x)}{mx}\ldots\ldots\ldots\ldots(5).$$

Equating (3) to 0, we have

$$[(a-x)^2+y^2]^{-\frac{5}{2}}\{3ny^2-n[(a-x)^2+y^2]\}=(x^2+y^2)^{-\frac{5}{2}}[m(x^2+y^2)-3my^2],$$

$$\text{or } \frac{[(a-x)^2+y^2]^{\frac{5}{2}}}{(x^2+y^2)^{\frac{5}{2}}}=\frac{n[2y^2-(a-x)^2]}{m(x^2-2y^2)}\ldots\ldots\ldots\ldots(6).$$

Solving (4) and (6), $\dfrac{(a-x)^5}{x^5}=\dfrac{-n(a-x)^2}{mx^2}$, which gives

$$\left\{\begin{array}{l} x=a \text{ or } 0 \text{ or } \dfrac{a}{1-\sqrt[3]{m+n}} \\ \text{and } y=0, \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

From (5) and (6) $\dfrac{n(a-x)}{mx}=\dfrac{n[2y^2-(a-x)^2]}{m(x^2-2y^2)}$, and $y^2=\dfrac{x(a-x)}{2}$......(8)

Instead of finding second differential coefficients, substitute from (7) in (1), $x=a$ and $x=0$, make $n=\infty$. $x=\dfrac{a}{1-\sqrt[3]{n+m}}$, makes $n=0$.

We can show that (8) does not produce any new condition for a maximum. To make y real x is not <0 nor $>a$.

If $x=a$ or 0, we have the values found in (7). Now for any value of x between 0 and a, y in (8) is seen to be finite, and n in (1) is also finite.

So $x=a$ or $x=0$ with $y=0$, producing the only infinite values of n indicate the positions of maximum intensity of illumination to be directly in front of either light.

PROBLEMS.

59. **Proposed by MOSES C. STEVENS, M. A., Department of Mathematics, Purdue University, Lafayette, Indiana.**

Solve $n\dfrac{d^2y}{dx^2}(x^2+y^2)^{\frac{1}{2}}=\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$.

[*From Forsyth's Differential Equations.*]

60. **Proposed by SETH PRATT, C. E., Assyria, Michigan.**

To remove $(1/a)$th of the volume of a sphere of a given radius by a conical hole, whose axis is the axis of a sphere, and whose vertex is at the surface of the sphere. Required the height of the cone and the diameter of its base.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

NOTE ON PROBLEM 26.

After carefully reading Dr. Martin's "Reply to Replies on Problem 26," we see no reason for changing our opinion respecting the solution we have been defending. We may, however, be led to agree with Dr. E. H. Moore, Dr. William

Hoover, and Prof. Henry Heaton, that there is no *correct* solution of the problem. That is to say, in so much as the problem is stated in the indefinite form, a solution taking any one of the elements of a triangle of which the area is a function will lead to a correct result. But it does seem to us that this statement, while it is true in general, does not apply in this case. It was asked of us during the summer whether anyone had sent in a solution assuming the altitude as the variable. Now we think it is quite clear that a solution which assumes the altitude of the triangle as the variable can, in no way be correct, for the solution would include not only right triangles, but oblique triangles as well. The result is

$$2\int_0^{\frac{1}{2}a} \tfrac{1}{2}apdp \div \int_0^{\frac{1}{2}a} dp=\tfrac{1}{4}a^2, \text{ where } p \text{ is the altitude.}$$

But if p is made a function of the angle at the center of the circle subtended by a side, the result will be $\dfrac{a^2}{2\pi}$. We think, however, that this controversy has been carried on long enough, and therefore it is desirable that it close without further discussion. EDITOR.

NOTE ON PROBLEM 29.

BY HENRY HEATON.

When the points to which r is measured are distributed symmetrically with respect to the minor axis the different radii vectores may be arranged in pairs such that the sum of the lengths of each pair will be $2a$. Hence, using Dr. Hoover's notation, m'' and m''' each equal a. m''' may be shown to equal a by the calculus, thus :

$$r=a-ex,\ d\theta=\frac{(a^2-e^2a^2)^{\frac{1}{2}}\,dx}{(a^2-x^2)^{\frac{1}{2}}}.$$

$$\text{Hence } m'''=\int_{-a}^{+a}\frac{(a-ex)(a^2-e^2x^2)^{\frac{1}{2}}\,dx}{(a^2-x^2)^{\frac{1}{2}}}\div\int_{-a}^{+a}\frac{(a^2-e^2x^2)^{\frac{1}{2}}\,dx}{(a^2-x^2)^{\frac{1}{2}}}$$

$$=a\int_{-a}^{+a}\frac{(a^2-e^2x^2)^{\frac{1}{2}}\,dx}{(a^2-x^2)^{\frac{1}{2}}}\div\int_{-a}^{+a}\frac{(a^2-e^2x^2)^{\frac{1}{2}}\,dx}{a^2-x^2}=a.$$

A fourth very obvious case of this problem is when the distances are measured at equal intervals of time.

$$\text{Then } m''''=\int rdA\div\int dA=\int_0^{\pi}\frac{r^3}{2}\,d\theta\div\int_0^{\pi}\frac{r^2}{2}d\theta$$

$$=\frac{a^2(1-e^2)^2}{b\pi}\int_0^{\pi}\frac{d\theta}{(1-e\cos\theta)^3}=\frac{a^2}{2b}\Big(3(4+e^2)(1-e^2)^{\frac{1}{2}}-10(1-e^2)^2\Big).$$

Corollary. Let $e=0$, then $m''''=a$, as it evidently should.

32. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the average area of the random sector whose vertex is a random point in a given circle.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let P be the random point. Through P draw the chords AC, DE forming the sector DPC. From A draw the diameter AB and the chord AD. Let $AB=2r$, $AP=z$, $\angle BAP=\varphi$, $\angle BAD=\theta$, area $DPC=u$, $\Delta=$required average. Then

$$u=r^2(\varphi-\theta-\tfrac{1}{2}\sin 2\theta+\tfrac{1}{2}\sin 2\varphi)-rz\cos\theta\sin(\varphi-\theta).$$

The limits of θ are $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; of φ, θ and $\frac{1}{2}\pi$; of z, 0 and $2r\cos\varphi=z'$.

$$\therefore\ \Delta=\frac{\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{\theta}^{\frac{1}{2}\pi}\int_{0}^{z'}u\,d\theta\,d\varphi\,z\,dz}{\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{\theta}^{\frac{1}{2}\pi}\int_{0}^{z'}d\theta\,d\varphi\,z\,dz}=\frac{2}{\pi^2r^2}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{\theta}^{\frac{1}{2}\pi}\int_{0}^{z'}u\,d\theta\,d\varphi\,z\,dz,$$

$$=\frac{2r^2}{3\pi^2}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{\theta}^{\frac{1}{2}\pi}[3\cos^2\varphi(2\varphi-2\theta-\sin 2\theta+\sin 2\varphi)-8\cos\theta\cos^3\varphi\sin(\varphi-\theta)]d\theta\,d\varphi$$

$$=\frac{r^2}{12\pi^2}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}(3\pi^2-12\pi\theta+\theta^2-16\cos^4\theta+4\sin^2\theta\cos^2\theta)d\theta=\frac{37\pi r^2}{144}-\frac{5r^2}{8\pi}.$$

Also solved by the *PROPOSER*.

33. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the average area of all regular polygons having a constant apothem.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let a=constant apothem, $2x$=side, 2θ=central angle of polygon.

$$\therefore\ \frac{\pi}{\theta}=\text{number of sides},\quad \frac{\pi a x}{\theta}=\text{area of polygon}.$$

$$\therefore\ \Delta=\text{average area}=\pi a\frac{\int_0^{a\sqrt{3}}\frac{x\,dx}{\theta}}{\int_0^{a\sqrt{3}}dx}=\frac{\pi}{\sqrt{3}}\int_0^{a\sqrt{3}}\frac{x\,dx}{\theta}$$

$$=\frac{\pi a^2}{\sqrt{3}}\int_0^{\frac{1}{3}\pi}\frac{\tan\theta\sec^2\theta\,d\theta}{\theta}\quad\text{where } x=a\tan\theta$$

$$=\frac{3\sqrt{3}a^2}{2}+\frac{\pi a^2}{2\sqrt{3}}\int_0^{\frac{1}{3}\pi}\left(\frac{\tan\theta}{\theta}\right)^2 d\theta$$

$$=\frac{3\sqrt{3}a^2}{2}+\frac{\pi a^2}{2\sqrt{3}}\int_0^{\frac{1}{3}\pi}(1+\tfrac{2}{3}\theta^2+\tfrac{17}{45}\theta^4+\tfrac{62}{945}\theta^6+\ldots\ldots)d\theta,$$

$$=\frac{3\sqrt{3}a^2}{2}+\frac{\pi^2a^2}{6\sqrt{3}}\left(1+\frac{2\pi^2}{81}+\frac{17\pi^4}{18225}+\frac{62\pi^6}{1607445}+\ \ldots\ldots\right)=3.8693a^2 \text{ nearly.}$$

Also solved by the *PROPOSER*.

. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

Two points are taken at random on the circumference of a semicircle. Find the chance that their ordinates fall on either side of a point taken at randon on the diameter.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let P be the random point on the diameter AE. Draw BP perpendicular to AE. Then one point must fall somewhere, as at C, on arc AB, the other somewhere, as at D, on arc BE. The chance thus obtained must be doubled as D might fall on AB and C on BE.

Let AO=unity, $\angle BOA=\theta$, $\angle COA=\varphi$, $\angle DOA=\psi$.

Then $OP=\cos\theta$. $\therefore d(OP)=-\sin\theta d\theta$.

Let p=required chance.

$$\text{Then } p=\frac{\int_0^\pi\int_0^\theta\int_\theta^\pi \sin\theta d\theta d\varphi d\psi}{\int_0^\pi\int_0^\pi\int_0^\pi \sin\theta d\theta d\varphi d\psi}=\frac{1}{\pi^2}\int_0^\pi\int_0^\theta\int_\theta^\pi \sin\theta d\theta d\varphi d\psi$$

$$=\frac{1}{\pi^2}\int_0^\pi(\pi\theta-\theta^2)\sin\theta d\theta=\frac{4}{\pi^2}.$$

PROBLEMS.

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48. Proposed by CHARLES E. MYERS, Canton, Ohio.

A attends church 4 Sundays out of 5; B, 5 Sundays out of 6; and C, 6 Sundays out of 7. What is the probability of an event that A and B will be at church and C will not?

49. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

In a circle whose radius is a, chords are drawn through a point distant b from the center. What is the average length of such chords, (1), if a chord is drawn from every point of the circumference, and (2), if they are drawn through the point at equal angular intervals?

EDITORIALS.

Prof. John N. Lyle, of Westminster College, has resigned his position on account of ill health, and is now living in Bentonville, Arkansas.

We shall be pleased to have our subscribers send us the names of persons likely to subscribe for the MONTHLY, in order that we may send such persons sample copies.

Any reader of the MONTHLY having a copy of *Salmon's Higher Plane Curves*, third edition, and wishing to sell the same, should write to us stating the price of the book.

We have only six complete sets of Volumes I and II, of the MONTHLY. Volume I will be sent to any address in the United States or Canada for $2.00; Volume II will be sent on receipt of $2.50.

Prof. Robert J. Aley, of Indiana University, is now studying mathematics in the University of Pennsylvania, having received a Mathematical Fellowship in that Institution last spring.

In our August-September number, we sent out bills to all those who are owing us. We hope that the matter of remittance may receive the attention of all those who are in arrears, as the MONTHLY is greatly in need of funds. All bills not paid by December 31st will be sent to an attorney for collection.

Prof. A. B. Nelson, of Centre College, Kentucky, says, in a letter of October 13th, "You deserve the thanks of mathematicians in this country for your self-sacrificing labors in behalf of our favorite science." We desire to thank Professor Nelson as well as many others who have thus expressed their appreciation of our labor. Surely it is a labor of love.

BOOKS AND PERIODICALS.

Elementary Solid Geometry and Mensuration. By Henry Dallas Thompson, D. Sc., Ph. D., Professor of Mathematics in Princeton University. 8vo. Cloth, 200 pages. Price, $1.25. New York: The Macmillan Co.

In this book, the author lays the foundation of his subject in clear cut and accurate *definitions* and well illustrated *postulates*. The diagrams are very fine, showing very accurately to the eye the relation of the points, lines, and planes. There are numerous original exercises scattered throughout the book. B. F. F.

The Elements of Algebra. Adapted for use in High Schools, Academies, and Colleges. By Lyman Hall, Graduate United States Military Academy, and Professor of Mathematics, Georgia School of Technology. 8vo. Cloth and Leather Back, 368 pages. Chicago: American Book Co.

This work is intended for beginners who have mastered the principles of any good common school Arithmetic. The familiar methods of arithmetic are preserved, in order to gradually convince the student that algebra is merely an extension of the mathematical knowledge he already possesses. *Preface.* B. F. F.

Trigonometry for Beginners. By the Rev. J. B. Lock, M. A., Fellow of Gonville and Caius College, Cambridge, Formerly Master at Eaton. Revised and Enlarged for the use of American Schools, by John A. Miller, A. M., Assistant Professor of Mathematics, Leland Stanford Jr. University, Professor (elect) of Mechanics and Mathematical Astronomy, Indiana University. Large 8vo. Cloth, 148 and 64 pages. Price, $1.10.

As we have not seen the original book, we do not know just how materially Professor Miller has changed it. He tells us in his Preface that it differs from the original, chiefly in the following particulars: (1) The subject matter of Chapter VII formerly followed that of Chapters VIII and IX; (2) the addition formulæ are proved for angles of any magnitude, and for more than two angles; (3) a chapter on Inverse Trigonometric Functions; and two chapters on Spherical Trigonometry have been added; (4) logarithmic and trigonometric tables have been inserted. Some of the trigonometrical formulæ are very neatly established by Geometrical Proof. B. F. F.

A School Algebra. Designed for use in High Schools and Academies. By Emerson E. White, A. M., LL. D., Author of "Series of Mathematics," "Elements of Pedagogy," "School Management," etc. 8vo. Cloth and Leather Back, 394 pages. Chicago: American Book Co.

The author's aim has been to prepare a school algebra that is pedagogically sound as well as mathematically accurate. Few educators will question Dr. White's ability to write a work pedagogically sound, but many mathematicians, upon examination of his treatment of *Undetermined Coefficients*, Chapter XXI., will question the mathematical accuracy of his text on algebra. His treatment of Undetermined Coefficients is that given in most algebras written during the last and present century. This demonstration is now pretty generally admitted to be incorrect, and correct demonstrations are being published in most recent works. However, upon the whole, the book is one well suited for the purpose for which it is written. B. F. F.

Elements of Geometry. By Andrew W. Phillips, Ph. D., and Irving Fisher, Ph. D., Professors in Yale University. Large 8vo. Cloth and Leather Back, 540 pages. Price, $1.75. New York: Harper & Bros.

There are several features in this work that make it especially interesting. Of these the most prominent are the beautiful diagrams. These are photo-engravings arranged side by side with skeleton drawings of geometrical figures. The photographs were taken from actual models recently constructed for use in the class-rooms of Yale University. In this respect the work excels anything that has yet appeared in this country. The work is characterized by clearness of presentation, both in the form of the diagrams and the natural and symmetrical methods of proof. The book closes with a short but very clear treatment of Modern Geometry. This will be helpful to those teachers who desire a knowledge of the three kinds of Geometries. We believe that this work is destined to be very extensively used throughout the country. B. F. F.

A History of Elementary Mathematics, with Hints and Methods of Teaching. By Florian Cajori, Ph. D., Professor of Physics in Colorado College. 8vo. Cloth, 304 pages. Price, $1.50. New York: The Macmillan Co.

The book is by no means an abridged edition of the author's *History of Mathematics*. It is an entirely new book giving a somewhat detailed account of the rise, struggle, and progress of Arithmetic, Algebra, and Geometry. The book should be read by all teachers of these subjects, and by mathematical students generally. B. F. F.

The Cosmopolitan. An Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year. Single numbes, 10 cents. Irvington-on-the-Hudson, New York.

The October number contains the following: A Summer Tour in the Scottish Highlands; The Story of a Child Trainer; The Perils and Wonders of a True Desert; A Modern Fairy Tale; Hofman's Object Lesson; Personal Recollections of the Tai-Ping Rebellion; The Modern Woman Out-of-Doors; The True History of our Cooks; To a Hyacinth Bulb (poem). B. F. F.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year. Single number, 25 cents. The Review of Reviews Co., New York.

In the September and October numbers of *The Review of Reviews*, the editor has given a remarkably fair and unprejudiced account of the progress of the present political campaign. It is a great satisfaction, after having read statements in the daily papers, which are believed to be misrepresenting, to go to *The Review of Reviews* and get the facts there given by its able editor. The November number contains a very able article on the "Summing Up of the Vital Issues of 1896," by Rev. Dr. Lyman Abbott. Also the question "Would Free Coinage Benefit Wage Earners?" is debated by Dr. Chas. B. Spahr and Prof. Richmond Mayo-Smith. This number also offers a remarkable symposium of current thought on "What Should be Done with Turkey?" The MONTHLY suggests in answer to this question, that Turkey be given a material and substantial roast by the civilized world.

B. F. F.

ERRATA.

After the word, ellipses, page 181, problem 60, insert, "passing through the foci of a given ellipse and having the tangents at the ends of the major axes for directrices."

Page 205, line 1, for "[illegible]" read [illegible].

Page 205, line 1, for "[illegible]" read [illegible].

Page 205, line 12, for "$4a^3$" read $2a^3$.

Page 206, line 3, for "$(5x^{30}$" read $(5x)^{30}$.

Page 217, line 15, for "y" read z.

Throughout the solution to problem 34, Mechanics, for "E_0^m" read F_0^m

// THE
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NUMBER AND FRACTIONS.

By J. K. ELLWOOD, A. M., Pittsburg, Pennsylvania.

A clear understanding of what *number* is and what gives rise to the number idea removes all difficulty from the grasping of the *fraction* idea.

Number does not inhere in objects, cannot be perceived by the senses; otherwise the mere presentation of 2, 3......n objects to the senses would give rise to the idea of number. There is in every sound mind a *measuring* instinct, which, in the nature of things, is just as essential to life and progress as is memory. Both the physical and ideal worlds are full of entities—vague wholes—which the mind must *measure* for the purpose of making them more definite. Measuring requires a "unit of measure." Naturally the first measurements made by a child are vague; as when he measures (counts) the chairs in a room, the marbles in his pocket, the fingers on his hand. His units of measure—chair, marble, finger—are indefinite, as are the results of his processes. A later stage involves *exact* measurements; i. e., an exactly defined unit of measure is used. A whole (of quantity), say a piece of cloth, is to be measured—made definite in value. A *yard* (exactly defined as 3 feet or 36 inches) is taken as the unit and applied (say) *ten* times. Then *ten* repetitions of the unit is the *number*. Considered by itself the *ten* is *pure number*, the result of a purely mental process; it expresses the *ratio* of the measured *quantity* to the measuring unit. Applied to the unit of measure, then ten expresses the numerical value of the measured quantity—10 yards of cloth. This ten yards, it is evident, is *quantity*, not number. It is what arithmetics erroneously call "concrete number." In this example the pure number indicates either of two things: (a) that the unit is taken *ten times*, or (b) that ten parts (units) are taken *one time*. It answers the question "how many?" Applied to the unit, it answers the question "how much?"

The number and unit of measure *together* give the absolute magnitude of the quantity; the number *alone* gives the relative value. Hence we may say that *number is the ratio of the quantity measured to the unit of measure.*

It is plain that *any* quantity may be used as a unit of measure. Measurement is more exact when this unit is itself made up of a definite number of equal parts—measured by some other unit, which may be called "primary" to distinguish it from the actual or direct unit of measure, which may be called "derived." Thus, if the unit of measure is three feet and it is taken ten times, we have the primary unit *one foot*, the derived unit *three feet*, and the number of derived units, *ten*. We have ten *threes*. To find the number of primary units we use multiplication, which gives thirty *ones*; the quantity is now more definite.

Again, in the quantity $5\times\$3$, the primary unit is \$1, the derived (direct, actual) unit \$3, five of which=15 primary units.

The derived unit is not necessarily a *multiple* of the primary unit; it may be one or more of its *equal parts*. Thus in $\$\frac{5}{2}$, the primary unit is, as above, \$1, while the derived unit is $\$\frac{1}{2}$, the number of them *five*. The fraction $\frac{5}{2}$ expresses the ratio of the measured quantity ($\$\frac{5}{2}$) to the primary unit (\$1). The numerator shows how many derived units make up the quantity, the denominator shows the relation between the derived and primary units. It is thus seen that the fraction involves no new idea. Its notation is more complete than that of the integer in that it defines the derived unit—makes *explicit* what is implied in the integral notation. This appears in the processes of finding the value of 5 hats (*a*) at \$3 each, (*b*) at $\$\frac{1}{2}$ each.

$$5\times\$3=5\times\overline{3\times\$1}=15\times\$1=\$15.$$
$$5\times\$\tfrac{1}{2}=5\times\overline{\tfrac{1}{2}\times\$1}=\tfrac{5}{2}\times\$1=\$\tfrac{5}{2}.$$

The denominator 2 shows the relation between the derived unit ($\$\frac{1}{2}$) and the primary unit (\$1). In \$15, however, there is nothing to show the relation between \$3 and \$1. (This is seen in $5\times\overline{3\times\$1}$). In no other respect does the fraction differ from the integer. Both 15 and $\frac{5}{2}$ express ratio to the primary unit \$1. The 15 shows the number of primary units, but not that of the derived units. The $\frac{5}{2}$ shows both; there are 5 derived units, $\frac{5}{2}$ primary units.

In view of these facts it appears that a correct definition of *number* includes that of *fraction*, which is simply a number whose notation gives a more complete statement of the mental processes by which number is constituted. For mathematical purposes *Newton's* definition cannot be much improved: "Number is the abstract ratio of one quantity to another quantity of the same kind." Ratio being a pure abstraction, the word "abstract" should be omitted. Euler says, "Number is the ratio of one quantity to another quantity taken as unit." Drs. McLellan and Dewey define number as, "The repetition of a certain magnitude used as the unit of measurement to equal or express the comparative value of a magnitude of the same kind."*

*In conclusion I wish to say that every live teacher should read "*The Psychology of Number*," It is clear that $\frac{1}{n}$ of any magnitude may be repeated as a unit just as well as $\frac{n}{n}$ or $\frac{8n}{n}$; it is equally plain that $\frac{m}{n}$ is as much an expression of ratio as is m. Hence each definition applies to fractions as well as integers.

It is neither necessary nor advisable to divide ("break") single things (individuals, as apples) into parts in order to get fractions. In counting the eggs in a dozen (e. g.) the wee bairn is on the border of the fairyland of fractions, though he may not be conscious of it. At any stage of his counting the result is either integral or fractional. Five eggs is integral with respect to the *unit* (1 egg); it is fractional with respect to the *unity* or whole (dozen)—5 out of 12, 5 *twelfths*. Five *half-yards* is just as integral as 5 *yards*. The *ratio* in each is *five*. But in $\frac{5}{2}$ yards the ratio is $\frac{5}{2}$; the fractional idea is present, owing to the denominator, which defines the unit of measure.

SOME TRIGONOMETRIC RELATIONS PROVED GEOMETRICALLY.

By F. H. PHILBRICK, C. E., Plaeville, Louisiana.

Most trigonometric formulæ may be proven geometrically in an elegant manner; and moreover, the relations between the trigonometric functions may be shown at a glance by means of the geometric figures. The results are all the more interesting, too, when proven also directly from first principles. For this reason the following exercises are offered.

For convenience, describe the arc AYX, and take the radius AC for the unit of measurement. Let the arc $AX=x$ and arc $AY=y$. Take M at the middle of XY, and draw lines as indicated.

Then $DY=\sin y$, $HX=\sin x$, $EM=\sin\frac{1}{2}(x+y)$, $KY=\sin\frac{1}{2}(x-y)$, $NX=\sin x-\sin y$, $NY=\cos y-\cos x$, $CE=\cos\frac{1}{2}(x+y)$, $CK=\cos\frac{1}{2}(x-y)$.

Now, $HX+DY=2KF=2EM\frac{KF}{EM}=2EM\cdot\frac{CK}{CM}=2EM.CK.$

That is, $\sin x+\sin y=2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$............(1).

by Drs. McClellan and Dewey. It is interesting in matter, vigorous and aggressive in style, refreshing in its originality, and scholarly in its conception and execution. It is in the 33d volume of the International Education Series, published by D. Appleton & Co., New York.

Again, $CH+CD=2CF=2CE.\frac{CF}{CE}=2CE\frac{CK}{CM}=2CE.CK$,

$$\text{or } \cos x+\cos y=2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y) \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The triangles CEM and XNY are similar;

$$\text{hence } \frac{NX}{NY}=\frac{CE}{CM}, \text{ or } NX=2CE\frac{\frac{1}{2}XY}{CM}=2CE.KY,$$

$$\text{that is, } \sin x-\sin y=\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y) \ldots\ldots \ldots\ldots (3).$$

Similarly, $\frac{NY}{XY}=\frac{EM}{CM}$, or $NY=2EM\frac{\frac{1}{2}XY}{CM}=2EM.KY$,

$$\text{or } \cos x-\cos y=-2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y) \ldots\ldots\ldots\ldots\ldots (4).$$

Equation (1) can be made very useful in computing trigonometric tables, as the writer intends subsequently to show.

Now let $AM=x$ and $MY=MX=y$. Then $AY=x-y$ and $AX=x+y$. We have $(CM)^2-(CK)^2=(CY)^2-(CK)^2=(KY)^2$. But $\frac{CM}{ME}=\frac{CK}{KF}=\frac{KY}{LY}$.

Therefore $(ME)^2-(KF)^2=(LY)^2=(KY)^2-(KL)^2$,

or $(KF)^2-(KL)^2=(ME)^2-(KY)^2$,

or $(KF+KL)(KF-KL)=(ME)^2-(KY)^2$,

or $HX\times DY=(ME)^2-(KY)^2$.

That is, $\sin(x+y)\sin(x-y)=\sin^2x-\sin^2y=\cos^2y-\cos^2x \ldots\ldots\ldots\ldots\ldots\ldots (5)$.

Again, $\frac{CM}{CE}=\frac{CK}{CF}=\frac{KY}{KL}$.

Therefore $(CE)^2-(CF)^2=(KL)^2=(KY)^2-(LY)^2$,

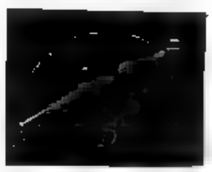
or $(CF)^2-(LY)^2=(CE)^2-(KY)^2$.

or $(CF-LY)(CF+LY)=CH\times CD=(CE)^2-(KY)^2$.

That is, $\cos(x+y)\cos(x-y)=\cos^2x-\sin^2y=\cos^2y-\sin^2x \ldots\ldots\ldots\ldots\ldots\ldots (6)$.

Let $DC=R$ the radius of a circle. Let the angle $CDB=2x$. Then $DAB=DBA=CBE=x$.

Then we have $\tan x=\frac{EC}{EB}$, also $\tan x=\frac{BE}{AE}$.



The product of these gives, $\tan^2x=\frac{CE}{AE}$, or $CE\times AE=(BE)^2$,

$$\text{or } \frac{EC}{AE}=\left(\frac{BE}{AE}\right)^2=\tan^2x.$$

Also, $\frac{EC}{BE}=\frac{\text{vers}2x}{\sin 2x}=\frac{1-\cos 2x}{\sin 2x}=\frac{\sin 2x}{1+\cos 2x}=\tan x$ [see above]........(7).

Then $1+\tan^2 x=1+\frac{EC}{AE}=\frac{AC}{AE}=\frac{2R}{AE}$,

$$1-\tan^2 x=1-\frac{EC}{AE}=\frac{AE-EC}{AE}=\frac{AC-2EC}{AE}=\frac{2(R-EC)}{AE}.$$

$\cot 2x=\frac{DE}{BE}$ and $\operatorname{cosec}2x=\frac{R}{BE}$. From these values we at once have,

$$\frac{2\tan x}{1+\tan^2 x}=\frac{2BE}{AE}\cdot\frac{AE}{2R}=\frac{BE}{R}=\sin 2x \quad\ldots\ldots(8).$$

$$\frac{2\tan x}{1-\tan^2 x}=\frac{2BE}{AE}\cdot\frac{AE}{2(R-EC)}=\frac{BE}{R-EC}=\frac{BE}{DE}=\tan 2x \quad\ldots\ldots(9).$$

$$\tan^2 x+2\cot 2x\tan x=\frac{EC}{AE}+\frac{2DE}{BE}\cdot\frac{BE}{AE}=\frac{EC+2DE}{AE}=\frac{AE}{AE}=1 \quad\ldots\ldots(10).$$

$$2\operatorname{cosec}2x\tan x-\tan^2 x=\frac{2R}{BE}\cdot\frac{BE}{AE}-\frac{EC}{AE}=\frac{2R-EC}{AE}=\frac{AE}{AE}=1 \quad\ldots\ldots(11).$$

$$\frac{1-\tan^2 x}{1+\tan^2 x}=\frac{2(R-EC)}{AE}\cdot\frac{AE}{2R}=\frac{R-CE}{R}=\frac{ED}{R}=\cos 2x \quad\ldots\ldots(12).$$

$$\operatorname{cosec}2x-\cot 2x=\frac{R-ED}{BE}=\frac{EC}{BE}=\tan x=\frac{1-\cos 2x}{\sin 2x}=\frac{\sin 2x}{1+\cos 2x} \quad\ldots\ldots(13).$$

$$\operatorname{cosec}2x+\cot 2x=\frac{R+ED}{BE}=\frac{AE}{BE}=\cot x=\frac{\sin 2x}{1-\cos 2x}=\frac{1+\cos 2x}{\sin 2x} \quad\ldots\ldots(14).$$

$$\frac{1+\sin 2x-\cos 2x}{1+\sin 2x+\cos 2x}=\frac{R+BE-ED}{R}\div\frac{R+BE+ED}{R}=\frac{EC+BE}{AE+BE}.$$

But $AE=\frac{(BE)^2}{EC}$; $\therefore\ \frac{CE+BE}{AE+BE}=\frac{EC+BE}{\frac{(BE)^2}{EC}+BE}$

$$=\frac{EC(EC+BE)}{BE(EC+BE)}=\frac{EC}{BE}=\tan x \quad\ldots\ldots(15).$$

Again, $\cos x=\frac{AE}{AB}$, also $\cos x=\frac{AB}{AC}$.

Twice the product of these gives $2\cos^2 x=\frac{2AE}{AC}=\frac{AE}{R}$.

Also $\cos 2x=\frac{DE}{R}$. $\quad 1+\cos 2x=\frac{DE+R}{R}=\frac{AE}{R}$. $\quad\therefore 1+\cos 2x=2\cos^2 x$......(16).

$\sin x=\frac{CB}{AC}=\frac{BC}{2R}$; also $\sin x=\frac{EC}{BC}$. Twice the product of these gives

$2\sin^2 x=\frac{EC}{R}$. $\quad 1-\cos 2x=\frac{R-ED}{R}=\frac{EC}{R}$. $\quad\therefore 1-\cos 2x=2\sin^2 x$..........(17).

To prove the "Tangent Proportion," let ABC be a plane triangle, the parts being represented as usual. Take $CE=CA$ and draw AEH. Draw BHK perpendicular to AH, to meet AC prolonged in K. Now considering the triangles ABC and ACE, the sum of the angles at A and E of the one is equal to the sum of the angles at A and B of the other. Hence $CAE+CEA=A+B$; and $CAE=CEA=BEH=\frac{1}{2}(A+B)$.

Also $BAE=A-\frac{1}{2}(A+B)=\frac{1}{2}(A-B)$. The angles at B and K of the triangle BCK are equal; for CBK is the complement of BEH or AEC, and BKC is the complement of the equal angle CAE. Hence $CK=CB=a$ and $AK=a+b$.

Now $\tan\frac{1}{2}(A-B)=\frac{BH}{AH}$ and $\tan\frac{1}{2}(A+B)=\frac{HK}{AH}$. $\quad\therefore \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{BH}{HK}$.

But $\frac{BH}{HK}=\frac{BE}{AK}=\frac{a-b}{a+b}$. $\quad\therefore \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{a-b}{a+b}$ ....(1).

From the triangle ABE, $\frac{BE}{AB}=\frac{\sin BAE}{\sin AEC}$, or $\frac{a-b}{c}=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}$..(2).

In the triangle AHK, $AH=AK\cos HAK=(a+b)\cos\frac{1}{2}(A+B)$.

In the triangle ABH, $AH=AB\cos BAH=c\cos\frac{1}{2}(A-B)$.

Equating, we have, $\frac{a+b}{c}=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}$ ....(3).

Equation (3) divided by (2) also gives (1).

TWO PERPENDICULARS TO A TRANSVERSAL.

By JOHN N. LYLE, Ph. D., Bentonville, Arkansas.

Do two perpendiculars to a transversal intersect?

Both Euclid and Lobatschewsky affirm that they do not. Euclid regards the two perpendiculars as equidistant, whilst Lobatschewsky considers them as diverging.

Experience confirms the view that the distance between the perpendiculars is a constant. As long as this is the case it is evident that intersection is impossible. If the perpendiculars do not approach each other within the range of observation and experience what would anology and induction indicate? Would they not unmistakably favor the hypothesis that the perpendiculars do not intersect beyond the limits of observation and experience? Our knowledge of *the here* and *the now*, if at all accurate, must assuredly count for something *elsewhere* and *tomorrow*.

But aside from conclusions based on purely empirical data and obtained by analogical and inductive processes the assumption that a straight line that has a beginning and an end is *infinite* involves contradiction and is therefore absurd. One end of each perpendicular is at the transversal. If these perpendiculars intersect each of them has two ends. But *two* ends is the distinctive characteristic of a finite straight line.

The further assumption that the intersection takes place at a hypothetical place called "infinity" does not remove the difficulty in the slightest. *Two* ends are still attributed to the *supposed* infinite line.

There is in reality a new difficulty and a very serious one, for the logical law of non-contradiction is violated.

The difficulty is not that the human mind by reason of its limited powers is unable to cognize an unlimited straight line and discover what will or will not take place "at infinity," but it is that the mind by reason of the logical law of non-contradiction can not cognize a line that is at the same time both unlimited and limited.

As a result of this brief investigation we find that there are insuperable difficulties, logical, geometrical, and philosophical, in the hypothesis that two perpendiculars to a transversal intersect at a supposed place called "infinity.".

Notwithstanding these difficulties in the way of this hypothesis many analysts daily and habitually accept it. They do make the "assumption that parallel lines, extended to an infinite distance, do intersect."

Euclid flatly contradicts this hypothesis in his statement that "parallels never meet however far they may be produced." In favor of Euclid's statement there is nothing in logic, science or geometry known to man that conflicts with it. I understand Mr. Drummond's protest to extend not only to Euclid's assumption but also to the assumption that Euclid contradicts.

If the analysts "can not comprehend the infinite" why do they employ the symbol of the infinite so freely in their equations and decide without hesitation so many questions against the Alexandrian geometer? The analysts make large use of the symbol ∞ in their equations. Do they or do they not comprehend the meaning of the symbolism employed? If they find ∞ incomprehensible, can they not obtain all legitimate results by the aid of *finite* quantities alone?

DEVELOPMENT OF SINθ AND COSθ.

By J. M. BANDY, Trinity College, Trinity, North Carolina.

In discussing the power of the calculus with my own students in Trinity College, I, several years ago, sprung the question "why can the trigonometric functions, sine and cosine, be developed by series?"

The calculus very readily furnished the series; but it did not expose the exponential nature of the functions.

The fact that the value of the functions can be expressed by series forced me to the conclusion that the reason existed in the nature of the functions themselves, and, therefore, they should yield this result directly.

Before proceeding to obtain the series directly from the functions, it will be necessary to produce a series involving an exponential function. The object thereafter will be to trace the law which connects sine and cosine with this exponential function.

We will develop $\left.\left(1+\frac{1}{x}\right)^x\right]_\infty$ which gives us a simple converging series. This series can be made to express an exponential function.

Denoting $\left.\left(1+\frac{1}{x}\right)^x\right]_\infty$ by e; that is, as x increases indefinitely, the *limiting value* of this function $\left.\left(1+\frac{1}{x}\right)^x\right]_\infty$ is e.

$\therefore\ e=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}$, etc.* From this we get

$$e^\theta=\left\{\left.\left(1+\frac{1}{x}\right)^x\right]_\infty\right\}^\theta=1+\theta+\frac{\theta^2}{1.2}+\frac{\theta^3}{1.2.3}+\text{etc.},\ \ldots\ldots\ldots\ldots\ldots(1),$$

$$e^{\frac{1}{x}}=\left\{\left.\left(1+\frac{1}{x}\right)^x\right]_\infty\right\}^{\frac{1}{x}}=1+\frac{1}{\infty},\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

---

*This gives $e=2.71828$, the Naperian base.

and $\log\left(1+\frac{1}{\infty}\right)=\frac{1}{\infty}\log e$ ....(3).

To expose the principles which connect $\sin\theta$ and $\cos\theta$ with the above equations, and thus show that they can be expressed by series.

By geometry, $\cos^2\theta+\sin^2\theta=1$ ....(4).

The first member of (4) may be expressed thus: $\cos^2\theta-(-\sin^2\theta)=1$.

(4), therefore, becomes $\cos^2\theta-(-\sin^2\theta)=1$.............................. (5).

Factoring first member of (5), we have,

$$(\cos\theta+\sin\theta\sqrt{-1})(\cos\theta-\sin\theta\sqrt{-1})=1 \ldots\ldots (6).$$

Taking log. of (6), we have $\log(\cos\theta+\sin\theta\sqrt{-1})+\log(\cos\theta-\sin\theta\sqrt{-1})=0$,

or $\log(\cos\theta+\sin\theta\sqrt{-1})=-\log(\cos\theta-\sin\theta\sqrt{-1})$..........................(7).

Denoting either member of (7) by y, we have,

$$\left.\begin{array}{r}\log(\cos\theta+\sin\theta\sqrt{-1})=y,\\ \text{and }\log(\cos\theta-\sin\theta\sqrt{-1})=-y,\end{array}\right\} \ldots\ldots (8).$$

$\therefore\ \cos\theta+\sin\theta\sqrt{-1}=10^y$, ...(9), and $\cos\theta-\sin\theta\sqrt{-1}=10^{-y}$.....(10).

Summing (9) and (10), $2\cos\theta=10^y+10^{-y}$..........................(11).

By trigonometry, $\cos^2\tfrac{1}{2}\theta=\tfrac{1}{2}(1+\cos\theta)=\tfrac{1}{4}(2+2\cos\theta)$

$=\tfrac{1}{4}(10^y+2+10^{-y})$, [from (11)]..............(12),

and $-\sin^2\tfrac{1}{2}\theta=\tfrac{1}{2}(\cos\theta-1)=\tfrac{1}{4}(2\cos\theta-2)=\tfrac{1}{4}(10^y-2+10^{-y})$, [from (11)]....(13).

Extracting square roots of (12) and (13),

$$\cos\tfrac{1}{2}\theta=10^{\frac{y}{2}}+10^{-\frac{y}{2}}, \ldots\ldots (14),$$

$$\text{and }\sin\tfrac{1}{2}\theta\sqrt{-1}=10^{\frac{y}{2}}-10^{-\frac{y}{2}} \ldots\ldots (15).$$

Adding (14) and (15), $\cos\tfrac{1}{2}\theta+\sin\tfrac{1}{2}\theta\sqrt{-1}=10^{\frac{y}{2}}$.......................(16).

Comparing (16) and (9), we see that θ may be changed into $\tfrac{1}{2}\theta$, provided that y is changed into $\tfrac{1}{2}y$. The same changes may, therefore, be made in (16): $\tfrac{1}{2}\theta$ may be changed into $\tfrac{1}{4}\theta$, if $\tfrac{1}{2}y$ is changed into $\tfrac{1}{4}y$. (16), therefore, becomes

$$\cos\tfrac{1}{4}\theta+\sin\tfrac{1}{4}\theta\sqrt{-1}=10^{\frac{y}{4}} \ldots\ldots (17).$$

Repeating this change, we have, $\cos\frac{1}{4}\theta+\sin\frac{1}{4}\theta\sqrt{-1}=10^{\frac{y}{4}}$ ....(18).

Thus we see that θ may be divided by any power of 2, however great, provided y is divided by the same power.

Let, then, $m=2^n$ ....(19).

We then have, $\cos\frac{1}{m}\theta+\sin\frac{1}{m}\theta\sqrt{-1}=10^{\frac{y}{m}}$ ....(20).

Taking log of (20), we have, $\log(\cos\frac{1}{m}\theta+\sin\frac{1}{m}\theta\sqrt{-1})=\frac{y}{m}$ ...... (21).

But when n in (19) becomes infinite, m becomes infinite.

$\therefore\ \cos\frac{1}{m}\theta$ in the limit equals 1, and $\sin\frac{1}{m}\theta\sqrt{-1}$ in the limit equals the arc. $\therefore$ (21) becomes $\log(1+\frac{\theta}{m}\sqrt{-1})=\frac{y}{m}$ ....(22).

But from (3), (22) becomes $\frac{\theta}{m}\sqrt{-1}\log e=\frac{y}{m}$, or $y=\theta\sqrt{-1}\log e$ .....(23).

Substituting this value of y in (8), $\log(\cos\theta+\sin\theta\sqrt{-1})=\theta\sqrt{-1}\log e$..(24),

and $\log(\cos\theta-\sin\theta\sqrt{-1})=-\theta\sqrt{-1}\log e$..............(25).

Whence $\cos\theta+\sin\theta\sqrt{-1}=e^{\theta\sqrt{-1}}$......................................(26),

and $\cos\theta-\sin\theta\sqrt{-1}=e^{-\theta\sqrt{-1}}$ ....(27).

Adding (26) and (27), and dividing by 2, $\cos\theta=\frac{1}{2}(e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}})$ ....(28),

by subtracting (27) from (26), and multiplying by $\sqrt{-1}$,

$\sin\theta=-\frac{1}{2}(e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}})\sqrt{-1}$..................(29).

(28) and (29) enable us to develop $\cos\theta$ and $\sin\theta$ in a series arranged according to the powers of θ. Since $(\theta\sqrt{-1})^2=-\theta^2$, $(\theta\sqrt{-1})^3=-\theta^3\sqrt{-1}$, $(\theta\sqrt{-1})^4=\theta^4$, the substitution of $\theta\sqrt{-1}$ for θ in (1), gives

$$e^{\theta\sqrt{-1}}=1+\theta\sqrt{-1}-\frac{\theta^2}{1.2}-\frac{\theta^3\sqrt{-1}}{1.2.3}+\frac{\theta^4}{1.2.3.4}+\frac{\theta^5\sqrt{-1}}{1.2.3.4.5}\ \ldots\ldots(30),$$

$$\text{and } e^{-\theta\sqrt{-1}}=1-\theta\sqrt{-1}-\frac{\theta^2}{1.2}+\frac{\theta^3\sqrt{-1}}{1.2.3}+\frac{\theta^4}{1.2.3.4}-\frac{\theta^5\sqrt{-1}}{1.2.3.4.5}\ \ldots\ldots(31).$$

Half the sum of (30) and (31) by (28) gives

$$\cos\theta = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \frac{\theta^6}{1.2.3.4.5.6} + \text{ etc.},$$

and half the difference of (30) and (31) by (29) gives

$$\sin\theta = \theta - \frac{\theta^3}{1.2.3} + \frac{\theta^5}{1.2.3.4.5} - \text{ etc.}$$

The above are the required series. It is hoped that the law connecting $\cos\theta$ and $\sin\theta$ has been made plain.

(28) and (28) are Euler's results reached in a different way.

From (28) and (29) Demoivre's Theorem, which enables us to obtain the n roots of $y^n+1=0$ and $y^n-1=0$, is derived.

November 4, 1893.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

63. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

I owe A \$100 due in 2 years, and \$200 due in 4 years; when will the payment of \$300 equitably discharge the debt, money being worth 6% ?

III. Solution by the PROPOSER.

Let x = equated time.

Now the amount of \$100 for $(x-2)$ years + the present worth of \$200 due $(4-x)$ years hence must = \$300.

$100+6(x-2)$ = amount of \$100 for $(x-2)$ years at 6%.

$\dfrac{10000}{62-3x}$ = present worth of \$200 due $(4-x)$ years hence at 6%.

$\therefore\ 100+6(x-2)+\dfrac{10000}{62+3x}=300.$

$\therefore\ x=3.31533+$ years = 3 years, 3 months, 24 days.

PROOF. \$107.89 = amount of \$100 for 1.31533 years at 6%.

\$192.11 = present worth of \$200 due 0.68467 year hence at 6%.

\$107.89 + \$192.11 = \$300.

QUERY: Will the answers prove as obtained to the solutions of this problem on page 238, Vol. III.?

64. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

If 27 men in 10 days of 7 hours each for \$375 dig a ditch 70 rods long, 25 feet wide, and 4 feet deep, how long a ditch 40 feet wide and 3 feet deep will 15 men dig in 16 days of 9 hours each for \$500?

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

The following method of solution I have found to be infallible for all problems of Compound Proportion.

The first ratio (simple) has for its antecedent the *quantity to be found*, and for its consequent the corresponding similar quantity of the problem; hence $x:70$.

We now reason as follows: Work, time, etc., as the case may be, being equal, can a *longer* or *shorter* ditch be dug—

(1). By digging it 40 feet wide than by digging it 25 feet wide? Evidently shorter; hence 25:40.

(2). By digging it 3 feet deep than by digging it 4 feet deep? Longer; hence 4:3.

(3). With 15 men than with 27 men? Shorter; hence 15:27.

(4). In 16 days than in 10 days? Longer; hence 16:10.

(5). By working 9 hours a day than by working 7 hours? Longer; hence 9:7.

(6). With \$500 than with \$375? Longer; hence 500:375.

$$\text{Whence, } x:70::\left\{\begin{array}{r} 25:40 \\ 4:3 \\ 15:27 \\ 16:10 \\ 9:7 \\ 500:375 \end{array}\right. \qquad \therefore x=88\tfrac{8}{9}.$$

Solved with same result by *P. S. BERG* and *EDWARD R. ROBBINS.*

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

There are two interpretations of the problem.

(1). The men are paid by the cubic foot; in this case the second lot should handle $\frac{500}{375}=\frac{4}{3}$ as much dirt as the first lot.

$$\therefore \tfrac{4}{3}\times\frac{70\times25\times4}{40\times3}=\tfrac{700}{9}=77\tfrac{7}{9} \text{ rods length of ditch.}$$

(2). Both ditches are dug by contract and the men are worked at their best all the time; in this case the amount received has nothing to do with the length of the ditch.

$$\therefore\left\{\begin{array}{c}27\\10\\7\end{array}\right\}:\left\{\begin{array}{c}70\\25\\4\end{array}\right\}=\left\{\begin{array}{c}15\\16\\9\end{array}\right\}:\left\{\begin{array}{c}x\\40\\3\end{array}\right\}.$$

$$\therefore x=\frac{70\times25\times4\times15\times16\times9}{27\times10\times7\times40\times3}=66\tfrac{2}{3} \text{ rods.}$$

[NOTE. 83⅓ is obtained by multiplying 66⅔ by 5/4.]

Solved with same result as in (1) by *FREDERICK R. HONEY.*

[NOTE. There seems to be some disagreement among our contributors as to the correct solution of this problem. I, however, agree with Mr. Gruber, and have used his method of solution for several years. For a more detailed statement of this method the reader is referred to my *Mathematical Solution Book*. EDITOR.]

65. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Bought April 4, 1894, 250 yards of broadcloth at $5.37½ per yard, less 12½ and 10% discount for cash payment. Sold September 5, 1894, at 15, 10, and 5% on *quoted price*, the cloth; and in settlement received a 90-day note which I had discounted at 5½%, October 19, 1894, by the First National Bank of Baltimore, Maryland. Reckoning 6% interest on the *money invested* in the cloth, what is the profit made?

I. Solution by P. S. BERG, Larimore, North Dakota, and G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

1.00−.12½=.87½, 1.00−.10=.90. ∴ 1.00×.87½×.90=78¾%.

$5.37½×.78¾×250=$1058.203125=cost.

From April 4th to September 5th is 5 months, 1 day, at 6%, $1. amounts to $1.025⅙. $1058.203125×1.025⅙=$1084.8336.

1.00+.15=1.15, 1.00+.10=1.10, 1.00+.05=105%.

1.15×1.10×1.05=132.825%.

$5.37½×1.32825×250=$1784.8359375.

From October 19th to December 8th, 50 days, at 5½%, $1.=.007$\frac{23}{36}$.

$1.00−$.007$\frac{23}{36}$=$.992$\frac{13}{36}$.

$1784.8359375×.992$\frac{13}{36}$=$1771.2017.

$1771.2017−$1084.8336=$686.3681 profit.

PROBLEMS.

69. Proposed by EDGAR H. JOHNSON, Professor of Mathematics, Emory College, Oxford, Georgia.

Every man in a certain group belongs to at least one of these classes: Methodists, Democrats, Farmers. In the group there are 10 Methodists, 12 Democrats. 18 Farmers; 8 men who are Methodists and Democrats, 4 who are Democrats and Farmers, 5 who are Methodists and Farmers. Finally, there are 2 men who are at the same time Methodists, Democrats and Farmers. Required the number of men in the group.

70. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

A owes me $100 due in 2 years, and I owe him $200 due in 4 years; when can I pay him $100 to settle the account equitably, money being worth 6%?

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

60. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Prove that the loci of the foci of variable ellipses passing through the foci of a given ellipse and having the tangents at the ends of the major axes for directrices form a pair of circles passing through the extremities of the major axis of the fixed ellipse and having for diameters the semi-latas rectum of the fixed ellipse.

Solution by the PROPOSER.

If the given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ....(1),

the equation to the required ellipse is of the form $\frac{x^2}{a_1^{\,2}}+\frac{(y-n)^2}{b_1^{\,2}}=1$ ....(2).

This passing through $(ae, 0)$, we have $\frac{a^2e^2}{a_1^{\,2}}+\frac{n^2}{b_1^{\,2}}=1$ ....(3).

The directrix of (2) is $x=\frac{a_1}{e_1}$..(4), e_1 being the eccentricity of (2), and $x=a$..(5) is the tangent to the given ellipse at the extremity of its major axis. Then

$\frac{a_1}{e_1}=a$............(6), or $a_1=ae_1$............(7), $a_1e_1=ae_1^{\,2}$...........(8).

Let (x', y') be the coordinates of the right hand focus of (2) in any one of its positions; then $a_1e_1=x'$............(9), $n=y'$...........(10), and by (8) and (9), $e_1^{\,2}=\frac{x'}{a}$, $1-e_1^{\,2}=\frac{a-x'}{a}$ ....(11).

Also by (7), $a_1^{\,2}=a^2e_1^{\,2}=ax'$ ....(12);

$\therefore b_1^{\,2}=a_1^{\,2}(1-e_1^{\,2})=x'(a-x')$.......................(13), and (3) becomes

$\frac{a^2e^2}{ax'}+\frac{y'^2}{x'(a-x')}=1$ ....(14).

Reducing, $x'^2+y'^2-a(1+e^2)x'=-a^2e^2$............(15), a circle whose center is on the axis of x, passing through $(a, 0)$, and having diameter $\frac{b^2}{a}$.

Also solved by *G. B. M. ZERR*.

61. Proposed by WILLIAM E. HEAL, Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

Let the bisectors of the angles A, B, C of a triangle intersect in O and meet the sides opposite A, B, C in A', B', C'. Prove that the perpendiculars form O on the sides of the triangle $A'B'C'$ are $p_1=\frac{rR}{d_1}$, $p_2=\frac{rR}{d_2}$, $p_3=\frac{rR}{d_3}$ where r, R are the radii of the inscribed and circumscribed circles of the triangle ABC and d_1, d_2, d_3 are the distances of the center of the circumscribed circle from the centers of the escribed circles.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Using trilinear coordinates, equation to CD is $\alpha-\beta=0$; to BE, $\alpha-\gamma=0$.

$$\therefore \left(\frac{2\Delta}{a+b}, \frac{2\Delta}{a+b}, 0\right), \left(\frac{2\Delta}{a+c}, 0, \frac{2\Delta}{a+c}\right),$$

are the coordinates of D, E.

$\therefore \beta+\gamma-\alpha=0$, is the equation to DE.

The distance from O, (r, r, r) from this line is,

$$p_1=\frac{r}{\sqrt{3abc+2\cos C-2\cos A+2\cos B}}$$

$$=\frac{r}{\sqrt{\frac{3abc+a^2c+b^2c-c^3-ab^2-ac^2-b^3+a^2b+bc^2+a^3}{abc}}}$$

$$=\frac{r}{\sqrt{\frac{abc+(a+b+c)(a+b-c)(a-b+c)}{abc}}}=\frac{r}{\sqrt{\frac{abc+8s(s-b)(s-c)}{abc}}}$$

$$=\frac{r}{\sqrt{\frac{abc(s-a)+8\Delta^2}{abc(s-a)}}}=\frac{r}{\sqrt{\frac{\frac{abc}{4\Delta}+\frac{2\Delta}{s-a}}{\frac{abc}{4\Delta}}}}=\frac{r}{\sqrt{\frac{R+2r_1}{R}}}=\frac{rR}{\sqrt{R^2+2Rr_1}}=\frac{rR}{d_1}.$$

Similarly $p_2=\frac{rR}{d_2}$, $p_3=\frac{rR}{d_3}$.

62. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Prove that two triangles are equal if they have two sides and *the median of one of them* equal, each to each.

Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland, and CHAS. C. CROSS, Laytonsville, Maryland.

Let $AB=A'B'$, $AC=A'C'$, $BD=B'D'$. $\triangle ABD=\triangle A'B'D'$, because all the sides are equal, each to each.

Then $\triangle BDC=\triangle B'D'C'$, having two sides and included angle of one= two sides and included angle of the other.

$\therefore$ $\triangle ABC=\triangle A'B'C'$.

Also solved by *EDWARD R. ROBBINS, M. A. GRUBER*, and *G. B. M. ZERR*.

66. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O., University, Mississippi.

A rectangular hyperbola cannot be cut from a right circular cone *if the angle at its vertex is less than a right angle.*

Solution by the PROPOSER.

Let the base and the axis of the cone coincide with the xy-plane and the z-axis respectively. Then if c denote the altitude of the cone and ϕ the angle which any one of its elements makes with the base, its equation is

$$(x^2+y^2)\tan^2\phi=(z-c)^2.$$

The equation of a plane through the y-axis and inclined at an angle θ to the xy-plane is

$$z=x\tan\theta.$$

The projection on the xy-plane of the intersection of the two surfaces is

$$(x^2+y^2)\tan^2\phi=(x\tan\theta-c)^2=x^2\tan^2\theta-2cx\tan\theta+c^2.$$

This becomes, when referred to rectangular axes in the plane of the section, the origin and y-axis being unchanged, $(x^2\cos^2\theta+y^2)\tan^2\phi=x^2\sin^2\theta-2cx\sin\theta+c^2$, or $x^2(\cos^2\theta\tan^2\phi-\sin^2\theta)+y^2\tan^2\phi+2cx\sin\theta-c^2=0$, which represents a rectangular hyperbola if $\tan^2\phi+\cos^2\theta\tan^2\phi-\sin^2\theta=0$. From this equation,

$$\sin^2\theta=\frac{2\tan^2\phi}{\tan^2\phi+1}=2\sin^2\phi, \text{ and } \sin\theta=\pm\sqrt{2}\sin\phi.$$

Since $\sin\phi$ cannot be greater than $\frac{1}{\sqrt{2}}$, ϕ cannot exceed 45°. Hence the angle at the vertex of the cone cannot be less than 90°.

Other solutions of this problem will appear in next issue.

PROBLEMS.

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67. Proposed by F. M. PRIEST, St. Louis, Mo.

Required: The length of a piece of carpet that is a yard wide with square ends, that can be placed diagonally in a room 40 feet long and 30 feet wide, the corners of the carpet just touching the walls of the room.

68. Proposed by LEONARD E. DICKSON, M. A., Ph. D., Formerly Fellow of Mathematics, University of Chicago; Chicago, Illinois.

Suppose a circle of unit radius divided at the points A, A_1, A_2, A_3, ... into n equal parts. [This division cannot in general be affected by geometry.] Through A draw the diameter OA and join O with $A_1, A_2, A_3, \ldots A_{\frac{n-1}{2}}$, where n is supposed to be odd.

Prove that $OA_1-OA_2+OA_3-OA_4+\ldots\pm OA_{\frac{n-1}{2}}$, every other chord being affected with the minus sign.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

36. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, New Windsor College, New Windsor, Maryland.

A vertical slit is made in the middle of the side of a rectangular box containing water. What is the time required to empty the box?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let a; b, h=length, width, and depth of box, c=width of slit, m=coefficient of contraction, z=distance of surface of water from bottom of box, x=distance of any elemental area of slit from bottom of box.

$\therefore$ The quantity discharged through the slit in an element of time is

$$Q=[mc\sqrt{2g}\int_0^z \sqrt{z-x}\,dx]dt=\tfrac{2}{3}mc\sqrt{2g}\,z^{\frac{3}{2}}\,dt=abdz.$$

$$\therefore t=\frac{3ab}{2mc\sqrt{2g}}\int_{h'}^{h}\frac{dz}{z^{\frac{3}{2}}}=\frac{3ab(\sqrt{h}-\sqrt{h'})}{mc\sqrt{2ghh'}}, \text{ for depth } (h-h').$$

When $h'=0$, t=infinity or it is impossible to absolutely empty the box.

II. Solution by the PROPOSER.

Let x=distance from base of box to any point in the vertical slit below surface of water.

Let y=distance from base of box to surface of water.

The velocity of discharge for point $x=\sqrt{2g(y-x)}$.

$\therefore\ dF = k\sqrt{2g(y-x)}dx$, where k=width of slit, and F=flow of water.

Whence $F = k\sqrt{2g}\int_0^y \sqrt{y-x}\,dx = \frac{2k\sqrt{2g}}{3}y^{\frac{3}{2}}$.

Call V the volume of water in the box at any instant.

Then $\frac{dV}{dt} = \frac{2k\sqrt{2g}}{3}y^{\frac{3}{2}}$. But $V = aby$, where a and b are the dimensions of base of box. $\therefore \frac{abdy}{dt} = \frac{2k\sqrt{2g}}{3}y^{\frac{3}{2}}$.

From which $t = \frac{3ab}{2k}\int_n^m y^{-\frac{3}{2}}dy = \frac{ab}{k\sqrt{2g}}\left[\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}}\right]$, m and n being the depths of water at beginning and end of time of discharge.

If $n=0$, or the box is emptied, $t=\infty$.

If $m=\infty$, $t = \frac{ab}{k\sqrt{2gn}}$; or the time to empty a box of infinite depth to a finite depth is finite.

87. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

A thin board, of which the elements are given, is balanced at the center but inclined at an angle. A sphere of known dimensions is put directly above the point of suspension and liberated. Find the motion of the system. That is, find (a) the time until the sphere leaves the board, (b) the ultimate angular velocity of the board.

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Take the horizontal line through the point making the greatest angle with the plane in its initial position as the axis of x, and the axis of y vertically downward through the same point. Let R and T be the normal and tangential reactions of the plane and sphere at any time t from the commencement of motion, θ and ϕ the angles of rotation of the sphere and of the plane, m, k, a the mass, radius of gyration, and radius of the sphere, and r=the distance the sphere has moved on the plane, and x and y the coordinates of the center of the sphere.

Resolving horizontally and vertically, and taking moments about the center of the sphere,

$$m\frac{d^2x}{dt^2} = R\sin\phi - T\cos\phi \quad\ldots\ldots(1),\qquad m\frac{d^2y}{dt^2} = mg - R\cos\phi - T\sin\phi \quad\ldots\ldots(2).$$

$$mk^2\frac{d^2\theta}{dt^2} = Ta \quad\ldots\ldots(3).\qquad \text{We also have } \theta = \frac{r}{a} + \phi \quad\ldots\ldots(4),$$

$$x = r\cos\phi + a\sin\phi \quad\ldots\ldots(5),\ \text{and}\ y = r\sin\phi - a\cos\phi \quad\ldots\ldots(6).$$

Eliminating T from (1) and (2), $\sin\phi\frac{d^2x}{dt^2}-\cos\phi\frac{d^2y}{dt^2}=\frac{R}{m}-g\cos\phi$ ....(7).

Eliminating T and R from (1), (2), and (3),

$$\cos\phi\frac{d^2x}{dt^2}+\sin\phi\frac{d^2y}{dt^2}=g\sin\phi-\frac{k^2}{a}\frac{d^2\theta}{dt^2} \quad \text{.........(8).}$$

From (4), $\frac{d^2\theta}{dt^2}=\frac{1}{a}\frac{d^2r}{dt^2}+\frac{d^2\phi}{dt^2}$ ....(9).

From (5) and (6), $\frac{d^2x}{dt^2}=\cos\phi\frac{d^2r}{dt^2}-2\sin\phi\frac{dr}{dt}\frac{d\phi}{dt}-r\cos\phi\frac{d\phi^2}{dt^2}$

$$-r\sin\phi\frac{d^2\phi}{dt^2}-a\sin\phi\frac{d\phi^2}{dt^2}+a\cos\phi\frac{d^2\phi}{dt^2} \quad \text{.........(10),}$$

$$\frac{d^2y}{dt^2}=\sin\phi\frac{d^2r}{dt^2}+2\cos\phi\frac{dr}{dt}\frac{d\phi}{dt}-r\sin\phi\frac{d\phi^2}{dt^2}$$

$$+r\cos\phi\frac{d^2\phi}{dt^2}+a\cos\phi\frac{d\phi^2}{dt^2}+a\sin\phi\frac{d^2\phi}{dt^2} \quad \text{.........(11).}$$

Eliminating $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2\phi}{dt^2}$ from (7) and (8),

$$2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}+a\frac{d\phi^2}{dt^2}=g\cos\phi-\frac{R}{m} \quad \text{.........(12),}$$

$$\frac{a^2+k^2}{a^2}\frac{d^2r}{dt^2}-r\frac{d\phi^2}{dt^2}+\frac{a^2+k^2}{a^2}\frac{d^2\phi}{dt^2}=g\sin\phi \quad \text{.........(13).}$$

(12) and (13) seem to indicate that one more condition at least should be given.

PROBLEMS.

46. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

"There was an old woman tossed up in a basket
Ninety times as high as the moon." *Mother Goose.*

Neglecting the resistance of the air, how long did it take the old lady to go up?

47. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

What is the focus of the convex surface of a plano-convex lens, index μ, which will converge parallel monochromatic rays to a given focus, the rays entering the lens on the plane side?

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

45. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

Solve the equation $x^3+y^2=a^2$.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Put $y=\frac{x(x-n^3)}{2n}$. Then we readily obtain $x^3+\left\{\frac{x(x-n^3)}{2n}\right\}^2=\left\{\frac{x(x+n^3)}{2n}\right\}^2$,

which is a general formula for finding *the sum of a cube aud a square equal to a square*, x and n representing any values. We have also the general condition, derived from the formula, $nx+y=a$. By taking $n=1$, and putting $x=$, consecutively, the natural numbers beginning with unity, we obtain a series of equations in which the consecutive values both of y and a form *the series of integral numbers the sum of any two consecutive terms of which is the square of their difference.* [Problem 43, page 370, Vol. II.]

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $y=mx$, then $x^3+m^2x^2=a^2$. $\therefore x+m^2=a^2/x^2=b^2$, $\therefore b^2-m^2=x$, where b and m can be any integers $b>m$. We append some values.

b	m	x	y	a
1	0	1	0	1
2	1	3	3	6
3	2	5	10	15
4	3	7	21	28
5	4	9	36	45
&c.	&c.	&c.	&c.	&c.

III. Solution by M. C. STEVENS, M. A., Department of Mathematics, Purdue University, Layfayette, Indiana.

If x be any integer and $y=\frac{x(x-1)}{2}$, then $x^3+y^2=\frac{x^4+2x^3+x^2}{4}=a^2$.

$\therefore a=\frac{x(x+1)}{2}$. If $x=1$, then $a=1$. If $x=2$, then $a=3$, and so on.
$y=0$ $\qquad$ $y=1$

IV. Solution by A. H. BELL, Box 184, Hillsboro, Illinois.

We write $x^3=a^2-y^2$. From the well known form

$mn=\left(\frac{m+n}{2}\right)^2-\left(\frac{m-n}{2}\right)^2$, if $x^3=mn$, the problem is answered.

Let m and n be 4 and 2; or 27 and 1; or 9 and 3; etc.; then $2^3+1^2=3^2$; $+13^2=14^2$; $3^3+3^2=6^2$; etc.

V. Solution by H. C. WILKES, Skull Run, West Virginia.

$x^3=(a+y)(a-y)$. Let $a+y=x^2$ and $a-y=x$, then $x^2+x=2a$, and $=-\frac{1}{2}\pm\sqrt{2a+\frac{1}{4}}$. Let a be any triangular number, and from the above formula, ntegral values for x, a, and y can be found.

VI. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Let $x=ky$. Then $x^3+y^2=a^2$ becomes $y^2\{k^3y+1\}=a^2$. This will be a square if $y=k^3+2$. $\therefore$ $y=k^3+2$, and $x=k(k^3+2)$ will be a solution, where k is any integer. If $k=1$, $y=3$, $x=3$ and $x^3+y^2=36$. If $k=2$, $y=10$, $x=20$, and $x^3+y^2=8100$, etc., etc.

VII. Solution by J. H. DRUMMOND, LL. D., Portland, Maine.

(A). If the problem is to be taken literally, $y=\sqrt{a^2-x^3}$ in which x may any number whose third power $<$ than a^2. But this does not give exact results.

(B). If it means that $x^3+y^2=\square$, let $x=my$ and we have $m^3y+1=\square=$ (say) b^2 and $y=(b^2-1)/m^3$ and $x=(b^2-1)/m^2$; but then $a=b(b^2-1)/m^3$, in which m and b may be any numbers greater than unity, but the value of a depends on x and y.

(C). By transposing $x^3=a^2-y^2$; take $x=a-y$, then $x^2=a+y$, and $a^2-2ay+y^2=a+y$, and $y=(2a+1\pm\sqrt{8a+1})/2$. As y must be less than a to make x positive, the sign of the radical term must be negative. It is readily seen that $a=n(n+1)/2$ makes $8a+1$ a square, and by reducing we get $y=n(n-1)/2$ and $x=n$, in which n may be any number.

(D). If the question means to find exact values of x and y for *any* value of a, I cannot solve it.

46. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

In $x^2+x\sqrt{xy}=a$......(1) and $y^2+y\sqrt{xy}=b$......(2) find such values of a find b as will make x and y integral; give a general solution.

I. Solution by the PROPOSER.

Take $y=m^2x$, and by combining the two equations and reducing we have, $\frac{b}{a}(m+1)=m^3(m+1)$ and consequently $m^3=\frac{b}{a}$.

From (1) we have $x=\pm\sqrt{\frac{a}{m+1}}$. Take $a=c^2$ and $m+1=d^2$ and substituting, we have $x=c/d$. To make this value integral, take $c=de$; then $x=e$, and $y=m^2x=e(d^2-1)^2$. But $a=c^2$, and $c=dx=de$. $\therefore$ $a=d^2e^2$; but $b=am^3=d^2e^2(d^2-1)^3$, in which a may be any whole number$>$1, and e any whole number.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

In order that $\sqrt{xy}$ be integral and rational, we put $x=rm^2$ and $y=rn^2$, r, m, and n being any integers. Whence we readily find that when $a=r^2m^3(m+n)$ and $b=r^2n^3(m+n)$, x and y are integral.

Now put $r=1$, $m=3$, and $n=2$, and we obtain $x^2+x\sqrt{xy}=135$ and $y^2+y\sqrt{xy}=40$; whence $x=9$ and $y=4$.

Put $r=2$, $m=2$, and $n=1$; then $x^2+x\sqrt{xy}=96$ and $y^2+y\sqrt{xy}=12$; whence $x=8$, and $y=2$.

III. Solution by A. H. BELL, Box 184, Hillsboro, Illinois.

The only condition to fill is to make $xy=\square$. Take $x=4$, $y=1$, and $a=24$, $b=8$, etc., etc.

IV. Solution by H. C. WILKES, Skull Run, West Virginia.

Let $m^2=x$, $n^2=y$. Then $m^3(m+n)=a$; $n^3(m+n)=b$. $\therefore$ To make x and y integral, a and b must have a common factor $(m+n)$. The remaining factors will be m^3 and n^3. Let $a=448$, $b=189$; then $x=16$, $y=9$. 7(64)$m=4$; 7(27)$n=3$.

V. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $P=x^{\frac{1}{2}}$; $Q=y^{\frac{1}{2}}$. Then $P^4+P^3Q=a$......(1). $Q^4+Q^3P=b$.....(2).

$$(1)\div(2),\ P=Q\sqrt[3]{a/b}.\quad \therefore P^2=x=\pm\frac{a^{\frac{2}{3}}}{\sqrt{a^{\frac{1}{3}}+b^{\frac{1}{3}}}},\ Q=y=\pm\frac{b^{\frac{2}{3}}}{\sqrt{a^{\frac{1}{3}}+b^{\frac{1}{3}}}}.$$

Let $a=\{\frac{1}{2}(m^2+n^2)\}^3$, $b=\{\frac{1}{2}(m^2-n^2)\}^3$.

$$\therefore x=\pm\frac{(m^2+n^2)^2}{4m},\quad y=\pm\frac{(m^2-n^2)^2}{4m}.$$

$$\text{Let } m=pn.\quad \therefore x=\pm\frac{n^3(p^2+1)^2}{4p},\quad y=\pm\frac{n^3(p^2-1)^2}{4p}.$$

Let $n=2p$. $\therefore x=\pm2p^2(p^2+1)^2$, $y=\pm2p^2(p^2-1)^2$.
$\therefore a=\{2p^2(p^2+1)\}^3$, $b=\{2p^2(p^2-1)\}^3$.

VI. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland

Let $y=m^2x$. Then $x^2(1+m)=a$, and $x^2m^3(1+m)=b$.

$$\therefore m=\frac{b^{\frac{1}{3}}}{a^{\frac{1}{3}}}.\quad x=\frac{a^{\frac{2}{3}}}{\sqrt{a^{\frac{1}{3}}+b^{\frac{1}{3}}}};\ y=\frac{b^{\frac{2}{3}}}{\sqrt{a^{\frac{1}{3}}+b^{\frac{1}{3}}}}.$$

$$\text{Let } a=p^3;\ b=q^3.\quad \text{Then } x=\frac{p^2}{\sqrt{p+q}};\ y=\frac{q^2}{\sqrt{p+q}}.$$

Let $p=2rs$; $q=r^2+s^2$. Then $x=\dfrac{4r^2s^2}{r+s}$; $y=\dfrac{(r^2+s^2)^2}{r+s}$.

Let $r=k+l$; $s=k-l$. Then $x=\dfrac{2(k^2-l^2)^2}{k}$; $y=\dfrac{2(k^2+l^2)^2}{k}$.

Let $l=\alpha k$. Then $x=2k^3(1-\alpha^2)^2$; $y=2k^3(1+\alpha^2)^2$.

Now $a=p^3=8r^3s^3=8(k^2-l^2)^3=8k^6(1-\alpha^2)^3$, and $b=q^3=(r^2+s^2)^3=8(k^2+l^2)^3=8k^6(1+\alpha^2)^3$, where α and k are integers.

PROBLEMS.

53. Proposed by A. H. BELL, Box 184, Hillsboro, Illinois.

Given $x^2-1141y^2=\mp3$ to find the least values of x and y in integers.

54. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

In the expression $2x^2-2ax+b^2$, find two series of values for x in integral terms of a and b.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

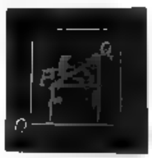
SOLUTIONS OF PROBLEMS.

35. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

Find the chance that the distance of two points within a square shall not exceed a side of the square. [From *Byerly's Integral Calculus.*]

I. Solution by ALWYN C. SMITH, The University of Colorado, Boulder, Colorado.

a is one side of the square; P and Q the two points; (x, y) the point P with O for origin; and r and ϕ the polar coordinates of Q, with P as origin. Then the favorable cases are

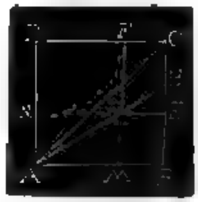


$$4\int_0^{\frac{1}{2}\pi}\int_0^a\int_0^{a-r\sin\phi}\int_0^{a-r\cos\phi}dx\,dy\,r\,dr\,d\phi=a^4(\pi-\tfrac{13}{6}).$$

All the cases $=a^2.a^2=a^4$. Therefore, $p=\pi-\frac{13}{6}$.

II. Solution by J. M. COLAW, A. M., Principal of High School, Monterey, Virginia.

Let a=a side of the square $ABCD$, and join A with any point P within the given square. Then as AP represents the distance and direction of the second point from the first, the area of the rectangle $PECF$ represents the number of ways the two points can be taken.



Let $AP=x$, $AH=x'$, and $\angle PAB=\theta$.

When $x'=a\sec\theta$, $PF=a-x\sin\theta$, $PE=a-x\cos\theta$.

$\therefore$ Area $PECF=(a-x\sin\theta)(a-x\cos\theta)$.

Hence the required chance is

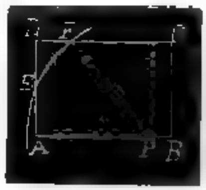
$$p_1=\frac{\int_0^{\frac{1}{4}\pi}\int_0^a (a-x\sin\theta)(a-x\cos\theta)x\,d\theta\,dx}{\int_0^{\frac{1}{4}\pi}\int_0^{x'} (a-x\sin\theta)(a-x\cos\theta)x\,d\theta\,dx}$$

$$=\frac{8}{a^4}\int_0^{\frac{1}{4}\pi}\int_0^a (a-x\sin\theta)(a-x\cos\theta)x\,d\theta\,dx$$

$$=\frac{1}{3}\int_0^{\frac{1}{4}\pi}(6-4\sin\theta-4\cos\theta+3\sin\theta\cos\theta)d\theta=\pi-\frac{13}{6}.$$

III. Solution by LEWIS NEIKIRK, Senior in the University of Colorado, Boulder, Colorado.

Take a rectangle $ABCD$, with sides $AB=b$, and $BC=a$, such that a is not greater than b; and consider the chance that the proposed distance shall exceed b. Let N be the number of favorable cases; then if a be increased infinitesimally, dN will be the number of new cases introduced by placing each point in turn on the differential slice along b while the other one traverses the mixtilinear area DEF.



That is, taking AP equal to x,

$$dN=4\Big[\int_{\sqrt{b^2-a^2}}^{b}\Big(ax-\frac{a}{2}\sqrt{b^2-a^2}-\frac{x}{2}\sqrt{b^2-x^2}-\frac{b^2}{2}\sin^{-1}\frac{x}{b}+\frac{b^2}{2}\cos^{-1}\frac{a}{b}\Big)dx\Big]da$$

$$=2\Big[2ab-ab\sqrt{b^2-a^2}-\frac{a^3}{3}-\frac{\pi b^2}{2}+b^2\cos^{-1}\frac{a}{b}\Big]da\text{; and,}$$

$$N=2\Big[a^2b^2+\frac{b}{3}\sqrt{(b^2-a^2)^3}-\frac{a^4}{12}-\frac{\pi ab^2}{2}+ab^2\cos^{-1}\frac{a}{b}-b^3\sqrt{b^2-a^2}\Big]+C.$$

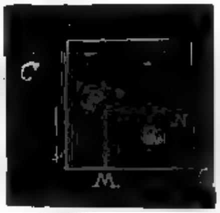
Since $N=0$ when $a=0$, $C=\frac{4b^4}{3}$; and,

$$N=2[a^2b^2+\frac{b}{3}\sqrt{(b^2-a^2)^3}-\frac{a^4}{12}-\frac{\pi ab^3}{2}+ab^3\cos^{-1}\frac{a}{b}-b^3\sqrt{b^2-a^2}+\frac{2b^4}{3}].$$

If now $a=b$, $N=b^4(\frac{13}{6}-\pi)$; and the whole number of cases $=b^4$. Hence the chance that the proposed distance shall exceed b is $\frac{13}{6}-\pi$; therefore, the chance that it will not exceed b is $\pi-\frac{13}{6}$.

IV. Solution by LEWIS NEIKIRK, Senior in the University of Colorado, Boulder, Colorado.

Let a be one side of the square, and O the origin. With center O and radius a describe a quadrant. Let P any point within the square (x, y) be one point, and Q be the other point. With center P and radius a describe the circle C. Now Q may be anywhere within the area common to this circle and the square. The favorable cases may then be found by confining Q within the rectangle xy while P traverses the entire square, and then taking four times the result. Hence,



$$p=\frac{4}{a^4}\left\{\int_0^a\int_0^{\sqrt{a^2-y^2}}xydxdy+\tfrac{1}{2}\int_0^a\int_{\sqrt{a^2-y^2}}^a[x\sqrt{a^2-x^2}+y\sqrt{a^2-y^2}+a^2\sin^{-1}\frac{x}{a}-a^2\cos^{-1}\frac{y}{a}]dxdy\right\}=\pi-\frac{13}{6}.$$

V. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

This problem affords a splendid test of the correctness of the general value for any convex area as demonstrated in problem 25, page 281, September-October MONTHLY.

Let AK, $AL=p$, $\angle LAB=\theta$, EF, $GH=C$.

For EF, $C=a\sec\theta$; the limits of p are $a\sin\theta$ to $a\cos\theta$.

For GH, $C=p\sec\theta\operatorname{cosec}\theta$; the limits of p are $a\sin\theta\cos\theta$ to $a\sin\theta$. The limits of θ are 0 to $\frac{1}{4}\pi$.

From problem 25,

$$\Delta=\frac{1}{3A^2}\int\int(C^3-3a^2C+2a^3)d\theta dp.$$

$$\therefore\quad \Delta=\frac{8}{3a^4}\int_0^{\frac{1}{4}\pi}\int_{a\sin\theta\cos\theta}^{a\sin\theta}(p^3\sec^3\operatorname{cosec}^3\theta-3a^2p\sec\theta\operatorname{cosec}\theta+2a^3)d\theta dp$$

$$+\frac{4}{3a^4}\int_0^{\frac{1}{4}\pi}\int_{a\sin\theta}^{a\cos\theta}(a^3\sec^3\theta-3a^3\sec\theta+2a^3)d\theta dp.$$

$$\Delta=\tfrac{1}{3}\int_0^{\frac{1}{4}\pi}(\tan\theta\sec^2\theta-3\sin\theta\cos\theta-6\tan\theta+8\sin\theta)d\theta$$

$$+\tfrac{1}{3}\int_0^{\frac{1}{4}\pi}(\sec^2\theta-\tan\theta\sec^2\theta-3+3\tan\theta+2\cos\theta-2\sin\theta)d\theta.$$

$\therefore\ \Delta=\frac{11}{3}-\pi.\quad p=1-\Delta=\pi-\frac{8}{3}=$required chance.

PROBLEMS.

44. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

What is the average length of all the chords that may be drawn from one extremity of the major axis of an ellipse if they are drawn at equal angular intervals?

45. Proposed by J. C. WILLIAMS, Boston, Massachusetts.

At the end of the fifth inning the base ball score stands 7 to 9. What is the probability of winning for either team?

46. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

Four men starting from random points on the circumference of a circular field and traveling at different rates, take random straight courses across it; find the chance that at least two of them will meet.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

38. Proposed by S. H. WRIGHT, M. D., A. M., Ph. D., Penn Yan, New York.

In latitude 42°30′ north $=\lambda$, at what angle with the horizon will the sun rise, its declination $=22^\circ$ north $=\delta$?

I. Solution by the PROPOSER.

Let BA be a portion of the equator, $CA=\delta$, a portion of a meridian passing through the sun at C when rising, and describing a small-circle arc CE, parallel with BA, and let BC be a portion of the horizon. Then the angles ECA, and BAC, each $=90^\circ$, because meridians cut the equator and circles of declination at right angles. Now $CBA=90^\circ-\lambda$, then $\sin BCA=\sin\lambda\sec\delta=\cos BCE$. $\therefore\ BCE=43^\circ 13' 37''=$required angle.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $x^2+z^2=R^2$ ....(1) be the equation to the horizon. Then, $x\cos\lambda+y\sin\lambda=R\sin\delta$ ....(2) is the equation to the plane of the sun's path. (1) and (2) intersect in the points

$$x=\frac{R\sin\delta}{\cos\lambda},\quad y=0,\quad z=\pm\frac{R}{\cos\lambda}\sqrt{\cos^2\lambda-\sin^2\delta}.$$

The equation for the tangent plane for z positive is,

$$x\sin\delta+z\sqrt{\cos^2\lambda-\sin^2\delta}=R\cos\lambda.$$

∴ We must find the angle ϕ between the two lines in space,

$$\left.\begin{aligned}x\sin\delta+z\sqrt{\cos^2\lambda-\sin^2\delta}=R\cos\lambda\\ x\cos\lambda+y\sin\lambda=R\sin\delta\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots(3),$$

$$\left.\begin{aligned}x\sin\delta+z\sqrt{\cos^2\lambda-\sin^2\delta}=R\cos\lambda\\ y=0\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots(4).$$

Let $s=\dfrac{\sin\delta}{\sqrt{\cos^2\lambda-\sin^2\delta}}$, $t=\cot\lambda$.

Then $\cos\phi=\dfrac{1+s^2}{\sqrt{(1+s^2)(1+s^2+t^2)}}=\sqrt{\dfrac{1+s^2}{1+s^2+t^2}}$. ∴ $\cos\phi=\dfrac{\sin\lambda}{\cos\delta}$.

∴ $\phi=43^\circ 13' 37''$.

III. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

Let HO represent the horizon, Z the zenith of the place of observation, EQ the equator, P the north pole, DL the diurnal circle of the sun, and S the position of the sun at rising. In the quadrantal spherical triangle ZPS, we have $ZS=90^\circ$, $ZP=90^\circ-\lambda$, $PS=90^\circ-\delta$.

We find $\cos ZSP=\sin\lambda/\cos\delta$, but $\angle ZSP$ is equal to the angle which a tangent at S of the circle DSL makes with the horizon.

∴ $\cos x=\dfrac{\sin\lambda}{\cos\delta}$, denoting by x the required angle. For the given concrete values we find $x=43^\circ 14'$.

[NOTE.—Prof. H. C. Whittaker obtained as the numerical result for the required angle 79 degrees, [illegible] minutes, [illegible] seconds, and Prof. E. W. Morrell obtained 73 degrees, 9 minutes, 4.3 seconds. The result given in the published solutions seems to us to be the correct one.]

39. Proposed by SETH PRATT, C. E., Assyria, Michigan.

The pendulum of a clock which gains 6 seconds in 1 hour and 13 minute, makes 6000 vibrations in 1 hour and $9\frac{1}{2}$ minutes. What is the length of the pendulum? And what length should it have to keep true time?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Regarding 1 hour, 13 minutes and 1 hour, $9\frac{1}{2}$ minutes as registered by a clock keeping correct time, $g=32.16$, $\pi=3.1416$, $t=\pi\sqrt{(l/g)}$. Then 1 hour, $9\frac{1}{2}$ minutes$=4170$ seconds.

$$\therefore\ t=\tfrac{4170}{6000}=\tfrac{139}{200}=\pi\sqrt{\frac{l}{g}}.\quad \therefore\ l=\frac{(139)^2 g}{(200\pi)^2}=1.57393\text{ ft.}=18.88716\text{ inches.}$$

1 hour, 13 minutes$=4380$ seconds.

$$\frac{4380\times200}{139}=\text{number of vibrations in 1 hour, 13 seconds.}$$

$$\therefore\ \frac{4380\times200}{139}=4386\text{ seconds.}$$

$$\therefore\ t'=\frac{4386\times139}{4380\times200}=\frac{731\times139}{730\times200}=\pi\sqrt{\frac{l'}{g}}.$$

$$\therefore\ l'=\frac{(731\times139)^2 g}{(730\times200\pi)^2}=1.578243\text{ feet.}$$

$\therefore\ l'=18.93892$ inches$=$length to keep true time.

II. Solution by E. W. MORRELL, Professor of Mathematics in Montpelier Seminary, Montpelier, Vermont.

1 hour and $9\frac{1}{2}$ minutes$=4170$ seconds. 4170 seconds$\div6000=.695$ seconds, the time of one vibration. From Mechanics $l=t^2g/\pi^2$, whence $l=18.886$ inches, the length of the pendulum. Again, 1 hour and 13 minutes$=4380$ seconds. $4380\div.695=876/.139=$number of vibrations in 1 hour and 13 minutes. As the pendulum gains 6 seconds in that time, $6\div(876/.139)=.834/876=.0095$, the time in seconds gained in one vibration.

$\therefore$ .695 seconds$+$.0095 seconds$=$.69595 seconds, the time of vibrations of pendulum to keep correct time. Hence by substitutions in the above formula $l=18.9379$ inches, the length of pendulum to keep true time.

[NOTE.—The results sent in with the problem by the Proposer were, 18.89685+ inches, and for true time .086086+ inches longer. Prof. P. S. Berg in his solution obtained for length of pendulum 18.837975 inches, and 22.898 inches as the length to keep true time. EDITOR.]

PROBLEMS.

49. Proposed by J. SCHEFFER, A. M., Hagerstown, Maryland.

Give a general proof that the centre of gravity, or centroid, determines that point from which the sum of the distances to all other points of a given area is the minimum.

50. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

Describe and compute the actual path traversed by the moon in July and August, 1896, taking into account the motion of the earth around the sun.

51. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A stock dealer traveled from his home H, due north across a lake L 40 miles wide to a city, and bought 156 horses and 177 mules for $23631; he then traveled farther due north to A, and bought at same price 468 horses and 235 mules for $52245; he then traveled from A due west 130 miles to B, and bought 120 cows; he then traveled due north to C, and bought 250 sheep; he then traveled from C due east 330 miles to D, and bought 300 goats,—paying 1-4 as much for cows as horses, and 1-9 as much for sheep as mules, and 1-2 as much for goats as sheep; at D he turned and traveled in a straight line to the city, a distance equal to the sum of the entire distance he traveled due north from his home H; he sold all his stock at a profit of 20%. How far did he travel from his home H the entire trip around and back to the city? What was the cost of each head of stock, and what was the entire gain?

52. Proposed by I. J. WIREBACK, M. D., St. Petersburg, Pennsylvania.

What is the volume of a segment of a right cone, whose diameter is 6 inches and perpendicular 9 inches? The section being parallel with the perpendicular of the cone and includes 1-4 of its circumference at the base.

NOTES.

NOTE ON ARTICLE IN AUGUST-SEPTEMBER NUMBER, VOL. III.

BY WARREN HOLDEN.

Referring to the demonstration on page 207 (current volume) without disputing the conclusion, allow me to submit the following considerations:

In Algebra, when zero is a factor in any term, the product is zero. Accordingly $0\times\infty=0$. In the course of the demonstration appears the expression $\frac{0\times1}{0}=\frac{0}{0}$, or the denominators being equal, $0\times1=0$. Would this result affect the conclusion in any way?

NOTE ON ELIMINATION.

BY J. C. CORBIN, PINE BLUFF, ARKANSAS.

The operation of elimination by addition and subtraction may often be shortened by the process and rule given below:

I. $5x+7y=43$.

$11x+9y=69$.

To eliminate y. $(9\times5-7\times11)x=9\times43-7\times69$. $\therefore x=3$.

To eliminate x. $(11\times7-5\times9)y=11\times43-5\times69$. $\therefore y=4$.

II. $21x+20y=165.$

$77x-30y=295.$

To eliminate x. $(3\times21+2\times77)y=3\times165+2\times295.$ $\because y=5.$

To eliminate y. $(11\times20+3\times30)x=11\times165-3\times295.$

This is, substantially, the Determinant method; but it is derived from the ordinary algebraic process by omitting all unessential work. The rule is: The difference (sum) of the products containing x (y) is equal to the difference (sum) of the numerical products.

EDITORIALS.

A few complete sets of Vol. I. and Vol. II. are still left. We will send Vol. I. to any address in the United States for $2., and Vol. II. for $2.50. Send in your order at once.

Prof. J. A. Calderhead, of Curry University, Pittsburg, Pennsylvania, sent in $3. as his subscription to the MONTHLY for 1896. We are very thankful for the material encouragement the friends of the MONTHLY are giving it.

A conference of the American Mathematical Society will convene in room 35 of Ryerson Physical Laboratory of the University of Chicago, at 10 o'clock, Thursday forenoon, December 31, 1896. It is expected that the conference will have three or four sessions and will adjourn on Friday, January 1, 1897. During the sessions of this conference some very important subjects will be discussed. Let every one interested in Mathematics attend this conference.

BOOKS AND PERIODICALS.

The Elements of Plane Geometry. By Charles A. Hobbs, A. M., Mathematical Master in the Volkmann School, Boston, Mass. 8vo. Cloth and Leather Back, 240 pages. Price, 75 cents. New York: A. Lovell & Co.

In this book the author has taken what seems to him to be a middle ground between the method of the students' following set demonstrations of a number of propositions and that of the students' producing all the argument in the course of a demonstration from original resources. There are 720 original propositions throughout the book besides many numerical exercises. The book is worthy the recognition of teachers. B. F. B.

Number and Its Algebra: A Syllabus of Lectures on the Theory of Number and Its Algebra Introductory to a Course in Algebra. By Arthur Lefevre, C. E., Instructor in Pure Mathematics, University of Texas. 8vo. Cloth, 230 pages. Boston: D. C. Heath & Co.

From only a cursory examination of this book we can say that it occupies a unique place in the literature of Mathematics. A careful reading of its contents by teachers will make the concept of numbers clear, and place their applications and the teaching of them on a solid foundation. B. F, F.

A Primer of the Calculus. By E. Sherman Gould, Member of American Society of Civil Engineers. 16mo. Boards, 92 pages. Price, 50 cents. New York: D. Van Nostrand Co.

This little work is a development of the infinitesimal Calculus as far as the first differentials of algebraic functions of one independent variable and their corresponding integrals. Its size permits it to be carried about in the coat pocket and thus the self-taught may have at his command a work which he may read and study during his leisure.

B. F. F.

Elements of the Differential Calculus. By Edgar W. Bass, Professor of Mathematics in the United States Military Academy. 12mo. Cloth, 354 pages. New York: John Wiley & Sons.

The author says: "This text-book has been prepared for the use of the cadets of the United States Military Academy who begin the subject with a knowledge of the elements of Algebra, Geometry, and Trigonometry which ranges from fair to excellent. * * * * My experience leads me to the belief that the more rigorous and comprehensive method of infinitesimals is suitable only for a treatise and not for a text-book intended for beginners."

The author has, therefore, laid the foundation of his book on the methods of limits —the most accurate and simple of all the methods of presentation. One among the many commendable features of the book is the numerous, beautiful, and accurate diagrams used to aid in establishing the various principles upon which the Calculus is based. In this respect, it will appeal most favorably to the beginner. The book is one I most heartily recommend, and it is to be hoped that the author will follow it up by an equally good work on the Integral Calculus. B. F. F.

List of Transitive Substitution Groups of Degree Twelve. By G. A. Miller, Ph. D., Göttingen, Germany. Extracted from The Quarterly Journal of Pure and Applied Mathematics, No. 111, 1896, pages 193—284.

Dr. Miller has given the subject of Substitution a great deal of study and he has written a number of articles on it. These various articles may be found in the leading Mathematical Journals of America and Europe. Those who are interested in this subject will find this article very helpful. B. F. F.

The Criterion for Two-Term Prismoidal Formulas. By Dr. George Bruce Halsted. Pamphlet, 14 pages.

This interesting and valuable paper was presented to the Texas Academy of Science at its meeting, April 5, 1896. It contains many historical references and gives a pretty full history of the development of that interesting formula. Write to Dr. Halsted for a copy.

B. F. F.

Projective Groups of Perspective Collineations in the Plane Treated Synthetically. Pamphlet, 34 pages.

A dissertation presented to the Faculty of the University of Kansas by Arnold Emch to attain the degree of Doctor of Philosophy. B. F. F.

The Outlook Illustrated Monthly Magazine, Number for October. Price, 10 cents. The Outlook Co., 13 Astor Place, New York.

This number contains a full accouut of Princeton's 150th Anniversary, by Henry Van Dyke, with pictures; The Boys' Republic, by Washington Gladden, with twelve pictures; William Morris: A Poet's Workshop, by R. F. Zueblin, with five pictures; The Founder of the Y. M. C. A., by Lord Kinnaird, with nine pictures. B. F. F.

Popular Astronomy. Edited by W. W. Payne and H. C. Wilson, Goodsell Observatory of Carlton College, Northfield, Minnesota.

The November number contains the following: The Teaching of Descriptive Astronomy; Sketch of Astronomical Work at Munich; Biography of Prof. H. A. Newton, New York Evening Post; The Theory of Probability—An Historical Sketch; The Moon; The Constitution and Function of Gases; The Twilight; The Fixed Stars; The Planets and Constellations for October; Variable Stars. B. F. F.

Prace Matematyczen-Fizyczne. Wydawane. Przez S. Dicksteina, Warsaw, Russia.

The Mathematical Gazette. Edited by F. S. Macauley, St. Paul's School, West Kensington, W. London, England. Price, 3s. per year.

The Gazette aims at satisfying a want felt by many students for a Journal of Elementary Mathematics and is especially intended to be useful to teachers. B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hudson.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year. Single number, 25 cents. The Review of Reviews Co., New York.

Errata in October Number.

Page 246, line 3, for "5^{n+1}" read 5^{n-1}.
Page 246, line 14, insert + before last term of (1).
Page 246, line 15, for "$4^{\frac{n-1}{2}}.5$" read $4^{\frac{n-2}{2}}.5$.
Page 246, line 19, insert + before last term in (2).
Page 247, line 12, for "$4626x^3$" read $4626x^5$, and for "×" read +.
Page 248, line 9, complete parenthesis after numerator of next to last term.
Page 250, problem 72 should read $2\sqrt{2+\sqrt{3}}/(4+\sqrt{6}-\sqrt{2})$.
Page 251, line 7 from bottom, for "$(-x)$" read $(-a)$.
Page 252, l. 20, read $R=[F(C^2-4AB)+AE^2+BD^2-CD^2]/(4AB-C^2)$.
Page 252, line 2 from bottom, reverse last mark of parenthesis after F.
Page 253, line 5, for "(Em^2-2k)" read $(Em^2-2k)y$.
Page 254, line 2, second = should be +.
Page 255, line 14, for "$n=$," etc. read u.
Page 255, line 18, for "(3)" read (2).
Page 256, line 1, in denominator, for "$\sqrt[3]{m+n}$" read $\sqrt[3]{n+m}$.

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LIE'S VIEWS ON SEVERAL IMPORTANT POINTS IN MODERN MATHEMATICS.

By G. A. MILLER, Ph. D., Göttingen, Germany.

It is generally admitted that America has contributed comparatively little towards the advancement of the science of mathematics. During the last twenty years there has been a rapidly increasing progress in this direction. Several European countries have also moved forward at a rapid rate during this period, so that our relative position is not improving as rapidly as might be desired.

The standard of general scholarship required for the higher degrees at our better institutions is comparatively high but the number of important discoveries does not yet correspond to this standard. In fact, the two are not apt to advance very far together, for the field of mathematics is so extensive that most are compelled to choose between a superficial acquaintance with the whole range of mathematical research and an exhaustive knowledge of only a few subjects.

In view of these facts it is natural that there should be many who strive to lead American mathematical talent to those newer regions which seem to offer the most fruitful fields of investigation. While there is a great difference of opinion with respect to these regions yet the most successful investigators are in the best possible position to judge in regard to them.

The view expressed by Klein during last year, in his address on *Arithmetizing Mathematics*, that Lie in Leipzig, Germany, and Poincaré in Paris, France, are the two most active mathematical investigators of the present day, is quite generally held. The following translation of a part of the introductory remarks of an article* published during last year by the former of these may therefore be

*Berichte der Koenigl. Saechs. Gesellschaft, 1895.

of considerable interest, as it contains the views of the author in regard to several important points in mathematics, especially in regard to the most important newer regions.

"In this century the concepts known as substitution and substitution group, transformation and transformation group, operation and operation group, invariant, differential invariant, and differential parameter, appear continually more clearly as the most important concepts of mathematics. While the curve as the representation of a function of a single variable has been the most important object of mathematical investigation for nearly two centuries from Descartes, while on the other hand, the concept of transformation first appeared in this century as an expedient in the study of curves and surfaces, there has gradually developed in the last decades a general theory of transformations whose elements are presented by the transformation itself while the series of transformations, in particular the transformation groups, constitute the object.

The general theory of transformations is a branch of analysis in the sense that it can be developed by purely analytic methods. It has however the material geometrical property that its operations are not only conceivable but directly intuitive to a large extent.

If we consider that the difference between the analytic and the synthetic methods exists in the fact that the synthesist reasons with concepts while the analyst operates with symbols, according to fixed rules, we may see an important property of the theory of transformations in this that its theorems can be developed in an elegant analytic as well as in a perspicuous even intuitively clear manner. It is due to this fact that the theory of transformations is considerably simpler than the theory of substitutions.

It should be added that different branches of mathematics have contributed to the development of the theory of transformations and that many parts of mathematics have already been considerably advanced by means of this theory.

The theory of differential equations is the most important branch of mathematics. Each department of physics presents problems which depend upon the integration of differential equations. In general, the theory of differential equations involves the road towards the explanation of all natural phenomena which require time. While this theory has an infinite practical value it has also a corresponding theoretic importance since it leads in a rational manner to the study of new important functions and classes of functions."

Gottingen, Germany, October 26, 1896.

NUMBER, COUNTING, MEASUREMENT.

By GEORGE BRUCE HALSTED, M. A. (Princeton), Ph. D. (Johns Hopkins), Professor of Mathematics in the University of Texas, Austin, Texas.

Counting is essentially prior to measuring, but also the primary number concept is essentially prior to counting and necessary to explain the meaning, cause and aim of counting. It is here maintained that integral number had not a metric origin, nor was metric in its original purpose ; that integral number did not involve the idea of ratio, that in fact it was enormously simpler than that very delicate concept, *ratio*. Number is primarily a quality of an artificial individual. The stress laid upon it, the importance attached to this quality comes first from the advantage of being able to identify one of these artificial individuals. By artificial is meant "of human make." The characteristic of these artificial individuals is that each, though made an individual, is conceived as consisting of other individuals.

The primitive function of number is to serve the purposes of identification. But again, counting, which consists in associating with each primitive individual in an artificial individual a distinct primitive individual in a familiar artificial individual, is thus itself essentially the identification, by a one-to-one correspondence, of an unfamiliar with a familiar thing. Thus primitive counting decides which of the familiar groups of fingers is to have its numeric quality attached to the unfamiliar group counted.

This primitive use of number in defining by identification is illustrated by an ordinary pack of playing cards, where the identification of King, Queen, and Knave is not more clearly qualitative and opposed to every mode of measurement than is the identification of ace, deuce, and tray ; and indeed that the King outvalues the Knave has more to do with measurement than the fact that the ace outvalues the tray.

Counting implies first a known series of groups, mental wholes each made up of distinct wholes ; secondly an unfamiliar mental whole ; thirdly the identification of the unfamiliar group by its one-to-one correspondence with a familiar group of the known series.

Absolutely no idea of a unit, of measurement, of amount, of value or even of equality is necessarily involved or indeed ordinarily used. One counts when one wishes to find out whether the same group of horses has been driven back at night that were taken out in the morning ; where counting is a process of identification which it would seem intentionally humorous or comical to try to connect fundamentally with any idea of a unit of reference or of some *value* to be ascertained, or of the setting off of a horse as a sample unit of value and then equating the total value to the number of such units. Such an *argumentum in circulo* may perhaps be funny, but it is neither fact nor mathematics. Mathematics afterwards defines numerical equality by means of one-to-one correspondence, which is absolutely distinct and away apart from the idea of ratio. We may say with perfect certainty that there is no implicit presence of the ratio idea in primitive number.

From the contemplation of the primitive individual in relation to the artificial individual spring the related ideas "one" and "many." An individual thought of in contrast to "a many" as not-many gives the idea of "a one." A many composed of "a one" and another "one" is characterized as "two." A many composed of "a one" and the special many "a two" is characterized as "three." And so on ; at first absolutely without counting, in fact before the invention of that patent process of identification now called counting. For a considerable period of its early life every child uses a number system consisting of only three terms, *one*, *two*, *many*, and no counting. As datum may be taken a psychical continuum, and distinctness may be found the outcome of a process of differentiation ; but what may be spoken of as the physically originated primitive individuals, however complete in their distinctness, have no numeric suggestion or quality. The intuitive but creative apperception and synthesis of a manifold must precede its conscious analysis which alone gives number. It is only to conceptual unities that the numeric quality pertains. Such conceptual unities are of human make and in a sense are not in nature, while on the other hand, though the world we consciously perceive is out and out a mental phenomenon, yet the primitive individuals, distinct things, while forming part of the artificial unities, exist in another way, in that they are subsisting somehow in nature as well as in conscious perception.

In reference to these fundamental matters some strange blunders have been made of late by eminent philosophers and teachers, not mathematicians.

The number-picture of a group is a selective photograph of the group, which takes or represents only one quality of the group, but takes that all at once.

This picture process only applies primarily to those particular artificial wholes which may be called discrete aggregates. But the overwhelming importance of the number-picture, primarily as a means of identification, led, after centuries of its use, to a human invention as clearly a device of man for himself as is the telephone. This was a device for making a primitive individual thinkable as a recognizable and recoverable artificial individual of the kind having numeric quality. This recondite device is measurement. Measurement is an artifice for making a primitive individual conceivable as an artificial individual of the group kind, and so having a number picture. The height of a horse, by use af the unit "a hand," is thinkable as a discrete aggregate and so has a number-picture identifiable by comparison with the standard set of pictures, that is by counting, as say 16.

In Euclid's wonderful Fifth Book a ratio is never a number. Newton, with the purpose of taking in the so-called surds or irrationals of arithmetic and algebra, assumed a ratio to be a number. Any continuity in his number-system comes then from the continuity in the magnitude whose ratio to a chosen unit for t magnitude is taken. He never gave any arithmetical or algebraic proof of continuity of any number-system.

Austin, Texas.

ʒW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from June-July Number.]

II. PROOFS RESULTING FROM STRAIGHT-LINE PROPERTIES OF THE CIRCLE.

XV. Let ABC be △ right-angled at C. With ıer extremity, as B, of the hypotenuse, as a center, l with a radius equal to the hypotenuse, describe a ıle. Produce the legs of the △ to chords. One of chords, as DE, will be a diameter.

Fig. 11.

Then $AC.CL=DC.CE$, or $b^2=(c-a)(c+a)$.

$\therefore c^2=a^2+b^2$.

XVI. Let ABC be △ right-angled at C. With either extremity, as B, of the hypotenuse, as the center, and with a radius equal to the adjacent leg, describe a circle. Produce the hypotenuse to a secant.

Fig. 12.

Then $\overline{AC}^2=AE.AD$, or $b^2=(c-a)(c+a)$.

$\therefore c^2=a^2+b^2$.

NOTE.—This method is given by Richardson in *Runkle's Mathematical Monthly*, No. II, 1859; also by Hoffmann, and, in a slightly different form, by Wipper, the latter stating that the proof is found in "Huberti Rudimenta Algebræ," Wurceb, 1762. It was known to the writers, however, independently of these sources.

XVII. Let ABC be a △ right-angled at C.

Case I. When the two legs are unequal.

With C as a center, and with the shorter leg, as , as a radius, describe a circle. Produce AC to a sec- . Draw CL perpendicular to AB.

Fig. 13.

Then $AD.AH=AE.AB$,

or $(b-a)(b+a)=c(c-2LB)$.

Substituting for LB any of its equivalents in ns of the sides of the given △, and reducing, we get, $c^2=a^2+b^2$.

Case 2. When the two legs are equal.

We easily pass, by the usual method of the theory of limits, from Case 1 to Case 2.

XVIII. Same as in XVII, except that the circle is described with the longer leg, as AC, as a radius. Then, produce all the sides to chords.

Then $AB.BL=BE.BD$,

or $c.BL=(b+a)(b-a)$...... ....(1).

Also, $AB : AH :: AC : AL$,

or. $c : 2b :: b : c+BL$,

whence, $c^2+c.BL=2b^2$ ....(2).

(1) in (2), $c^2=a^2+b^2$.

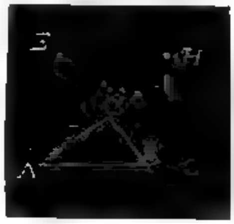


Fig. 14.

When the legs are equal, we pass from the case given as suggested in Case 2 of XVII.

XIX. Same as in XVII, except that both cases are treated alike, and the circle is described with a radius equal to the perpendicular from C to AB. Then produce the legs to secants, and draw CD.

Then $\overline{AD}^2=AH.AE=b^2-\overline{CD}^2$;

$\overline{BD}^2=a^2-\overline{CD}^2$;

also, $2AD.DB=2\overline{CD}^2$.

Adding, $c^2=a^2+b^2$.

Fig. 15.

XX. Let ABC be a $\triangle$ right-angled at C. Produce either leg, as AC, through C, making $CD=AC$. Join BD. Circumscribe a circle about $\triangle$ ABD, and produce BC to a diameter.

Then $\overline{BC}^2=AB.BD-AC.CD$,

or $a^2=c^2-b^2$.

$\therefore c^2=a^2+b^2$.

XXI. Fig. 16.

$AB.BD=BE.BC$,

or $c^2=a^2+a.CE=a^2+b^2$.

Fig. 16.

NOTE.—The last two are special cases of familiar propositions, and are given by various writers.

(To be Continued.)

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

64. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

If 27 men in 10 days of 7 hours each for \$375 dig a ditch 70 rods long, 25 feet wide, and 4 feet deep, how long a ditch 40 feet wide and 3 feet deep will 15 men dig in 16 days of 9 hours each for \$500?

III. Solution by the PROPOSER.

Mr. Gruber's method is all right except the *assumption* that the length of the ditch increases as the price paid. The \$375 pays for 1890 hours' labor; at the same rate, \$500 would pay for 2520 hours' work. But there are only 2160 hours worked. Hence, the *efficiency* must be increased $\frac{1}{6}$. That is, the ditch will be $66\frac{2}{3}$ rods $\times\frac{7}{6}=77\frac{7}{9}$ rods long.

Or, in another light: Since 1890 hours' labor are worth \$375, 2160 hours' work, at same wages, are worth $\$428\frac{4}{7}$. But they get \$500, an increase of $\frac{1}{6}$ as before.

In this problem the *time* is limited—fixed—hence the only thing that can vary is the *efficiency* of the workmen. And it seems plain that it must increase as the *hourly* price increases—not as the *gross* price. Suppose

2 men in 1 day of 10 hours for \$20 dig x rods, and

3 men in 2 days of 10 hours for \$40 dig y rods. What is the ratio of y to x?

Can the *efficiency*, or productiveness, be found without considering the *hourly* wages?

66. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Brown adds $m=10\%$ of water to the pure wine he buys, and then sells the mixture at a price $n=10\%$ greater than the cost price of the pure wine. What is his rate per cent. of profit?

Solution by E. W. MORRELL, Professor of Mathematics in Montpelier Seminary, Montpelier, Vermont.

Let 100%=cost of the wine. Then 110% of $110\%=121\%$, the selling price of the mixture. Hence, $121\%-100\%=21\%$, the gain.

67. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A agreed to work a year for \$300 and a suit of clothes. At the end of five months he left, receiving for his wages \$60 and the clothes. What was the suit worth?

Solution by P. S. BERG, Larimore, North Dakota.

Since he received \$300 and a suit of clothes for a year, for one month he received \$25 and $\frac{1}{12}$ suit of clothes, and for five months he received \$125 and $\frac{5}{12}$ suit of clothes. He received \$60 and the clothes, hence \$60+suit of clothes= \$125+$\frac{5}{12}$ suit of clothes, or $\frac{7}{12}$ suit=\$65. Whence once suit=\$111$\frac{3}{7}$.

Also solved by *E. W. MORRELL* and *JAMES F. LAWRENCE.*

68. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

The population of a city is annually increasing $m=2\frac{1}{2}\%$. If the population now is $P=68921$, what was it $n=3$ years ago? At this rate of increase, what will the population be $n=3$ years hence?

Solution by P. S. BERG, Larimore, North Dakota.

Let 100%=what the population was 3 years ago. Then the population at present is $(100\%+2\frac{1}{2}\%)^3$. Hence $(100\%+2\frac{1}{2}\%)^3=68921$. Whence 100%= 64000, the population 3 years ago. In 3 years hence the population will be $(100\%+2\frac{1}{2}\%)^3$ of 68921, or 74220.378765625.

69. Proposed by EDGAR M. JOHNSON, Professor of Mathematics, Emery College, Oxford, Georgia.

Every man in a certain group belongs to at least one of these classes: Methodists, Democrats, Farmers. In the group there are 10 Methodists, 12 Democrats. 13 Farmers; 3 men who are Methodists and Democrats, 4 who are Democrats and Farmers, 5 who are Methodists and Farmers. Finally, there are 2 men who are at the same time Methodists, Democrets and Farmers. Required the number of men in the group.

I. Solution by J. C. CORBIN, Pine Bluff, Arkansas.

Using obvious abbreviations, we can form the following table in which each small letter denotes a man:

Methodists.	Democrats.	Farmers.
a, *b*	*a*, *b*	*a*, *b*
c, *d*, *e*, *f*, *g*	*h*, *i*, *j*, *k*	*h*, *i*, *j*, *k*
l, *m*, *n*	*l*, *m*, *n*	*r*, *s*
	o, *p*, *q*	

Counting each letter once only, gives 19; 10 in the first column, 12 in the second column, and 13 in the third column.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas, and FREDERICK R. HONEY, Ph. B., New Haven, Connecticut.

Methodists.	Democrats.	Farmers.	Total.
3	3	0	3
0	4	4	4
5	0	5	5
2	2	2	2
10	9	11	14
0	3	2	5
10	12	13	19

∴ 19 men in the group.

Also solved by *E. W. MORRELL, JAMES F. LAWRENCE* and *P. S. BERG.*

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

66. Proposed by A. H. BELL, Box 184, Hillsboro, Illinois.

Solve the equations:

$$a^2x=(2x^2-a^2)\sqrt{x^2+y^2} \quad\ldots\ldots(1),$$

$$b^2y=(2y^2-b^2)\sqrt{x^2+y^2} \quad\ldots\ldots(2).$$

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $x=r\cos\theta$, $y=r\sin\theta$. Then the equations become

$$a^2\cos\theta=2r^2\cos^2\theta-a^2 \quad\ldots\ldots(1),$$

$$b^2\sin\theta=2r^2\sin^2\theta-b^2 \quad\ldots\ldots(2).$$

Eliminating r^2 from (1) and (2) we get

$$\frac{b^2(1+\sin\theta)}{1+\cos\theta}=a^2\tan^2\theta \quad\ldots\ldots(3).$$

Now $\sin\theta=\dfrac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$, $\cos\theta=\dfrac{1-\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$. $\therefore$ $b^2(1+\tan\frac{1}{2}\theta)^2=2a^2\tan^2\theta$,

$$\text{or } b(1+\tan\tfrac{1}{2}\theta)=\pm\sqrt{2}\,a\tan\theta \quad\ldots\ldots(4).$$

Let $z=\tan\frac{1}{2}\theta$; then (4) becomes

$$z^3+z^2-\left(1\mp\frac{2\sqrt{2}a}{b}\right)z=1 \quad\ldots\ldots(5).$$

Let $u=z-\frac{1}{3}$; then (5) becomes

$$u^3-\tfrac{1}{3}\left(2\mp\frac{3\sqrt{2a}}{b}\right)u=\tfrac{2}{27}\left(8\pm\frac{9\sqrt{2a}}{b}\right) \quad\ldots\ldots(6).$$

When a and b are known we can find u from (6), after which z and r and finally x and y become known.

II. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

As in preceding solution,

$$a^2\cos\theta=2r^2\cos^2\theta-a^2 \quad\ldots\ldots(3),$$

$$b^2\sin\theta=2r^2\sin^2\theta-b^2 \quad\ldots\ldots(4).$$

$\therefore\ 2r^2\cos^2\theta=a^2(1+\cos\theta)\ldots\ldots(5)$, and $2r^2\sin^2\theta=b^2(1+\sin\theta)\ldots\ldots(6)$.

Dividing (5) by (4), $\tan^2\theta=\frac{b^2}{a^2}\left(\frac{1+\sin\theta}{1+\cos\theta}\right)=\frac{b^2}{a^2}\left(\frac{\sec\theta+\tan\theta}{\sec\theta+1}\right)\ldots\ldots(7)$.

$\therefore\ a^2\tan^2\theta\sec\theta+a^2\tan^2\theta=b^2\sec\theta+b^2\tan\theta\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$.

$\therefore\ (a^2\tan^2\theta-b^2)\sec\theta=b^2\tan\theta-a^2\tan^2\theta\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$.

Squaring (9) and substituting for $\sec^2\theta$ its value $1+\tan^2\theta$, performing operations indicated and arranging with reference to $\tan\theta$,

$$a^4\tan^6\theta-2a^2b^2\tan^4\theta+2a^2b^2\tan^3\theta-2a^2b^2t^2+b^4=0\ldots\ldots\ldots(10).$$

Transposing the three middle terms and subtracting $2a^2b^2\tan^3\theta$,

$$a^4\tan^6\theta-2a^2b^2\tan^3\theta+b^4=2a^2b^2(\tan^4\theta-2\tan^3\theta+\tan^2\theta)\ \ldots\ldots(11).$$

Extracting square root, $a^2\tan^3\theta-b^2=\pm ab(\tan^2\theta-\tan\theta)\sqrt{2}\ldots\ldots\ldots(12)$.

$$\text{Whence }\tan\theta=\frac{\pm d\sqrt{2}}{3}+\frac{1}{3a}\left[\frac{b^2}{2}(9a\pm4b\sqrt{2})+\frac{3b}{2}(\pm24a^3b\sqrt{2}-39a^2b^2\pm24ab^3\sqrt{2})^{\frac{1}{2}}\right]^{\frac{1}{3}}$$

$$+\frac{1}{3a}\left[\frac{b^2}{2}(9a\pm4b\sqrt{2})-\frac{3b}{2}(\pm24a^3b\sqrt{2}-39a^2b^2\pm24ab^3\sqrt{2})^{\frac{1}{2}}\right]^{\frac{1}{3}}.$$

From equation (5), $x=a\sqrt{\frac{1+\cos\theta}{2}}=a\cos\frac{1}{2}\theta$, and from equation (6)

$$y=b\sqrt{\frac{1+\sin\theta}{2}}=b\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\theta)=\frac{b}{\sqrt{2}}(\cos\tfrac{1}{2}\theta-\sin\tfrac{1}{2}\theta).$$

If $a=b$, from (12), $\tan\theta=1$ or $\frac{\pm\sqrt{2}-1}{2}\pm\frac{1}{2}(\pm2\sqrt{2}-1)^{\frac{1}{2}}$.

III. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

Dividing (1) by (2) and putting $y=tx$, we obtain

$$x^2=\frac{a^2b^2(1-t)}{2t(a^2t-b^2)},\ \text{and then }y^2=\frac{a^2b^2t(1-t)}{2(a^2t-b^2)}.$$

Substituting these in (1), we obtain finally the equation

$$t^6-\frac{2b^2}{a^2}t^4+\frac{2b^2}{a^2}t^3-\frac{2b^2}{a^2}t^2+\frac{b^4}{a^4}=0.$$

Solving this for numerical values of a and b, we get the values of x and y om the above expressions.

The same equation may be arrived at by putting $x=r\cos\theta$, $y=r\sin\theta$. The iven equation then changes into $a^2\cos\theta=2r^2\cos^2\theta-a^2$; $b^2\sin\theta=2r^2\sin^2\theta-b^2$. dding, we get $a^2\cos\theta+b^2\sin\theta=2r^2-(a^2+b^2)$, whence $r^2=\frac{1}{2}[a^2\cos\theta+b^2\sin\theta$ $-a^2+b^2]$. Also, $r^2=\frac{1}{2}.\frac{a^2\cos\theta+a^2}{\cos^2\theta}$.

Equalizing, changing into the tangent function, the latter being denoted y t, we obtain the same equation as above.

IV. **Solution by H. C. WILKES, Skull Run, West Virginia.**

Putting $x^2+y^2=s^2$; then from (1), $a^2(x+s)=2sx^2$, and from (2), $^2(y+s)=2sy^2$. Any rational value for s will give integral [?] fractional values r a^2 and b^2. Let $s=5$, $a^2=45/4$, and $b^2=160/9$; $s=13$, $a^2=325/9$, and $^2=3744/25$; $s=17$, $a^2=2176/25$, and $b^2=3825/16$.

67. **Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, noxville, Tennessee.**

Prove that $\cos\frac{n\pi}{7}+\cos\frac{3n\pi}{7}+\cos\frac{5n\pi}{7}=\frac{1}{2}$ or $-\frac{1}{2}$, according as n is *odd* r *even*, [and not a multiple of 7].

I. **Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.**

$$\sin\frac{2n\pi}{7}=2\sin\frac{n\pi}{7}\cos\frac{n\pi}{7}.$$

$$\sin\frac{4n\pi}{7}-\sin\frac{2n\pi}{7}=2\sin\frac{n\pi}{7}\cos\frac{3n\pi}{7}.$$

$$\sin\frac{6n\pi}{7}-\sin\frac{4n\pi}{7}=2\sin\frac{n\pi}{7}\cos\frac{5n\pi}{7}.$$

$$\therefore \cos\frac{n\pi}{7}+\cos\frac{3n\pi}{7}+\cos\frac{5n\pi}{7}=\frac{\sin\frac{6}{7}n\pi}{2\sin\frac{1}{7}n\pi}=\frac{\sin(n\pi-\frac{1}{7}n\pi)}{2\sin\frac{1}{7}n\pi}$$

$=-\frac{1}{2}\cos n\pi=\pm\frac{1}{2}$, according as n is odd or even.

II. **Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.**

Employing the well-known formula

$$\sum_{n=1}^{n=n}\cos[a+(n-1)b]=\frac{\cos[a+\frac{1}{2}(n-1)b]\sin\frac{1}{2}nb}{\sin\frac{1}{2}b},$$

and putting $b=2a$, $n=3$, we have $\cos a+\cos 3a+\cos 5a=\dfrac{\cos 3a\sin 3a}{\sin a}=\frac{1}{2}\dfrac{\sin 6a}{\sin a}$.

But either $\frac{1}{2}\dfrac{\sin 6a}{\sin a}=\frac{1}{2}\dfrac{\sin 6a+\sin a}{\sin a}-\frac{1}{2}=\dfrac{\sin\frac{7}{2}a.\cos\frac{5}{2}a}{\sin a}-\frac{1}{2}$.

or, $\frac{1}{2}\dfrac{\sin 6a}{\sin a}=\frac{1}{2}\dfrac{\sin 6a-\sin a}{\sin a}+\frac{1}{2}=\dfrac{\cos\frac{7}{2}a.\sin\frac{5}{2}a}{\sin a}+\frac{1}{2}$.

Putting $a=\frac{1}{7}n\pi$, we get in the former case $\dfrac{\sin\frac{1}{2}n\pi\cos\frac{5}{14}n\pi}{\sin\frac{1}{7}n\pi}-\frac{1}{2}$, and in the latter $\dfrac{\cos\frac{1}{2}n\pi\sin\frac{5}{14}n\pi}{\sin\frac{1}{7}n\pi}+\frac{1}{2}$. If n is *even*, $\sin\frac{1}{2}n\pi=0$, if *odd*, $\cos\frac{1}{2}n\pi=0$.

Q. E. D.

III. Solution by OTTO O. CLAYTON, A. B., Fowler, Indiana.

Unite 1st and 3rd terms of the left member ; then by factoring, we have,

$$(2\cos\tfrac{2}{7}n\pi+1)\cos\tfrac{3}{7}n\pi=\tfrac{1}{2}\text{ or }-\tfrac{1}{2}.$$

Substituting for $(2\cos\frac{2}{7}n\pi+1)$, we have $\dfrac{\sin\frac{3}{7}n\pi\cos\frac{3}{7}n\pi}{\sin\frac{1}{7}n\pi}=\frac{1}{2}$ or $-\frac{1}{2}$, from which $\frac{1}{2}\dfrac{\sin\frac{6}{7}n\pi}{\sin\frac{1}{7}n\pi}=\frac{1}{2}.\dfrac{\sin-\frac{1}{7}n\pi}{\sin\frac{1}{7}n\pi}=\frac{1}{2}$ or $-\frac{1}{2}$.

This being an identical equation the problem is proved ; for ratio

$$\frac{\sin-\frac{1}{7}n\pi}{\sin\frac{1}{7}n\pi}=1\text{ or }-1,\text{ according as } n \text{ is odd or even.}$$

IV. Solution by JOHN B. FAUGHT, A. M., Instructor in Mathematics in Indiana University, Bloomington. Indiana.

The equation (1), $(\cos\theta+i\sin\theta)^7=-1$, *i. e.*, $\cos 7\theta+i\sin 7\theta=-1$, is clearly satisfied when θ has either of the following values : $\frac{1}{7}\pi$, $\frac{3}{7}\pi$, $\frac{5}{7}\pi$, $\frac{7}{7}\pi$, $\frac{9}{7}\pi$, $\frac{11}{7}\pi$ and $\frac{13}{7}\pi$.

$\therefore$ (2), $(\cos n\theta+i\sin n\theta)^7=(-1)^n$ is satisfied by $\theta=\frac{1}{7}\pi$, $\frac{3}{7}\pi$, $\frac{5}{7}\pi$, $\frac{7}{7}\pi$, $\frac{9}{7}\pi$, $\frac{11}{7}\pi$ or $\frac{13}{7}\pi$, or $n\theta=\frac{1}{7}n\pi$, $\frac{3}{7}n\pi$, $\frac{5}{7}n\pi$, $\frac{7}{7}n\pi$, $\frac{9}{7}n\pi$, $\frac{11}{7}n\pi$ or $\frac{13}{7}n\pi$.

But (3), $(\cos n\theta+i\sin n\theta)^7=\cos^7 n\theta+7i\cos^6 n\theta\sin n\theta-21\cos^5 n\theta\sin^2 n\theta$ $-35i\cos^4 n\theta\sin^3 n\theta+35\cos^3 n\theta\sin^4 n\theta+21i\cos^2 n\theta\sin^5 n\theta-7\cos n\theta\sin^6 n\theta-i\sin^7 n\theta$ $=(-1)^n$.

$\therefore$ (4), $\cos^7 n\theta-21\cos^5 n\theta\sin^2 n\theta+35\cos^3 n\theta\sin^4 n\theta-7\cos n\theta\sin^6 n\theta=(-1)^n$.

Or (5), $64\cos^7 n\theta-112\cos^5 n\theta+56\cos^3 n\theta-7\cos n\theta-(-1)^n=0$, of which

$\cos\frac{1}{7}n\pi$, $\cos\frac{3}{7}n\pi$, $\cos\frac{5}{7}n\pi$, $\cos\frac{7}{7}n\pi$, $\cos\frac{9}{7}n\pi$, $\cos\frac{11}{7}n\pi$ and $\cos\frac{13}{7}n\pi$ are the ratio.

Now $\cos\frac{7}{7}n\pi=\mp1$, according as n is *odd* or *even*, and $\cos\frac{13}{7}n\pi=\cos\frac{1}{7}n\pi$; $\cos\frac{9}{7}n\pi=\cos\frac{5}{7}n\pi$.

Hence we have (6), $(\cos n\theta\pm1)(64\cos^6 n\theta\mp64\cos^5 n\theta-48\cos^4 n\theta\pm48\cos^3 n\theta+8\cos^2 n\theta\mp\cos n\theta+1)=0$, according as n is *odd* or *even*.

$\therefore$ (7), $2(\cos\frac{1}{7}n\pi+\cos\frac{3}{7}n\pi+\cos\frac{5}{7}n\pi)=\pm\frac{7}{7}=\pm1$. Or $\cos\frac{1}{7}n\pi+\cos\frac{3}{7}n\pi+\cos\frac{5}{7}n\pi=\pm\frac{1}{2}$, according as n is *odd* or *even*.

We might deduce a number of equally interesting results, thus,

$$(\cos\tfrac{1}{7}n\pi.\cos\tfrac{3}{7}n\pi.\cos\tfrac{5}{7}n\pi)^2=\tfrac{1}{64}.$$

$\therefore$ $\cos\frac{1}{7}n\pi.\cos\frac{3}{7}n\pi.\cos\frac{5}{7}n\pi=\pm\frac{1}{8}$, when n is either *odd* or *even*, etc.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

63. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O., University of Mississippi.

A rectangular hyperbola cannot be cut from a right circular cone if the angle at its vertex is less than a right angle.

II. Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, New Jersey.

Assume axes of coördinates at right angles.

(a) The equation of the surface of a cone with axis of z as axis of cone, and origin at the vertex of cone is

$$x^2+y^2-z^2\tan^2\tfrac{1}{2}v=0\ldots\ldots\ldots\ldots(1)$$

where v=angle at vertex.

(b) The equation of a plane to same axes and origin as above, in terms of its direction cosines and perpendicular from origin is

$$lx+my+nz=\phi\ldots\ldots\ldots\ldots(2).$$

Eliminate z between (1) and (2), and then we have the conic

$$(n^2-l^2\alpha^2)x^2-2lmxy\alpha^2+y^2(n^2-m^2\alpha^2)+2pl\alpha^2x+2pm\alpha^2y-p^2\alpha^2=0\,..(3)$$

in which α is substituted for $\tan\frac{1}{2}v$.

In order that this conic may be an equilateral hyperbola, the angle between its asymptotes

$$(n^2-l^2\alpha^2)x^2-2lm\alpha^2xy+(n^2-m^2\alpha^2)y^2=0$$

must be a right angle, the condition for which is (for rectangular axes)

$$n^2-l^2\alpha^2+n^2-m^2\alpha^2=0, \text{ or } \alpha^2=2n^2/l^2+m^2 \;\ldots\ldots\ldots\ldots(4).$$

Now, in order that the plane above considered shall cut out the hyperbola, the angle whose direction cosine is n must be less than $\frac{1}{2}v$; that is to say, l and m must both be less than n. Hence, l^2+m^2 is necessarily less than n^2+n^2 or $2n^2$; or the fraction (4) is an improper fraction, whence $\alpha^2(=\tan^2\frac{1}{2}v)$ is greater than unity. This establishes that $\frac{1}{2}v$ is greater than 45°, and v is greater than 90°. Q. E. D.

III. Solution by GEORGE LILLEY, LL. D., Portland, Oregon.

Let ABF be a section of the cone made by the plane of the paper passing through its axis AM; $OPQNH$ any section of the cone made by a plane perpendicular to the plane ABF. Pass a plane through P at right angles to AM cutting the plane ABF in DE. Draw OL parallel to BF, and HK parallel to AF. Let $\angle MAB=\alpha$, $\angle AOH=\theta$, $AO=c$, $OH=x$, and $HP=y$.

$$HP^2=HD\times HE. \quad HE=\frac{x\sin\theta}{\cos\alpha},$$

$$DH=LK=2c\sin\alpha-\frac{x\sin(\theta+2\alpha)}{\cos\alpha}.$$

$$\therefore\; y^2=\frac{2c\sin\theta\sin\alpha}{\cos\alpha}x-\frac{\sin\theta\sin(\theta+2\alpha)}{\cos^2\alpha}x^2.$$

The section represented by the equation is any hyperbola when $\theta+2\alpha$ is greater than 180°. Comparing the equation with $y^2=\frac{2b^2}{a}x+\frac{b^2}{a^2}x^2$, we have

$$\frac{2b^2}{a}=\frac{2c\sin\alpha\sin\theta}{\cos\alpha}, \quad -\frac{b^2}{a^2}=\frac{\sin\theta\sin(\theta+2\alpha)}{\cos^2\alpha}.$$

$$\therefore\; 2a=\frac{c\sin2\alpha}{\sin(\theta+2\alpha)}, \quad b^2=\frac{c^2\sin^2\alpha}{\sin(\theta+2\alpha)}.$$

$e^2=\dfrac{a^2+b^2}{a^2}=\dfrac{1-\sin^2(\theta+\alpha)-2\sin^2\alpha}{\cos^2\alpha}$, where e is the excentricity of the hyperbola. Or $1=\dfrac{1-\sin^2(\theta+\alpha)-2\sin^2\alpha}{e^2\cos^2\alpha}$.

$e^2\cos^2\alpha$ must not be greater than unity. But $e^2=2$; therefore, $\cos^2\alpha$ must not be greater than $\frac{1}{2}$, and α must not be less than 45°. Hence, the angle at the vertex of the cone must not be less than a right angle; therefore, it is greater than a right angle.

It may, however, be equal to that angle.

Note on the angle between the asymptotes of the hyperbola.

Let ϕ=the angle between the asymptotes, and we have $\sec\frac{1}{2}\phi=e$, where e is the excentricity of the hyperbola.

$\sec^2\frac{1}{2}\phi\cos^2\alpha=e^2\cos^2\alpha$, or $\dfrac{\cos^2\alpha}{\cos^2\frac{1}{2}\phi}=e^2\cos^2\alpha$, but $e^2\cos^2\alpha$ must not be be greater than unity, see solution of problem 63. Hence, $\cos\frac{1}{2}\phi$ must not be less than $\cos\alpha$ and α must not be less than $\frac{1}{2}\phi$; or the angle at the vertex of the right circular cone, from which the hyperbola is cut, must not be less than the angle between the asymptotes. *It may, however, be equal to that angle.*

IV. Solution by W. E. CARTER, Professor of Mathematics in Centenary College of Louisiana, Jackson, Louisiana.

If the axis of the cone be the axis z; h, the distance of the vertex from the origin, and θ the semi-angle at the vertex, the equation of the cone is

$$(x^2+y^2)\tan^2(90^\circ-\theta)=(h-z)^2.$$

The section of this cone made by a plane through the axis y is a conic section, and if the angle which the plane makes with xy be ϕ and the curve of intersection be referred to axes in its own plane, its equation is

$$y^2\tan^2(90^\circ-\theta)+x^2\cos^2\phi[\tan^2(90^\circ-\theta)-\tan^2\phi]+2hx\sin\phi-h^2=0.$$

If this is a rectangular hyperbola, then

$$\tan^2(90^\circ-\theta)=\cos^2\phi[\tan^2\phi-\tan^2(90^\circ-\theta)].$$

$\therefore\ \tan\phi=\pm\dfrac{\sqrt{2}\sin(90^\circ-\theta)}{\sqrt{\cos^2(90^\circ-\theta)}}$. But ϕ is real.

$\therefore\ 90^\circ-\theta<45^\circ.\quad \therefore\ -\theta<-45^\circ.\quad \therefore\ 2\theta>90^\circ.$

An hyperbola may also be cut from this cone by a plane parallel to the axis z. Its equation then is, if the cutting plane is $y=a$,

$$(x^2+a^2)\tan^2(90^\circ-\theta)=(h-z)^2.$$

If this be a rectangular hyperbola,

$$\tan^2(90^\circ-\theta)=1 \qquad \tan(90^\circ-\theta)=1 \qquad (-1 \text{ makes } \theta \text{ negative}).$$

$$\therefore 90^\circ-\theta=45^\circ. \quad \therefore \theta=45^\circ. \quad \therefore 2\theta=90^\circ.$$

This problem was also solved in an excellent manner by *G. B. M. ZERR*.

64. Proposed by WILLIAM E. HEAL, Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

Let the bisectors of the angles A, B, C of a triangle meet the sides opposite A, B, C in A', B', C'. Let AA', BB', CC' meet the sides of the triangle $A'B'C'$ in A'', B'', C''. Let this process continue indefinitely. Express the sides and angles of the triangle $A^{(n)}B^{(n)}C^{(n)}$ in terms of the sides and angles of the original triangle ABC.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Using trilinear coordinates we have $\beta-\gamma=0$, $\gamma-\alpha=0$, $\alpha-\beta=0$ for the equations to AA', BB', CC' respectively.

$$\left(0, \frac{2\Delta}{b+c}, \frac{2\Delta}{b+c}\right), \quad \left(\frac{2\Delta}{a+c}, 0, \frac{2\Delta}{a+c}\right),$$

$$\left(\frac{2\Delta}{a+b}, \frac{2\Delta}{a+b}, 0\right)$$

are the coordinates of A', B', C' respectively.

$\therefore \alpha+\beta-\gamma=0$, $\alpha+\gamma-\beta=0$, $\beta+\gamma-\alpha=0$ are the equations to $A'B'$, $A'C'$, $B'C'$ respectively.

$$\left(\frac{4\Delta}{2a+b+c}, \frac{2\Delta}{2a+b+c}, \frac{2\Delta}{2a+b+c}\right), \left(\frac{2\Delta}{a+2b+c}, \frac{4\Delta}{a+2b+c}, \frac{2\Delta}{a+2b+c}\right),$$

$$\left(\frac{2\Delta}{a+b+2c}, \frac{2\Delta}{a+b+2c}, \frac{4\Delta}{a+b+2c}\right)$$

are the coordinates of A'', B'', C'' respectively.

$\therefore \alpha+\beta-3\gamma=0$, $\alpha+\gamma-3\beta=0$, $\beta+\gamma-3\alpha=0$ are the equations to $A''C''$, $A''C''$, $B''C''$ respectively.

$$\left(\frac{4\Delta}{2a+3b+3c}, \frac{6\Delta}{2a+3b+3c}, \frac{6\Delta}{2a+3b+3c}\right), \left(\frac{6\Delta}{3a+2b+3c}, \frac{4\Delta}{3a+2b+3c},\right.$$

$$\frac{6\Delta}{3a+2b+3c}\Big), \Big(\frac{6\Delta}{3a+3b+2c}, \frac{6\Delta}{3a+3b+2c}, \frac{4\Delta}{3a+3b+2c}\Big)$$

are the coordinates of A''', B''', C''' respectively.

$\therefore$ $3\alpha+3\beta-5\gamma=0$, $3\alpha+3\gamma-5\beta=0$, $3\beta+3\gamma-5\alpha=0$ are the equations to $A'''B'''$, $A'''C'''$, $B'''C'''$ respectively.

$$\Big(\frac{12\Delta}{6a+5b+5c}, \frac{10\Delta}{6a+5b+5c}, \frac{10\Delta}{6a+5b+5c}\Big), \Big(\frac{10\Delta}{5a+6b+5c}, \frac{12\Delta}{5a+6b+5c},$$

$$\frac{10\Delta}{5a+6b+5c}\Big), \Big(\frac{10\Delta}{5a+5b+6c}, \frac{10\Delta}{5a+5b+6c}, \frac{12\Delta}{5a+5b+6c}\Big)$$

are the coordinates of A'''', B'''', C'''' respectively.

$\therefore$ $5\alpha+5\beta-11\gamma=0$, $5\alpha+5\gamma-11\beta=0$, $5\beta+5\gamma-11\alpha=0$ are the equations to $A''''B''''$, $A''''C''''$, $B''''C''''$ respectively.

In what follows, the upper signs are used for m odd, and the lower for m even. The mth term of the series 1, 3, 5, 11, 21, 43, 85, etc., is $\frac{1}{3}(2^m\pm1)$.

$$\therefore\Big(\frac{4\Delta(2^{m-1}\mp1)}{2(2^{m-1}\mp1)a+(2^m\pm1)(b+c)}, \frac{2\Delta(2^m\pm1)}{2(2^{m-1}\mp1)a+(2^m\pm1)(b+c)},$$

$$\frac{2\Delta(2^m\pm1)}{2(2^{m-1}\mp1)a+(2^m\pm1)(b+c)}\Big), \Big(\frac{2\Delta(2^m\pm1)}{2(2^{m-1}\mp1)b+(2^m\pm1)(a+c)},$$

$$\frac{4\Delta(2^{m-1}\mp1)}{2(2^{m-1}\mp1)b+(2^m\pm1)(a+c)}, \frac{2\Delta(2^m\pm1)}{2(2^{m-1}\mp1)b+(2^m\pm1)(a+c)}\Big),$$

$$\Big(\frac{2\Delta(2^m\pm1)}{2(2^{m-1}\mp1)c+(2^m\pm1)(a+b)}, \frac{2\Delta(2^m\pm1)}{2(2^{m-1}\mp1)c+(2^m\pm1)(a+b)},$$

$$\frac{4\Delta(2^{m-1}\mp1)}{2(2^{m-1}\mp1)c+(2^m\pm1)(a+b)}\Big)$$

are the coordinates of A^m, B^m, C^m respectively.

$\therefore$ $\frac{1}{3}(2^m\pm1)(\alpha+\beta)-\frac{1}{3}(2^{m+1}\mp1)\gamma=0$, $\frac{1}{3}(2^m\pm1)(\alpha+\gamma)-\frac{1}{3}(2^{m+1}\mp1)\beta=0$, $\frac{1}{3}(2^m\pm1)(\beta+\gamma)-\frac{1}{3}(2^{m+1}\mp1)\alpha=0$, (1, 2, 3) are the equations to A^mB^m, A^mC^m, B^mC^m respectively.

From (1) and (2), (1) and (3), (2) and (3) respectively, we get

$$\tan A^m = \frac{3\{2^{m+1}(2^{m-1}\mp1)\sin A+2^m(2^m\pm1)(\sin B+\sin C)\}}{3(2^{2m}-1)+2(5.2^{2m-1}\mp2^m+1)\cos A-2(2^m\pm1)(2^{m-1}\mp1)(\cos B+\cos C)}$$

$$\tan B^m = \frac{3\{2^{m+1}(2^{m-1}\mp1)\sin B + 2^m(2^m\pm1)(\sin A+\sin C)\}}{3(2^{2m}-1)+2(5.2^{2m-1}\mp2^m+1)\cos B - 2(2^m\pm1)(2^{m-1}\mp1)(\cos A+\cos C)}$$

$$\tan C^m = \frac{3\{2^{m+1}(2^{m-1}\mp1)\sin C + 2^m(2^m\pm1)(\sin A+\sin B)\}}{3(2^{2m}-1)+2(5.2^{2m-1}\mp2^m+1)\cos C - 2(2^m\pm1)(2^{m-1}\mp1)(\cos A+\cos B)}$$

Let A=area of $A^{(m)}B^{(m)}C^{(m)}$, p=perpendicular from $C^{(m)}$ on $A^{(m)}B^{(m)}$.

$$\therefore\ A = [27abc\triangle.2^m] \div [\{2(2^{m-1}\mp1)a + (2^m\pm1)(b+c)\}$$
$$\{2(2^{m-1}\mp1)b + (2^m\pm1)(a+c)\}\{2(2^{m-1}\mp1)c + (2^m\pm1)(a+b)\}]$$

$$p = [9\triangle.2^{m+1}] \div [\{(2^m\pm1)(a+b)+2(2^{m-1}\mp1)c\}$$
$$\sqrt{3(2^{2m+1}+1)+2(2^m\pm1)(2^{m+1}\mp1)(\cos A+\cos B) - 2(2^m\pm1)^2\cos C}].$$

But $A = \frac{1}{2}pA^{(m)}B^{(m)}$. $\therefore\ A^{(m)}B^{(m)} = 2A/p$.

$\therefore\ A^{(m)}B^{(m)}$

$$= \frac{3abc\sqrt{3(2^{2m+1}+1)+2(2^m\pm1)(2^{m+1}\mp1)(\cos A+\cos B)-2(2^m\pm1)^2\cos C}}{\{2(2^{m-1}\mp1)a+(2^m\pm1)(b+c)\}\{2(2^{m-1}\mp1)b+(2^m\pm1)(a+c)\}}$$

$$A^{(m)}C^{(m)} = \frac{3abc\sqrt{3(2^{2m+1}+1)+2(2^m\pm1)(2^{m+1}\mp1)(\cos A+\cos C)-2(2^m\pm1)^2\cos B}}{\{2(2^{m-1}\mp1)a+(2^m\pm1)(b+c)\}\{2(2^{m-1}\mp1)c+(2^m\pm1)(a+b)\}}$$

$$B^{(m)}C^{(m)} = \frac{3abc\sqrt{3(2^{2m+1}+1)+2(2^m\pm1)(2^{m+1}\mp1)(\cos B+\cos B)-2(2^m\pm1)^2\cos A}}{\{2(2^{m-1}\mp1)b+(2^m\pm1)(a+c)\}\{2(2^{m-1}\mp1)c+(2^m\pm1)(a+b)\}}$$

All principles necessary to understand the above solution will be found in the chapter on "Trilinear Coordinates" in Todhunter's Conic Sections.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

53. Proposed by O. D. SMITH, A. M., Professor of Mathematics, Alabama Polytechnic Institute, Auburn, Alabama.

Solve the differential equation, $dy/dx=y(x-y)/x(x+y)$, and show that $x=y\log(xy)$.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland; C. W. M. BLACK, Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts; and P. S. BERG, Larimore, North Dakota.

Clearing of fractions we obtain after transposing two terms

$$y(xdy+ydx)=x(ydx-xdy).$$

Dividing by y^2, we have

$$xdy+ydx=xy.\frac{ydx-xdy}{y^2}, \text{ or, } d(xy)=xy.d\left(\frac{x}{y}\right). \quad \therefore \frac{d(xy)}{xy}=d\left(\frac{x}{y}\right).$$

Integrating, $\log(axy)=(x)+(y)$, whence $x=y\log(axy)$.

The result given is not general enough, the constant having been left out of consideration.

II. Solution by W. W. LANDIS, A. M., Department of Mathematics and Astronomy in Dickinson College, Carlisle, Pennsylvania; F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, New Jersey; J. OWEN MAHONEY, B. E., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tennessee; G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas; and A. H. HOLMES, Brunswick, Maine.

Let $y=vx$, then the equation becomes

$$v+\frac{xdv}{dx}=\frac{x(1-v)}{1+v}, \text{ or } 2v^2+\frac{x(1+v)dv}{dx}=0. \quad \therefore \frac{1+v}{2v^2}dv+\frac{dx}{x}=0.$$

The variables are separable, whence

$$\frac{dv}{2v^2}+\frac{dv}{2v}+\frac{dx}{x}=0. \quad \text{Integrating, } \frac{1}{2v}=\tfrac{1}{2}\log v+\log x.$$

$\therefore 1/v=\log v+\log x^2=\log(vx^2)$.

$\therefore x/y=\log(xy)$, or $x=y\log(xy)$, when no constant is added, or $x=y\log(xy)+Cy$ where C is an arbitrary constant.

III. Solution by M. C. STEVENS, M. A., Mathematical Department, Purdue University, Lafayette, Indiana; HENRY HEATON, M. Sc., Atlantic, Iowa; JOHN B. FAUGHT, A. M., Instructor in Mathematics in Indiana University, Bloomington, Indiana; and J. C. GREGG, A. M., Brazil, Indiana.

Put $y=vx$ and we have

$$v+x\frac{dv}{dx}=\frac{vx^2-v^2x^2}{x^2+vx^2}=\frac{v-v^2}{1+v}=v-\frac{2v^2}{1+v}.\quad \text{Whence, } \frac{1+v}{v^2}dv+\frac{2dx}{x}=0.$$

Integrating, $-1/v+\log(vx^2)+C=0$, or $x/y=\log(xy)+C$, and $x=y\log(xy)+Cy$.

The C should not be omitted unless the conditions of the question giving rise to the equation are such as to make it zero.

IV. Solution by H. C. WHITTAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

Let $y=x^pv^q$ and substitute in the given equation and we obtain

$$\frac{dv}{dx}=\frac{(1-p)v-(1+p)x^{p-1}v^{q+1}}{q^x(1+x^{p-1}v^q}.$$

This will reduce to a simple form if we take $p=1$ and $q=-1$, giving

$$\frac{dv}{dx}=\frac{2}{x(1+v^{-1})},\text{ or } dv(1+v^{-1})=2x^{-1}dx.$$

$v+\log v=\log x^2$; $x/y+\log(x/y)=\log x^2$.
$x/y=\log x^2-\log(x/y)=\log(xy)$, whence $x=y\log(xy)$.

64. Proposed by J. SCHEFFER, A. M., Hagerstown, Maryland.

A certain solid has a square, side$=a$, for its base, and all parallel sections are squares, the two sections through the middle points of the opposite side of the square are semi-circles, however. Find surface, volume, and center of gravity of each.

I. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

The length of a side of a parallel section distant x from the base is $(a^2-4x^2)^{\frac{1}{2}}$. If dx be the distance between two parallel sections, the distance between two corresponding sides is $adx/(a^2-4x^2)^{\frac{1}{2}}$. Hence the surface

$$S=4\int_0^{\frac{1}{2}a} adx=2a^2\text{ ; the volume } V=\int_0^{\frac{1}{2}a}(a^2-4x^2)dx=\tfrac{1}{3}a^3\text{ ;}$$

the distance of the center of gravity of the surface from the base is

$$\frac{1}{2a^2}\int_0^{\frac{1}{2}a} axdx=\tfrac{1}{8}a\text{ ;}$$

and the distance of the center of gravity of the volume from the base is

$$\frac{3}{a^3}\int_0^{\frac{1}{2}a} x(a^2-4x^2)dx=\tfrac{3}{16}a.$$

II. Solution by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering in the State A. M. College, College Station, Texas.

Take the intersection of the planes of the circular sections as the axis of z, the origin being in the center of the base. Then since the radius of each circle is $\frac{1}{2}a$ we shall have for the projection of one fourth of the elementary area intercepted between two planes parallel to the base, and distant dz from each other, upon the plane of one of the circles,

$$dS.\cos\theta=\sqrt{\tfrac{1}{4}a^2-z^2}.dz,$$

where θ is the angle made by this elementary area with the plane of projection.

But $\cos\theta=\dfrac{\sqrt{\frac{1}{4}a^2-z^2}}{\frac{1}{2}a}$, and the whole surface is $S=4\int_0^{\frac{1}{2}a} adz=2a^2$....(1).

The center of gravity of S is distant from the base

$$z_1=\frac{4\int_0^{\frac{1}{2}a} azdz}{2a^2}=\tfrac{1}{4}a \text{........................(2).}$$

For the volume, taking planes parallel to the base,

$$V=\int_0^{\frac{1}{2}a} 2\sqrt{\tfrac{1}{4}a^2-z^2}.2\sqrt{\tfrac{1}{4}a^2-z^2}.dz=\int_0^{\frac{1}{2}a}(a^2-4z^2)dz=a^3/3 \text{.........(3),}$$

and its center of gravity above the base is

$$z_2=\frac{\int_0^{\frac{1}{2}a}(a^2-4z^2)zdz}{\frac{1}{3}a^3}=\tfrac{3}{16}a \text{....................(4).}$$

The figure will be a cloistered arch formed by the intersection of two right semi-cylinders.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $x^2+z^2=\frac{1}{4}a^2$......(1), $y^2+z^2=\frac{1}{4}a^2$......(2) be the equations to the cylinders which form the groin. From (1) $dz/dx=-x/z$, $dz/dy=0$.

$$S=\int\int\sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}\,dxdy=8\int_0^{\frac{1}{2}a}\int_0^x\sqrt{1+\frac{x^2}{z^2}}\,dxdy$$

$$=4a\int_0^{\frac{1}{2}a}\int_0^x\frac{dxdy}{z}=4a\int_0^{\frac{1}{2}a}\int_0^x\frac{dxdy}{\sqrt{\frac{1}{4}a^2-x^2}}=4a\int_0^{\frac{1}{2}a}\frac{xdx}{\sqrt{\frac{1}{4}a^2-x^2}}=2a^2.$$

$$V=\int\int\int dzdxdy=4\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}\sqrt{a^2-4z^2}}\int_0^{\frac{1}{2}\sqrt{a^2-4z^2}}dzdxdy=\int_0^{\frac{1}{2}a}(a^2-4z^2)dz=\tfrac{1}{3}a^3.$$

$$\text{Center of graving of surface}=\frac{\int\int zdS}{\int\int dS}=\frac{2}{a}\int_0^{\frac{1}{2}a}\int_0^x dxdy=\tfrac{1}{4}a.$$

$$\text{Center of gravity of volume}=\frac{\int\int\int zdzdxdy}{\int\int\int dzdxdy}=\frac{3}{a^3}\int_0^{\frac{1}{2}a}z(a^2-4z^2)dz=\tfrac{3}{16}a.$$

IV. **Solution by J. C. GREGG, A. M., Brazil, Indiana.**

Let the given figure represent a section of the solid through the middle point of two opposite sides of the base. We have $r=a/2$, and the equation of the circle EDF is $x^2+y^2=r^2\ldots.(1)$, and $AC^2=(2y)^2=4(r^2-x^2)=A_x=$a section parallel to the base, and for the volume

$$V=4\int_0^r(r^2-x^2)dx=\tfrac{8}{3}r^3=\tfrac{1}{3}a^3.$$

The surface may be considered to be generated by the sides of a section parallel to the base, and we have for the surface,

$$S=4\int 2yds=4\int_0^r 2\sqrt{r^2-x^2}.\frac{rdx}{\sqrt{r^2-x^2}},\ =8r\int_0^r dx=8r^2=2a^2.$$

For the center of gravity of the volume,

$$\bar{x}=\frac{\int x(A_x)dx}{V},\ =\frac{4\int_0^r x(r^2-x^2)dx}{\frac{8}{3}r^3},\ =\frac{3}{2r^3}\int_0^r x(r^2-x^2)dx=\tfrac{3}{8}r=\tfrac{3}{16}a.$$

For the center of gravity of the *curved* surface we have,

$$\bar{x}=\frac{4\int 2xyds}{S},\ =\frac{8r\int_0^r xdx}{8r^2}=\frac{1}{r}\int_0^r xdx,\ =\tfrac{1}{2}r,\ =\tfrac{1}{4}a.$$

For the center of gravity of the *whole* surface, since the curved surface is *twice* that of the base we have, $\bar{x}=\tfrac{2}{3}.\tfrac{1}{4}a=\tfrac{1}{6}a$.

Also solved by *H. C. WHITAKER*, *C. W. M. BLACK*, and the *PROPOSER*.

Professors Black and Scheffer used "side$=2a$" as in Problem 47, instead of *side*$=a$, and hence their results did not agree with those in the published solutions. The results obtained were: Volume$=8a^3/3$, surface$=8a^2$, center of gravity of volume$=3a/8$, and center of gravity of surface$=\tfrac{1}{3}a$. See problem 42 for two additional SOLUTIONS for surface and volume. EDITOR.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

28. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A prolate spheroid of revolution is fixed at its focus; a blow is given it at the extremity of the axis minor in a line tangent to the direction perpendicular to the axis major. Find the axis about which the body begins to rotate. [From *Loudon's Rigid Dynamics*.]

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

The general equations of motion are:

$$\left.\begin{array}{l} A\omega_x-(\Sigma mxy)\omega_y-(\Sigma mxz)\omega_z=L \\ B\omega_y-(\Sigma myz)\omega_z-(\Sigma myx)\omega_x=M \\ C\omega_z-(\Sigma mzx)\omega_x-(\Sigma mzy)\omega_y=N \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ (1).$$

The equation to the ellipsoid with focus as origin is $a^2y^2+a^2z^2+b^2x^2=2aeb^2x+b^4$. Now $\Sigma mxy=\Sigma mxz=\Sigma myz=0$. $\therefore$ (1) reduce to

$$\left.\begin{array}{l} A\omega_x=L \\ B\omega_y=M \\ C\omega_z=N \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2).$$

Let $2aeb^2x+b^4-b^2x^2=a^2c^2$. Then

$$A=\Sigma m(y^2+z^2)=\mu\int_{-a(1-e)}^{a(1+e)}\int_{-c}^{c}\int_{-\sqrt{c^2-y^2}}^{\sqrt{c^2-y^2}}(y^2+z^2)dxdydz$$

$$=\frac{4\mu}{3}\int_{-a(1-e)}^{a(1+e)}\int_0^c \{3y^2\sqrt{c^2-y^2}+(c^2-y^2)\sqrt{c^2-y^2}\}dxdy$$

$$=\frac{\pi\mu}{2a^4}\int_{-a(1-e)}^{a(1+e)}(2aeb^2x+b^4-b^2x^2)^2dx=\tfrac{8}{15}\mu\pi ab^4.$$

$$B=C=\Sigma m(x^2+y^2)=\mu\int_{-a(1-e)}^{a(1+e)}\int_{-c}^{c}\int_{-\sqrt{c^2-y^2}}^{\sqrt{c^2-y^2}}(x^2+y^2)dxdydz$$

$$=4\mu\int_{-a(1-e)}^{a(1+e)}\int_0^c (x^2+y^2)\sqrt{c^2-y^2}\,dxdy$$

$$=\frac{\mu\pi}{4}\int_{-a(1-e)}^{a(1+e)}(4c^2x^2+c^4)dx=\tfrac{8}{15}\mu\pi a^3b^2(1+2e^2).$$

Let the blow$=P$ be struck perpendicular to the plane (xy), then the ments of the impulsive forces about the axes are $L=Pb$, $M=Pae$, $N=0$.
These in (2) give

$$\left.\begin{array}{l}\tfrac{8}{15}\mu\pi ab^4\omega_x=Pb\\ \tfrac{8}{15}\mu\pi a^3b^2(1+2e^2)\omega_y=Pae\\ \omega_z=0\end{array}\right\}\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\therefore\frac{\omega_y}{\omega_x}=\frac{e^2}{1+2e^2}\cdot\frac{b}{ae}.$$

Let F be the focus, O the center of the ellipsoid. Then on the minor in the plane (xy), take $OE=\frac{e^2}{1+2e^2}.b$, then will FE be the axis required.

Let $a=5$, $b=4$. $\therefore e=\tfrac{3}{5}$. $\therefore OE=\tfrac{9}{43}b=\tfrac{36}{43}$. The resultant angula locity will be

$$\frac{15P}{8\mu\pi a^2b^2e}.OF=\frac{P}{512\mu\pi}.OF=\frac{3P\sqrt{1993}}{22016\mu\pi},\text{ when } a=5,\ b=4.$$

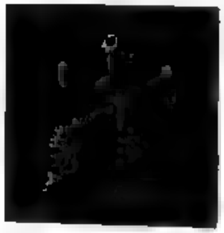
30. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Spring- Missouri.

A person whose height is a and weight W stands in a swing whose length is l. Sup- ng the initial inclination of the swing to the vertical is α and that the person always ches when in the highest position and stands up when in the lowest, his center of grav- noving through a distance b measured from lower part of swing upward, find how much arc is increased after n complete vibrations.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington,

Let RS be the path of the center of gravity from extreme position to verti- TU the path from vertical to other extreme position.

$OP=l$, $RP=k$, (say), $TS=b$; $\angle ROS=\alpha$, $\angle TOU$ $_1$, etc.



The energy acquired by swing in passing from S is $(l-k)(1-\cos\alpha)W$.

When it has passed to V the energy is $k-b)(1-\cos\alpha_1)W$.

By conservation of energy

$$(l-k-b)(1-\cos\alpha_1)W=(l-k)(1-\cos\alpha)W.$$

$$\text{Whence } 1-\cos\alpha_1=\frac{l-k}{l-k-b}(1-\cos\alpha).$$

In passing back to original position

$$1-\cos\alpha_2=\frac{l-k}{l-k-b}(1-\cos\alpha_1)=\left(\frac{l-k}{l-k-b}\right)^2(1-\cos\alpha).$$

$$\text{For two complete vibrations } 1-\cos\alpha_4=\left(\frac{l-k}{l-k-b}\right)^4(1-\cos\alpha).$$

In like manner for n complete vibrations

$$1-\cos\alpha_{2n}=\left(\frac{l-k}{l-k-b}\right)^{2n}(1-\cos\alpha) \text{ or } \sin(\tfrac{1}{2}\alpha_{2n})=\left(\frac{l-k}{l-k-b}\right)^{n}\sin(\tfrac{1}{2}\alpha),$$

:h enables us to compute the increase in amplitude.

II. Solution by the PROPOSER.

Let O be the point of suspension of the swing, S the position of the center ravity of the man when crouching, and T its position when the man is stand- and Q the lower end of the swing.

Let $OQ=l$, $SQ=k$, the distance from lower end of swing to the center of gravity of the man when he is crouching, $ST=b$, $\angle QOP=\alpha$, and $\angle QOV=\theta_0$.

Then the velocity, v, of the man at the point Q, $=\sqrt{2gl(l-k)(1-\cos\alpha)}$, *Bowser's Analytical Mechanics*, page 350, Article 192. Hence, his energy due to his weight $=\frac{W}{2g}v^2=W(l-k)(1-\cos\alpha)$. This energy will carry him to V and equals $W(l-k-b)(1-\cos\theta_0)$, since the man rises at the point Q. Since he crouches at the point V, his energy at the point Q on his return is $W(l-k)(1-\cos\theta_0)$. This energy will carry him to a point to the left of S, and the energy expended will be $W(l-k-b)(1-\cos\theta_1)$, where θ_1 is the angle between the vertical and the swing.

According to the principle of the conservation of energy, we have,

$$W(l-k)(1-\cos\alpha)=W(l-k-b)(1-\cos\theta_0)\ldots\ldots\ldots\ldots(1),$$

$$W(l-k)(1-\cos\theta_0)=W(l-k-b)(1-\cos\theta_1)\ldots\ldots\ldots\ldots(2),$$

$$W(l-k)(1-\cos\theta_1)=W(l-k-b)(1-\cos\theta_2)\ldots\ldots\ldots\ldots(3),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$W(l-k)(1-\cos\theta_{2n-1})=W(l-k-b)(1-\cos\theta_{2n-1})\ldots\ldots(2n).$$

Multiplying these equations together member for member, and solving for $1-\cos\theta_{2n}$, we have,

$$1-\cos\theta_{2n}=\left(\frac{l-k}{l-k-b}\right)^{2n}(1-\cos\alpha),\text{ or }\sin^2(\tfrac{1}{2}\theta_{2n})=\left(\frac{l-k}{l-k-b}\right)^{2n}(\sin^2\tfrac{1}{2}\alpha).$$

$$\text{Whence, }\sin\tfrac{1}{2}\theta_{2n}=\left(\frac{l-k}{l-k-b}\right)^{n}\sin\tfrac{1}{2}\alpha,\text{ or }\theta_{2n}=2\sin^{-1}\left[\left(\frac{l-k}{l-k-b}\right)\sin\tfrac{1}{2}\alpha\right].$$

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

42. Proposed by E. B. ESCOTT, Chicago, Illinois.

Required all the parallelograms whose sides a, b and diagonals c, d, are rational.

II. Solution by W. F. KING, Ottawa, Canada.

The conditions are, $2(a^2+b^2)=c^2+d^2$, with the condition that a, b, c and b, d shall be capable of forming triangles (sum of any two sides greater than third side). That is, if we suppose $a>b$, $c>d$, $a+b$ must be greater than c, $a-b<d$. These two conditions again are the same, for if $a+b>c$ and $2(a^2+b^2)=c^2+d^2$, $2(a^2+b^2)-(a+b)^2<d^2$ or $a-b<d$.

Let us suppose the numbers involved to be integers. We have $c^2+d^2=2(a^2+b^2)=(a+b)^2+(a-b)^2$. If $c=a+b$, $d=a-b$, the parallelogram vanishes, the angle opposite to the diagonal c becoming 180°. But if not, we have a number c^2+d^2 which is resolvable into the sum of two squares in another way. Hence, as may easily be proved, a^2+b^2 is resolvable into factors, which, by a theorem in the Theory of Numbers, are (whether prime or composite) each expressible as the sum of two squares. Also every prime number of the form $4n+1$ is expressible as the sum of two squares, and every number which is the sum of two squares is the product of prime factors of the form $4n+1$. (The only even prime $2=1^2+1^2$ or a power thereof may also be a factor, which case will be considered further on).

Hence we have a rule to find a and b so as to make c and d rational.

Form a^2+b^2 by multiplying together two or more of the various prime numbers of the form $4n+1$, such as 5, 13, 17, 29, etc.

The product may be expressed in two ways at least as the sum of two squares. Thus we shall have $f^2+g^2=h^2+k^2$.

$\therefore$ $2(f^2+g^2)=(h^2+k)^2+(h-k)^2$, which gives a solution by putting $f=a$, $g=b$, $h+k=c$, $h-k=d$, provided that, (following the condition for a possible triangle) $f-g<h-k$.

If $f-g>h-k$, we must take h and k for a and b ; $f+g$ and $f-g$ for c and d. Then $2(h^2+k^2)=(f+g)^2+(f-g)^2$.

That is, of the two equal sums into which the product has been resolved, take that for a^2+b^2 which has the less difference between its components.

For example, multiply 5 by $13=65$.

$65=8^2+1^2=7^2+4^2$ and $7-4=3<8-1$. Hence $a=7$, $b=4$, $h=8$, $k=1$, and $2(7^2+4^2)=(8+1)^2+(8-1)^2=9^2+7^2$.

$\therefore$ $c=9$, $d=7$. And 7, 4, 9 ; 7, 4, 7 are possible sides for triangles. The components of the product can readily be found from the components of the prime factors, thus:

Let $N=(p^2+q^2)(r^2+s^2)=p^2r^2+2pqrs+q^2s^2+q^2r^2-2pqrs+p^2s^2=p^2r^2-2pqrs+q^2s^2+q^2r^2+2pqrs+p^2s^2=(pr+qs)^2(qr-ps)^2=(pr-qs)^2+(qr+ps)^2$.

For example, to resolve 65 into the sum of two squares.

$65=13\times5=(3^2+2^2)(2^2+1^2)$. Here $pr+qs=3\times2+2\times1=8$.

$qr-ps=2\times2-3\times1=1$. $pr-qs=3\times2-2\times1=4$,

$qr+ps=2\times2+3\times1=7$. $\therefore$ $65=8^2+1^2=4^2+7^2$.

A third factor, $r_1^2+s_1^2$, can be introduced by putting $pr+qs=p_1$, $qr-ps=q_1$, and multiplying out $(p_1^2+q_1^2)(r_1^2+s_1^2)$ as before.

Observe that this gives two forms for the product, and two more would be got by putting $pr-qs=p_1$, $qr+ps=q_1$, so that with three factors there will be four forms for the product. These forms may be taken any two together giving $4.3/1.2=6$ solutions for c and d.

In the preceding it has been assumed that a and b have no common factor. If they have one (which may be any number whatever) the preceding investigation will still hold. But such factors of the common measure of a and b as are not primes of the form $4n+1$ will re-appear as common factors of c and d. It is to be noted that if $a^2+b^2=2\times$ a single prime factor a^2+b^2 can be expressed as the sum of two squares in one way only, viz: $a^2+b^2=2(p^2+q^2)=(p+q)^2+(p-q)^2$. For the factors of a^2+b^2 are p^2+q^2 and r^2+s^2 where $r=s=1$, and in multiplying $(p^2+q^2)(r^2+s^2)$ the two expressions $(qr+ps)^2+(pr-qs)^2$ and $(pr+qs)^2+(qr-ps)^2$ become identical when $r=s=1$, and these two cannot be equated to a third and different sum of two squares without factoring p^2+q^2, which is by supposition a prime. Hence when $\frac{1}{2}(a^2+b^2)$ is a prime the solution fails, for we get $2(a^2+b^2)=4(p^2+q^2)=(2p)^2+(2q)^2=(a+b)^2+(a-b)^2$, which does not give a parallelogram. So also when a^2+b^2 is a product of a prime by any odd power of 2. An even power of 2 may however be used, for example,

$$a^2+b^2=260=2^2\times5\times13=2^2(3^2+2^2)(2^2+1^2)=(6^2+4^2)(2^2+1^2)=16^2+2^2$$
$$=14^2+8^2 \text{ and } 2(a^2+b^2)=2(14^2+8^2)=2(16^2+2^2)=18^2+14^2=c^2+d^2.$$

The above discussion made on the assumption that a, b, c, d are integers, is readily extended to give solutions in rational fractions.

Thus $1885=5\times13\times29=42^2+11^2=34^2+27^2$.

$\therefore\ 7^2+(\frac{11}{6})^2=(\frac{17}{3})^2+(\frac{9}{2})^2$ and $2\{(\frac{17}{3})^2+(\frac{9}{2})^2\}=(\frac{61}{6})^2+(\frac{7}{6})^2$.

Note on solution of problem 37, page 151. The failure to give the least values in my solution was due to solving $x_1{}^2-40y_1{}^2=1$; by continued fractions, we obtain positive integral values for x and y, but y does not enter into the required values directly; hence may be fractional. This point was overlooked.

To obtain all these values, let $x=(x_2)+(z_2)$, $y=(y_2)+(z_2)$, and then $x_2{}^2-z_2{}^2=40y_2{}^2=10y_2{}^2$. $(x_2+z_2)(x_2-z_2)=10y_2{}^2$. Let $y_2{}^2=p^2q^2$ and $10=$any two factors.

$$\left.\begin{array}{l} x_2+z_2=p^2 \text{ or } 2p^2 \\ x_2-z_2=10q^2 \text{ or } 5q^2 \end{array}\right\}$$

Add and subtract, then

$$\left.\begin{array}{l} x_2=p^2+10q^2 \text{ or } 2p^2+5q^2 \\ z_2=\mp p^2\pm10q^2 \text{ or } \mp2p^2\pm5q^2 \\ y_2=2pq \end{array}\right\}$$

p and q taken at pleasure will give an infinite number of values, integral and fractional. Mr. Gruber's list is correct. A. H. Bell.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

Note on Problem 33. By Henry Heaton, Atlantic, Iowa.

The result obtained in the published solution of this problem cannot be correct.

The area of the regular pentagon is $3.6327a^2$. The area of each of the infinite number of regular polygons whose apothem is a and number of sides greater than five, is less than $3.6327a^2$, while that of only two, the square and triangle, is greater. Hence the average area of all regular polygons with apothem a is less than $3.6327a^2$. Hence the result obtained in the published solution ($3.8693a^2$) is too large.

In a similar manner it may be shown that any result larger than $a^2\pi$ is too large, while it is evident that any result smaller than that is too small.

37. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

Required the average area of all triangles two of whose sides are a and b.

I. Solution by the PROPOSER.

It is well known that every triangle consists of six parts, three sides and three angles, and one side with any two other parts determines the triangle.

In constructing this triangle we may use all possible values, first, of the included angle C, second, of the third side, c, third, of the angle A, and fourth, of the angle B. This gives us four cases.

I. Put angle $C=\theta$. Then $A_1=\dfrac{ab}{2}\int_0^\pi \sin\theta d\theta \div \int_0^\pi d\theta=\dfrac{ab}{\pi}$.

II. Put side $c=x$. Then

$$A_2=\tfrac{1}{4}\int_{a-b}^{a+b} [(a+b)^2-x^2]^{\frac{1}{2}}\,[x^2-(a-b)^2]^{\frac{1}{2}}\,dx.$$

$$\div\int_{a-b}^{a+b}dx=\frac{a+b}{12b}\left\{ (a^2+b^2)E\left[\left(\frac{2\sqrt{ab}}{a-b}\right),\tfrac{1}{2}\pi\right]-(a-b)^2F\left[\left(\frac{2\sqrt{ab}}{a+b}\right),\tfrac{1}{2}\pi\right]\right\}.$$

(To integrate this expression put $x=[(a+b)^2-4ab\sin\theta]^{\frac{1}{2}}$).

III. Put angle $A=\theta$, b being $<a$, then

$$A_3=\tfrac{1}{2}b\int_0^\pi[b\cos\theta+(a^2-b^2\sin^2\theta)^{\frac{1}{2}}]\sin\theta d\theta\div\int_0^\pi d\theta=\frac{ab}{2\pi}+\left(\frac{a^2-b^2}{4\pi}\right)\log_e\left(\frac{a+b}{a-b}\right).$$

IV. Put angle $B=\theta$. For every value of B there are two triangles whose average arc is $\frac{1}{2}a^2\sin\theta\cos\theta$. Hence,

$$A_4=\tfrac{1}{2}a^2\int_0^{\sin^{-1}\frac{b}{a}}\sin\theta\cos\theta d\theta\div\int_0^{\sin^{-1}\frac{b}{a}}.d\theta=\tfrac{1}{2}b^2\div\sin^{-1}\frac{b}{a}.$$

COROLLARY. If $b=a$, $A_1=a^2/\pi$, and $A^2=a^2/3$. These are double the values found in the solutions of problem 26, as they evidently should be. The values of A_3 and A_4 do not hold when $b=a$, for the reason that while the sum of the areas remains the same the number of triangles is reduced one-half at the moment that $b=a$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let x=third side, A=area, Δ=average area.

$$\therefore A=\tfrac{1}{4}\sqrt{(a+b)^2-x^2}\sqrt{x^2-(a-b)^2}.$$

$$\Delta=\int_{a-b}^{a+b}Adx\div\int_{a-b}^{a+b}dx=\frac{1}{2b}\int_{a-b}^{a+b}Adx.$$

Let $(a+b)^2-x^2=4aby^2$, $\dfrac{4ab}{(a+b)^2}=e^2$.

$$\therefore \Delta=\frac{2a^2b}{a+b}\int_0^1\frac{y^2\sqrt{1-y^2}dy}{\sqrt{1-e^2y^2}}=\tfrac{1}{2}a(a+b)\int_0^1\frac{y^2\sqrt{1-e^2y^2}dy}{\sqrt{1-y^2}}$$

$$-\frac{a(a-b)^2}{2(a+b)}\int_0^1\frac{y^2dy}{\sqrt{1-y^2}\sqrt{1-e^2y^2}}=\frac{a+b}{12b}\{(a^2+b^2)E(e)-(a-b)^2F(e)\}.$$

III. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C., and A. P. REED, Clarence, Missouri.

The area of triangle is $\triangle=\frac{1}{2}ab\sin\theta$.

Hence, average area $=\frac{1}{2}ab\int_0^\pi\sin\theta d\theta\div\int_0^\pi d\theta=\frac{1}{2}ab\Big[-\cos\theta\Big]_0^\pi\div\pi=\dfrac{ab}{\pi}$.

38. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Two arrows are sticking in a circular target: show that the chance that their distance is greater than the radius of the target is $3\sqrt{3}/4\pi$. [From *Todhunter's Integral Calculus*, page 335.]

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington,

Let Q be the position of one arrow. Call the radius of target R, and let $=\rho$.

The area $PNSR=2R^2\cos^{-1}\frac{\rho}{2R}-\rho\sqrt{R^2-(\frac{1}{2}\rho)^2}$.

Then the chance that the second arrow is within above region is

$$\frac{2}{\pi}\cos^{-1}\frac{\rho}{2R}-\frac{\rho}{\pi R^2}\sqrt{R^2-(\frac{1}{2}\rho)^2}.$$

The chance that the first arrow is at a distance ρ from the center is

$$\frac{2\pi\rho d\rho}{\pi R^2}=\frac{2\rho d\rho}{R^2}.$$

The chance that the *two* arrows are as indicated above is

$$\frac{4}{\pi R^2}\cos^{-1}\frac{\rho}{2R}.\rho d\rho-\frac{2}{\pi R^4}\sqrt{R^2-(\frac{1}{2}\rho)^2}\,\rho^2 d\rho.$$

The sum of all such chances is

$$\frac{4}{\pi R^2}\int_0^R \rho\cos^{-1}\frac{\rho}{2R}d\rho-\frac{2}{\pi R^4}\int_0^R \sqrt{R^2-(\frac{1}{2}\rho)^2}\,\rho^2 d\rho=1-\frac{3\sqrt{3}}{4\pi}.$$

$\therefore$ Chance that the second arrow is *without* the region $PNSR$ is

$$1-\left(1-\frac{3\sqrt{3}}{4\pi}\right)=\frac{3\sqrt{3}}{4\pi}.$$

II. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

Let O be the center of the target and A the position of one of the arrows. n if the distance between the arrows is greater than radius, a, the other arrow must lie outside the arc ? drawn from A as center and radius a. If $AO=x$, area of the surface outside of the arc BEC is

$$S=2a^2\sin^{-1}\left(\frac{x}{2a}\right)+\frac{1}{2}x(4a^2-x^2)^{\frac{1}{2}}.$$

The probability that the one arrow is at the dis- e x from the center is $2\pi xdx\div a^2\pi=2xdx\div a^2$. The probability that the other is on the surface outside the arc BEC is $S+a^2\pi$. Hence the required probability is

$$P=\frac{2}{a^4\pi}\int_0^a Sxdx. \quad \text{Put } x=2a\sin\theta.$$

$$\text{Then } P=\frac{16}{\pi}\int_0^{\frac{1}{2}\pi}(\theta+\sin\theta\cos\theta)\sin\theta\cos\theta=3\sqrt{3}/4\pi.$$

III. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let P, Q be the arrows, $SQ=x$, $PQ=y$, $ST=u$, $OR=z$, $\angle DOB=\theta$, $OA=a$.

An element of area at Q is $dzdx$; at P, $yd\theta dy$.

The limits of x and 0 are $u-a$; of y, $u-x$ and a, and doubled; of z, 0 and $\frac{1}{2}a\sqrt{3}$, and doubled; of θ, 0 and $\frac{1}{2}\pi$, and doubled. Δ=chance, $u=2\sqrt{a^2-z^2}$.

$$\therefore \Delta=\frac{8}{\pi^2a^4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}a\sqrt{3}}\int_0^{u-a}\int_a^{u-x}d\theta dzydxdy$$

$$=\frac{4}{\pi^2a^4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}a\sqrt{3}}\int_0^{u-a}\{(u-x)^2-a^2\}d\theta dzdx$$

$$=\frac{4}{3\pi^2a^4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}a\sqrt{3}}(u^3+2a^3-3a^2u)d\theta dz=\frac{3\sqrt{3}}{2\pi^2}\int_0^{\frac{1}{2}\pi}d\theta=\frac{3\sqrt{3}}{4\pi}.$$

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

34. Proposed by THOS. U. TAYLOR, C. E., M. C.; Department of Engineering, University of Texas, Austin, Texas.

Given a variable parallelogram $ABCP$, where P remains fixed. A moves on an irregular plane curve (closed) and C moves on another irregular plane curve (closed) whose plane is parallel to the plane of (A) curve. The generator PC moves completely around and returns to its initial position, AB always moving parallel to PC, and, of course, returns to its initial position. If distance between planes (A) and (C)=h, show by elementary mathematics and without using theorem of Koppe that volume of solid generated by variable parallelogram $ABCP=\frac{1}{2}h$ (area generated by AP+area generated by BC).

Solution by the PROPOSER.

Let (A)=area generated by PA ; (B)=area curve generated by B ; (C) =area curve generated by C.

Project area (A) orthogonally on plane of (B) and (C). Then by Elliott's Extension of Holditch's Theorem

$$S=x(A)+y(B)-xy(C).$$

where $x+y=1$, and x and y are the radii in which the section S divides the generator. Make $x=y=\frac{1}{2}$.

$$\therefore S_{\frac{1}{2}}=\tfrac{1}{2}(A)+\tfrac{1}{2}(B)-\tfrac{1}{4}(C).$$

But by Newton's formula, V=volume of whole solid

$$=\tfrac{1}{6}H\{(A)+4S_{\frac{1}{2}}+(B)\}=H\{\tfrac{1}{2}[(B)+(A)]-\tfrac{1}{6}(C)\}.$$

Volume of cone$=\frac{1}{3}H(C)$. $\therefore$ Volume generated by

$$APCB=\tfrac{1}{2}H\{(A)+[(B)-(C)]\}=\tfrac{1}{2}H\{\text{area } AP+\text{area } BC\}.$$

34. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, California ; P. O., Sebastopol, California.

To an observer whose latitude is 40 degrees north, what is the sidereal time when Fomalhout and Antares have the same altitude: taking the Right Ascension and Declination of the former to be 22 hours, 52 minntes, —30 degrees, 12 minutes; of the latter, 16 hours, 28 minutes, —26 degrees, 12 minutes ?

II. Solution (continued) by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

$h=130°\ 4'\ 57''$ for upper meridian.

$\therefore\ 180°-130°\ 4'\ 57''=49°\ 55'\ 3''=h$, for lower meridian.

$\therefore\ h=3$ hours, 19 minutes, 40.2 seconds.

$\therefore$ sidereal time for equal altitudes on latitude 40° south is $a-h=19$ hours, 32 minutes, 19.8 seconds.

$a-h-12=7$ hours, 32 minutes, 19.8 seconds is sidereal time on upper meridian at same moment.

Dr. S. Hart·Wright communicated to me the following rather startling discovery which is probably responsible for the problem : The arc of a great circle passing to and between the two stars actually passed through the Nadir. Now when the stars are of equal altitudes they are equally distant from the Nadir as well as from the Zenith.

$\therefore$ The arc between them$=82°\ 51'\ 52.5''$ must be bisected, each being $41°\ 25'\ 56\frac{1}{4}''$.

These facts, if they had been stated, would have made the problem quite simple.

See problem and solution in August-September number.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

71. Proposed by J. C. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

A man owes me \$200 due in 2 years, and I owe him \$100 due in 4 years; when can he pay me \$100 to settle the account equitably, money being worth 6%?

72. Proposed by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana.

Though the length of my field is 1-7 longer than my neighbor's, and its quality is 1-9 better, yet as its breadth is 1-4 less, his is worth \$500 more than mine. What is mine worth? *Encyclopedia Brittanica.*

73. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.

I would like to change problem 70, Arithmetic, to read as follows and have it proposed for solution:

A owes me \$100 due in 2 years, and I owe him \$200 due in 4 years. When can I pay him \$100 to settle the account equitably, money being worth 6%, and *the interest to draw interest until the time of settlement?*

Solve by simple arithmetic without the aid of algebraic symbols.

74. Proposed by JOHN T. FAIRCHILD, Principal of Crawfis College, Crawfis College, Ohio.

When U. S. bonds are quoted in London at 108¾ and in Philadelphia at 112¼, exchange \$4.89½, gold quoted at 107, how much more was a \$1000 U. S. bond worth in London than in Philadelphia?

ALGEBRA.

74. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.

Solve according to the conditions given:

$$\sqrt{x+1}+\sqrt{x}=\frac{3}{\sqrt{1+x}}.$$

First, square without transposing and then solve; second, transpose $\sqrt{x+1}$ and then solve. Obtain the same roots as in the first way of solving.

75. Proposed by B. F. BURLESON, Oneida Castle, New York.

Mr. B's farm is in shape a quadrilateral, both inscriptible and circumscriptible, and contains an area of $k=10752$ square rods. The square described on the radius of its inscribed circle contains $r^2=2304$ square rods; while the square described on the radius of its circumscribed circle contains an area of $R^2=7345$ square rods. Required the lengths of the sides of his farm.

76. Proposed by E. B. ESCOTT, Fellow in Mathematics, University of Chicago, Chicago, Illinois.

Prove the identities

$$2-\sqrt{2}=\tfrac{1}{2}+\frac{1}{2^2.3}+\frac{1}{2^3.3.17}+\frac{1}{2^4.3.17.577}\cdots\cdots$$

$$\frac{5-\sqrt{5}}{2}=1+\tfrac{1}{3}+\frac{1}{3.7}+\frac{1}{3.7.47}+\frac{1}{3.7.47.2207}\cdots\cdots$$

77. Proposed by G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, ew Hampshire.

Solve the equation, $(6x^2+x-3)^2-48^2=(x+15)^2$.

GEOMETRY.

69. Proposed by WILLIAM SYMMONDS, M. A., Professor of Mathematics and Astronomy in Pacific Colge, Santa Rosa, California; P. O., Sebastopol, California.

To divide a square card into right-lined sections in a manner, that a rectngle of a given breadth can be formed from the sections; likewise, form a square 'om a rectangular card.

70. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio niversity, Athens, Ohio.

Prove that the locus of the center of the circle which passes through the ertex of a parabola and through its intersections with a normal chord is the parbola $2y^2=ax-a^2$, the equation to the given parabola being $y^2=4ax$.

71. Prove by pure geometry: A perpendicular at the middle point, M_a, of he side BC of the triangle ABC meets the circumcircle in A'. On this perpenicular A'' and A''' are taken so that $M_aA''=M_aA'$ and $A''A'''=AH$. (H is the rthocenter of triangle ABC). Prove that A''' is on the circumcircle. *Anon.*

72. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washingn, D. C.

If a line with its extremities upon two curves move in any manner whatver, (the line may vary in length), and P a point upon the line which divides it n the ratio $m:n$ describe a curve, the area of this curve will be given by he formula—

$$A=\frac{(m^2+nm)A_1+(n^2+mn)A_2-mnA_3}{(m+n)^2}.$$

73. Prove by pure geometry: (1) A', B', and C' are the middle points f the arcs BC, CA, and AB respectively. With these points as centers, circles re described passing through B and C, C and A, and A and B respectively. rove that these circles intersect in O, the center of the incircle of the triangle BC; (2), that O, the center of the incircle, is Nagel's point of the triangle ormed by joining the middle points of the sides. *Anonymous.*

CALCULUS.

61. Proposed by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, uisiana.

If $r=a\sin n\theta$ is the polar equation of a curve, show (1) that the curve consists of n or loops according as n is an odd or an even integer; (2) that its area is ¼ or ½ of the cirmscribing circle according as n is an add or an even integer.

62. Proposed by A. H. HOLMES, Brunswick, Maine.

A bucket is in the form of a frustum of a cone having its smaller end as a base. It a inches in diameter at base and b inches in diameter at top, and its perpendicul height is c inches. It contains water the perpendicular height of which is $\frac{1}{2}c$ inches. What is the greatest height, from the plane on which the vessel rests, to which the surface of the water will rise when the bucket is overturned, no allowance being made for the thickness of the material of the bucket.

63. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

What is the volume removed by boring an auger hole radius R through a right cylinder radius R, the center of the auger hole to pass at a distance c from the axis of the cylinder and inclined to the axis at an angle α ?

64. Proposed by E. S. LOOMIS, A. M., Ph. D., Professor of Mathematics, High School, Cleveland, Ohio.

Find volume and surface generated by revolving about y, the catenary $y=\frac{1}{2}a(e^{x\backslash a}+e^{-x\backslash a})$, from $x=0$ to $x=a$. [*Osborne's Calculus*, page 255, example 8.]

MECHANICS.

48. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, Texas.

Two equal heavy rings connected by a string passing over a peg at the focus of a conic section will be in equilibrium at all points on the curve.

49. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

A rectangular stick of timber of known dimensions is placed upon a platform of given height in a vertical position with the center above the edge of platform, and slightly displaced from the vertical. Where and in what manner will it strike the ground.

50. Proposed by J. SCHEFFER, A. M., Hagerstown, Maryland.

A plane quadrilateral $ABCD$ in the vertical wall of a cistern, filled with water, has its four vertices A, B, C, D at the distances 10 feet, 4 feet, 5 feet, and 7 feet respectively, from the surface of the water. The projections of AB, BC, and CD upon the surface are respectively 2 feet, 3 feet, and 1 foot. Find the pressure of the water upon the quadrilateral, and the position of the center of mean pressure.

51. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

"Swift of foot was Hiawatha.
He could shoot an arrow from him
And run forward with such fleetness
That the arrow fell behind him!
Strong of arm was Hiawatha;
He could shoot ten arrows upward
Shoot them with such strength and swiftness
That the tenth had left the bowstring
Ere the first to earth had fallen." *Longfellow.*

Assuming Hiawatha to have been able to shoot an arrow every second and to have aimed when not shooting vertically so that the arrow might have the longest range; what was Hiawatha's time in a hundred yards ?

AVERAGE AND PROBABILITY.

47. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

What is the average length of the chords that may be drawn from one extremity of the major axis of an ellipse to every point of the curve ?

48. Proposed by P. H. PHILBRICK, C. E., Pineville, Louisiana.

A, B, C, D, and E play with dice, each throwing three, three successive times, for a stake a. A, B, and C throw; C throwing the highest, 52. What is his expectation ?

49. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A square whose side is $2a$ and an equilateral triangle whose altitude is $3a$ are fastened together at their centers, but otherwise free to move. If they are thrown on a floor at random, what is the average area common to both ?

50. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Find (1), the average length of all straight lines having a given direction, between 0 and a; (2), the average length of chords drawn from one extremity of the diameter a of a semi-circle to all points in the semi-circumference; and (3), find the average area of all triangles formed by a straight line of constant length a sliding between two straight lines at right angles.

[Solutions of these problems should be sent to the editors of the respective departments on or before February 1, 1897.]

EDITORIALS.

Our valued contributor, Prof. O. W. Anthony, has been elected Professor of Mathematics in the Columbian University, Washington, D. C.

James F. Lawrence, I. F. Yothers, G. B. M. Zerr, J. C. Corbin, Frederick R. Honey, H. C. Wilkes, and Nelson S. Roray should have received credit for solving Nos. 66, 67, 68, and 69, Department of Arithmetic. O. W. Anthony should have received credit for solving No. 63, Department of Geometry. We wish to state again that all solutions, to receive credit, should be sent to the proper editor; but this remark does not apply to the above persons.

The MONTHLY will soon begin its fourth volume. Will not every one of its old subscribers try and secure one new subscriber for the coming year? Send us names of persons likely to subscribe and we shall take pleasure in sending them sample copies. Persons sending us three new subscribers and remitting us $6.00 will receive a years subscription as a premium.

Some of our readers have suggested that we publish in groups portraits of our contributors. If this suggestion meets the approval of our contributors, we shall be pleased to receive photos which we will have grouped by one of the best artists in Springfield, and shall furnish the plates at cost to us. We shall be pleased to hear from the contributors to the MONTHLY in reference to this matter.

A letter from Dr. Halsted dated November 27th, 1896, says, "For four months I was buried in the uttermost parts of Hungary, Russia, and Siberia, and am just getting used to English again. I made many important finds and had many strange experiences." There are few other Americans whose travels in Russia would have been as important to the Non-Euclidean Geometry as this trip of Dr. Halsted's. He is already working on some very important translations which will soon be made known for the first time to English speaking mathematicians.

BOOKS AND PERIODICALS.

Elements of Mechanics, Including Kenematics, Kinetics, and Statics, with Applications. By Thomas Wallace Wright, M. A., Ph. D., Professor in Union College. 8vo. Cloth, 372 pages. Price, $2.50. New York: D. Van Nostrand Company.

This is a completely rewritten edition of the author's Text-book of Mechanics. The same general plan has been followed, but many changes in detail have made, so the book comes before the public with a new name. In this book much use is made of the graphical method; machines are discussed in great detail; the important subjects of oscillation and rotation have been treated with more fullness than is usual in an elementary treatise. Numerous well chosen problems are appended to the discussion, while at the end of each chapter is added a series of examination questions. Historical notes are freely interspersed to add a more live interest to the subject. This is a very excellent book and we very heartily recommend it to teachers desiring a good work on Mechanics. B. F. F.

The Elements of Physics. A College Text-book. By Edward L. Nichols and William S. Franklin. In three volumes, Vol. II. Electricity and Magnetism. 8vo. Cloth, ix and 272 pages. Price, $1.50. New York: The Macmillan Co.

In the study of this excellent work a knowledge of the elementary principles of the calculus and quaternions is required. This fact will preclude its use in many colleges. The authors recognizing, however, that there is a growing tendency among the best colleges to increase the requirements in mathematics, these colleges realizing that the discipline received from the study of mathematics is not excelled by any other branch of study, have not slurred over certain parts of Physics containing *real and unavoidable difficulties.* Nor have those portions containing these difficulties been omitted, but they have been faced frankly; the statements involving them having been reduced to the simplest form which is compatible with accuracy. Colleges in which only one course is offered in Physics should at once so adjust their courses of study as to make it possible to use a text-book such as the one before us, as a course of Physics pursued in accordance with the plan of this work will be of infinitely more value both from a practical and an educational point of view, than two or three popular courses requiring only a knowledge of Elementary Algebra and Geometry. B. F. F.

Elements of Plane and Spherical Trigonometry. By C. W. Crockett, Professor of Mathematics and Astronomy, Rensselaer Polytechnic Institute, Troy, New York. Large 8vo. Cloth, 192 pages and 120 pages of tables. Price, $1.25. New York and Chicago: American Book Company.

This work is fully up to the standard of good text-books. It contains a full course in Plane and Spherical Trigonometry; in fact, all that is needed in a course in the best schools and colleges. There are many examples and illustrations. The typographical and mechanical execution of the work is first-class. B. F. F.

Darwinism and Non-Euclidean Geometry. Reprint from the Bulletin de La Société Physico-Mathématique de Kasan. Tome VI. No. 3—4. By Dr. George Bruce Halsted. Pamphlet, 4 pages.

This interesting article seems to have been written by Dr. Halsted while visiting at Kasan in July and August of last summer. In his travels he explored many libraries and made many important finds. B. F. F.

The Maine Farmer's Almanac for 1897.

Through the courtesy of Prof. William Hoover, of Athens, Ohio, we received a copy of this noted little Almanac, which, among other important and useful information, contains two pages devoted to Mathematical Questions and Solutions. The price of the Almanac is 10 cents. B. F. F.

Prismoidal Formulae, with Special Derivation of Two-Term Formulae. By Thomas U. Taylor, C. E. (University of Virginia), M. C. E. (Cornell), Associate Professor of Applied Mathematics, University of Texas. Pamphlet, 55 pages.

This paper, which was read before the Texas Academy of Science, March, 1896, adds some valuable material to the literature of Prismoidal Formulae. B. F. F.

Mathematical Questions and Solutions. From the "Educational Times," with an Appendix. Edited by W. J. C. Miller, B. A. Vol. LXV., 8vo. Boards, 128 pages. Francis Hodgson, 89 Farringdon Street, E. C., London.

This valuable reprint contains solutions of about 165 problems. Our readers who secure it will find many interesting problems with their solutions. The price is 5s., 3d., postpaid. J. M. C.

Elementary Hydro-Statics. University Tutorial Series. By William Briggs and G. H. Bryan. Cloth, 208 pages. Price, 50 cents. New York: W. B. Clive, 65 Fifth Avenue.

This work is written in a suggestive and attractive manner. In scope and in method it is admirably adapted to class use as an elementary text. In the examples results are deduced from first principles, and thus the student is not lead to rely on memory for his formulae. The new features are good, the examples are numerous and well selected, and the topical index convenient and useful. J. M. C.

Inductive Manual of Straight Line and Circle. By William J. Meyers, Professor of Mathematics, State Agricultural College of Colorado. Published by the Author, Fort Collins, Colorado, 1896. 113 pages. Price, 60 cents.

The fundamental idea of the book seems to be to furnish the student the tools and material, and by the aid of helpful questions where needed, to have him work up his ideas for himself, in all cases leaving some actual work and thought to the student himself. As distinguishing features we notice: A constant effort to keep prominent the connection between geometrical relations and their applications in the arts; the early introduction and use of the notions of locus and of symmetry; distinction between the obverse and reverse of plane figures; and the closeness of relation between regular chains, polygons, and the circle. There are numerous exercises and problems. It must be left to actual trial to determine its adaptation to class use. J. M. C.

The Alumni Bulletin of the University of Virginia, for November, contains an appreciative sketch, with portrait, of our esteemed subscriber, Professor Charles Scott Venable, LL. D., who lately retired from the head professorship of mathematics at the University of Virginia, a position he has held for over thirty years. J. M. C.

We have received the following valuable papers, in pamphlet form, from Dr. Artemas Martin, editor of the *Mathematical Magazine:* "About Cube Numbers whose Sum is a Cube Number"; About Biquadrate Numbers whose Sum is a Biquadrate Number"; Notes about Square Numbers whose Sum is either a Square or the Sum of other Squares"; On Fifth-Power Numbers whose Sum is a Fifth Power"; and "Solutions of the 'Duck' Problem." Those interested in the subjects of which these papers treat cannot afford to miss them.

The last number of the *Magazine*, issued in May, 1896, contains the paper on Biquadrate Numbers, and the second installment of that on Cube Numbers. Three interesting problems are solved and ten new ones are proposed. J. M. C.

The following periodicals have been received: Journal de Mathématiques Elémentaires, (1er December, 1896); American Journal of Mathematics, (October, 1896); The Mathematical Gazette, (October, 1896); L' Intermédiare des Mathématiciens, (November, 1896); Miscellaneous Notes and Queries, (December); The Kansas University Quarterly, (October, 1896), The Monist, (October, 1896); Bulletin of the American Mathematical Society, (December, 1896); The Educational Times, (November, 1896); The Mathematical Review, (July, 1896); The Mathematical Magazine, (No. 10, issued in May, 1896); Annals of Mathematics, (September, 1896). J. M. C.

ERRATA.

On page 221, for "numbers" read *terms* in line 16.

In solution of problem 42, page 220, the part under Example 2, reading, "For $p-q$, $a=9/2$, $b=13/2$, etc.," should be under Example 1, to tally with "for $p+q$, etc."

Page 234, line 5, for "ρ^{-1}" read $\rho^{-\frac{1}{2}}$.

Page 234, line 5, for $\sqrt{\rho+\rho}$" read $\rho+\sqrt{\rho}$.

Page 243, line 5, omit decimal point in denominator.

Page 258, in Figure, read D for "B" and B for "D".

Page 259, multiply the numerator of the right hand member in the value of p by 2.

Page 288, problem 38, the figure is wrong. The arc CE should be *parallel* to BA, as the solution says. Also, BC, which is an arc of the horizon, should be in a level plane.

THE

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DEVOTED TO THE
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OF NOTED MATHEMATICANS, ETC.

EDITED BY
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∵ The numbers in parenthesis refer to the pages where the problems, solutions, and icles are found.

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BIOGRAPHY.

DE MORGAN.

BY GEORGE BRUCE HALSTED.

AUGUSTUS DE MORGAN is fortunately a well-known character. The world has an excellent sketch of him by his pupil Jevons in the Encyclopædia Britannica, from which I copy literally some of the preliminary biographic data here given. If I am able to add anything specific new to this part of the paper, it is from a life-long study of his works, and ıy conversations about him with the great Sylvester, who knew him very mately.

Augustus De Morgan was born in June, 1806, in India. On his mother's he was descended from James Dodson, F. R. S., author of the Anti-Logamic Canon and other mathematical works of merit, and a friend of De Moivre. Augustus lost one eye in his early infancy, and this prevented his joining in usual games. He read algebra "like a novel," and pricked out equations on school pew instead of listening to the sermons.

When 16 years old he entered Trinity College, Cambridge, and studied his hematics partly under the tuition of Airy, making many friends, including teachers Whewell and Peacock. In 1825 he gained a Trinity Scholarship.

But De Morgan's attention was by no means confined to mathematics, and love of wide reading somewhat interfered with his success in the mathemat- tripos, in which he took fourth place in 1827, before he had completed his ; year.

He was prevented from taking his M. A. degree, or from obtaining a fellowship, by his conscientious objection to signing the theological tests then required from masters of arts and Fellows at Cambridge. Jevons says, "a strong repugnance to any sectarian restraint upon the freedom of opinion was one of De Morgan's most marked characteristics throughout life."

A career in his own university being thus closed against him, on the twenty-third of February, 1828, he was elected Professor of Mathematics in University College, London, and began lecturing at the early age of 22 years.

But the regents of this College claimed the right of dismissing a professor without assigning reasons, and acted upon this principle in dismissing the Professor of Anatomy. Immediately De Morgan resigned in protest. After the regulations were changed he was invited to resume his chair, was reappointed, and served for the next 30 years. His dislike to honorary titles led him to refuse the offer of LL. D. from the University of Edinburgh.

In 1866 a discussion arose as to the true interpretation of the principle of religious neutrality avowedly adopted by the College. De Morgan held that any consideration of a candidate's ecclesiastical position or creed or lack of creed was inconsistent with the principle. In protest against a violation of this principle he resigned a second time in a letter dated November 10th, 1866. He was always remarkably free from any touch of sordid self-interest.

As a teacher, De Morgan was particularly gifted. A voluminous writer on mathematics, he contributed essentially to those expansions of the fundamental concepts which have rendered possible the new algebras, such as Quaternions, and have generalized the whole idea of a mathematical algorithm, until now an American produces a thesis entitled "Mathematics the Science of Algorithms."

His logical work alone would give De Morgan lasting fame. Here he stands alongside his immortal contemporary, Boole. The eternally memorable year in the history of Logic was 1847, in which George Boole issued "The Mathematical Analysis of Logic, being an Essay toward a Calculus of Deductive Reasoning," von Staudt published his "Geometrie der Lage," a mathematics utterly freed from any idea of quantity, a mathematics strictly qualitative, and De Morgan printed his fundamental treatise, called "Formal Logic; or, the Calculus of Inference, Necessary and Probable."

The great memoirs produced in 1850, 1858. 1860, 1863 are preserved, if buried, in the inaccessible "Cambridge Philosophical Transactions." De Morgan's great combination of logical with mathematical learning, and his prominent position in London, the great metropolis, made him the man to whom resorted all the Circle-Squarers, Angle-Trisectors, Perpetual-Motionists, Triangle-Angle-sumers. Adding this curious experience to his great bibliographic knowledge of what had been attempted in that way in the past, he formed a large book called "A Budget of Paradoxes," which is one of the most interesting treatises ever written on what may be called extended fallacies. This charming book has steadily advanced in market price until I find it cited in Macmillan's Catalogue No. 245 (1086) at fifty shillings per copy.

It was De Morgan who first gave a thorough treatment of contrary, negative, or contradictory terms, though in just the sense I have heard babies use them while learning our language. But remember it is Darwinism that has since taught the world to learn at the feet of babes. Bain says: "According to the true view of contrariety, as given by De Morgan, the negative is a remainder, gained by the subtraction of the positive from the universe; the negative of X is $U-X$, and may be symbolized by a distinct mark, x; whence X and x are the opposites under a given universe; not $-X$ is x, and not$-x$ is X."

Of the separation of logic and mathematic De Morgan says: "The effect has been unfortunate. . . . The sciences of which we speak may be considered either as disciplines of the mind, or as instruments in the investigation of nature and the advancement of the arts. In the former point of view their object is to strengthen the power of logical deduction by frequent examples; to give a view of the difference between reasoning on probable premises and on certain ones by the construction of a body of results which in no case involve any of the uncertainty arising from the previous introduction of what may be false; to establish confidence in abstract reasoning by the exhibition of processes whose results may be verified in many ways; to help in enabling to acquire correct notions of generalization; to give caution in receiving that which at first sight appears good reasoning; to instill a correct estimate of the powers of the mind by pointing out the enormous extent of the consequences which may be developed out of a few of its most fundamental notions; and to give the luxury of pursuing a study in which self-interest cannot lay down premises nor deduce conclusions.

As instruments in the investigation of nature and the advancement of the arts it is the object of these two sciences to find out truth in every matter in which nature is to be investigated, or her powers and those of the mind to be applied to the physical progress of the human race, or their advancement in the knowledge of the material creation."

De Morgan was fond of laughing at the metaphysicians. He says: "We know all about *can* and *cannot* from our cradles; we never feel the same assurance about *is* and *is not.*

A philosopher, in a dark age, may determine to set out with a knowledge of the naturally possible and impossible; but not even a philosopher ever pretended to set out with a knowledge of the existent and non-existent."

Aristotle and all the old logicians said that the whole of the middle term must be taken in at least one of the premises. As they put it, the middle term must be distributed at least once in the premises, otherwise the minor term may be compared with one part and the major with another part of it. From

Some men are poets,
Some men are Indians,

nothing follows. But the Aristotelians were wrong, as De Morgan clearly showed in his doctrine of Plurative Judgments. For example, if we have given the premises,

Most men are uneducated,
Most men are superstitious,

according to Aristotle we are not warranted in drawing any conclusion; for the middle term is men, and in neither premise is anything said about all men.

But, in point of fact, we can draw the perfectly valid conclusion,

Some uneducated men are superstitious.

Again Aristotle is contradicted by numerically definite judgments. In these there is inference when the quantities of the middle term *in the two premises together* exceed the whole quantity of that term.

Lambert first thought of this principle. De Morgan reconceived it and extended its use.

Suppose we grant the premises,

Two-thirds of all adults are women.

The number of women who have been married is never greater than the *total* number of men. It follows that half the entire number of women are single.

Still, easy and certain as such reasoning is, it may be difficult to a logician trained only in the traditional logic.

In a Princeton "Manual of Logic" the only numerically definite syllogism given was erroneous, and stood so for years. I stated this to the author, and in the latest stereotyped edition it has been changed. The Syllogism he gave was as follows:

"60 out of every 100 are unreflecting.
"60 out of every 100 are restless.
"Therefore, 20 out of every 100 restless persons are unreflecting."

After pointing out to him the fault in what he had been teaching for years, the following has been substituted:

"60 out of this 100 are unreflecting.
"60 out of this 100 are restless.
" ∴ 20 restless persons are unreflecting."

But De Morgan's greatest work was connected with his development of the Logic of Relatives, independently discovered by Robert Leslie Ellis after reading Boole's "Laws of Thought."

One of De Morgan's last memoirs, in the tenth volume of the "Cambridge Transactions," was on the Logic of Relations, which is, in the mathematical sense, a far-reaching generalization of the old logic. In our modern mathematics everything is generalized as far as possible. Every study of a generalization gives additional power over the particular. We need to go beyond and look back from an elevation.

Any first-rate mathematician working in logic would attempt to generalize, and, in fact, Boole generalized the scholastic logic in a manner entirely different from De Morgan. In De Morgan's view of the subject, the purely formal proposition with judgment wholly void of matter, is seen in "There is the probability x that X is in the relation L to Y." The syllogism is the determination of the relation which exists between two objects of thought by means of the relation in which each of them stands to some third object which is the middle term. The pure general form of the syllogism, when its premises are absolutely asserted, is as follows: X is in the relation L to Y, Y is in the relation M to Z; therefore X is in the relation "L of M," compounded of L and M, to Z. In ordinary logic the actual composition of the relation is made by our consciousness of its *transitive* character. A relation is transitive when, being compounded with itself, it reproduces itself; that is, L is transitive when every L of L is L; for example "brother."

Thus De Morgan broke away from that paltry narrowness which asserts that our minds in pure thinking can use nothing but the relation of identity; from the Jevons sophism that thought cannot move because all thought is the substitution of identicals.

So we see that in logic, as in mathematics, we may develop a whole system of theorems about symbols which are to be used in a given manner; and then to make this whole system true of a desired relation or subject matter we have only to show that this relation or subject matter fulfills the few fundamental principles of the system.

De Morgan treated of convertible and inconvertible relatives, repeating relatives; non-repeating relatives, transitive and intransitive relatives, and inaugurated a general system.

To what tremendous estate this system has grown may be seen in the 649 pages of Ernst Schroeder's Treatise, "Algebra und Logik der Relative;" Leipzig, Teubner, 1895, on whose first page, as founder of the system, stands the name of Augustus De Morgan.

ON THE SOLUTION OF THE QUADRATIC EQUATION.

By G. A. MILLER, Ph. D., Paris, France.

One of the most important applications of substitution groups occurs in the theory of the solution of algebraic equations. It seems desirable that a fairly complete discussion of the solution of the quadratic equation should precede the study of this application. It is hoped that this discussion will not be without interest in itself even if the facts with which we have to deal are well known. As we shall need a clear idea of the *domain of rationality* we shall first develop several elementary concepts which involve the notion of groups and naturally lead to the more general concept of the domain of rationality.

Let us first consider the totality of numbers (T_1) formed by all the positive integers,* each positive integer occurring once and only once. By adding

*We shall throughout confine our attention to the finite. Not only are the numbers to be regarded as finite but the number of times that a given operation is to be performed is also to be considered finite.

any one of these to itself or to any other number in T_1 we obtain no new number. We may therefore say that T_1 forms a *group* (G_1) *with respect to addition*. It is evident that T_1 also forms a group (G_2) with respect to multiplication. Each of these two groups contains an indefinite number of subgroups.* To every subgroup of G_1 there corresponds a subgroup of G_2 which contains the same numbers but the converse is not true. For instance, all the positive integral powers of any prime number form a subgroup of G_2† but they do not form a subgroup of G_1.

We shall not enter upon the discussion of the subgroups found in G_1 and G_2 but only call attention to a few of the most simple ones. All the even positive integers clearly form a subgroup of both of these groups. We may inquire what is the smallest subgroup that contains any given positive integer a. In G_2 this subgroup is a^α, $\alpha=1, 2, 3, \ldots\ldots$ while in G_1 this subgroup is βa, $\beta=1, 2, 3, \ldots\ldots$ Hence unity is a subgroup of G_2 but not of G_1. As G_1 contains no subgroup that involves unity we may say that it is generated by this element. G_2 is generated by all the prime numbers together with unity.

The totality of negative integers (T_2 also forms a group with respect to addition but it does not form a group with respect to multiplication. It is clear that the smallest group with respect to multiplication that contains T_2 must also contain T_1. That is, $T_1+T_2=T_3$ is a totality which forms a group with respect to multiplication. T_3 also forms a group with respect to each of the operations addition and subtraction. The group with respect to subtraction does not contain any subgroup involving either T_1 or T_2, that with respect to multiplication contains a subgroup that involves T_1 but none that involves T_2, while that with respect to addition contains a subgroup that includes T_1 and also one that includes T_2.

Among the numbers which are now in common use those included in T_1 are perhaps of special importance as is also indicated by the fact that they are frequently called the natural numbers. In considering the groups which certain totalities of numbers form with respect to given operations it is therefore of special importance to inquire into the smallest groups that contain T_1. We have already seen that with respect to subtraction this smallest group includes other numbers than those contained in T_1. Similarly we observe that with respect to division‡ this smallest group includes an additional totality of numbers, viz: the fractions whose numerators and denominators are positive integers.

Instead of inquiring into the smallest totality of numbers that contains T_1 and forms a group with respect to a given operation we may also inquire into the smallest totality that contains T_1 and forms a group with respect to each of a given number of different operations. For instance, the smallest totality

*The term subgroup is used, as usual, to represent a group that is contained in the larger group under consideration.

†For brevity we shall say that certain elements of a group form a subgroup instead of saying that these elements form a group if they are combined according to the same operation as the elements of the group.

‡As we have excluded the infinite we are not allowed to divide by 0. We may regard this as an impossible operation in the region to which we confine our observations.

(T_4) that contains T_1 and forms a group with respect to each of the four fundamental operations—addition, subtraction, multiplication, and division—is that which is formed by all the rational numbers. If we form the smallest totality that contains unity and forms a group with respect to each of these four operations we evidently arrive again at T_4. On this account the totality of all the rational numbers is generally called the domain unity. This is the simplest domain of rationality.*

It would be of interest to consider all the different types of subgroups of the groups formed by T_4 with respect to the given fundamental operations. We shall not enter into this field as the matter has probably been sufficiently developed for our present purposes. We would remark, however, that a careful study of these matters seems to us to be one of the simplest roads towards forming a clear notion of groups as well as of the domain of rationality.

The operations which we have thus far considered may be represented by the simple equations:

$$a+b=x, \quad a-b=x, \quad a\times b=x, \quad a\div b=x.$$

We have seen that x is always a rational number when both a and b are such numbers. In other words, we have seen that when we take a and b from the totality of numbers represented by T_4, x will also belong to this totality, but when we take a and b from one of the other totalities of numbers that have been considered—T_1, T_2, T_3—we cannot affirm that x belongs to the same totality in all the equations.

At an early stage in the development of mathematics it became necessary to solve equations of a higher than the first degree. One of the great difficulties which presented itself at this point was the fact that T_4 does not form a group with respect to the algebraic operations whose degree exceeds one. As long as these operations can be represented by a binomial equation of the form

$$x^n=a$$

n being a positive integer and a being a positive rational number, the difficulty was not much greater than that which had to be overcome at preceding stages, for the extension of the totality of numbers in such a way as to include irrational numbers does not seem an irrational adventure, even if the association of these numbers with matters of observation is not so direct and evident as it had been in the preceding cases.

*The totality of rational functions with integral coefficients of a given number of determinate or indeterminate independent quantities (R, R', R'', ...) is called, after Kronecker, a domain of rationality. In other words, it is the smallest totality that contains these quantities and forms a group with respect to the four fundamental operations. In the domain unity we have clearly only one such quantity and this is determinate, viz: $R=1$. We shall soon consider a domain of one indeterminate quantity or parameter. Sometimes the given R's are defined as indeterminate parameters. According to this definition the domain unity contains no parameter. This domain is clearly generated by any rational finite number except 0. It may therefore be called the domain 2, 3, ... as well as the domain 1.

A much more serious difficulty presented itself when it was required to introduce the operation indicated by the general quadratic equations

$$x^2+bx+c=0. \qquad (A).$$

Even if we take b and c from the totality T_1 it often happens that we can not find any number among those that have been considered which comes near towards satisfying this equation. Hence the totality of real numbers (T_5) can clearly not form a group with respect to this operation. With respect to the four fundamental operations T_5 forms a group which contains T_4 as a subgroup.

It should however not be inferred that none of the numbers that have been considered form a group with respect to the operation (A). In order that a single number (α) may have this property it is necessary and sufficient that $x=\alpha$ satisfies the equation

$$x^2+\alpha x+\alpha=0.$$

Hence there are two numbers ($-\frac{1}{2}$ and 0) each of which forms a group with respect to the quadratic equation. 0 also forms a group with respect to the four fundamental operations but $-\frac{1}{2}$ does not have this property. It should also be observed that when $\alpha=-\frac{1}{2}$ it is not the only value of x that satisfies the given equation and that the term group has therefore to used in a somewhat restricted sense when we say that $-\frac{1}{2}$ forms a group with respect to the quadratic equation.

But, even if it was known that certain special numbers form a group with respect to the operation (A), and, what is more important, that x belongs to the totality of numbers T_5 for a large number of types of (A) yet the matter remained in an unsatisfactory state as long as no totality of numbers was known which includes T_5 and forms a group with respect to (A). The struggle for light on this point was a long one, reaching far into our century. We cannot enter into a history of this struggle. It must suffice to state that the triumph was largely due to the elegant geometrical representation of the complex number by means of points in the plane. This was not the first nor last assistance that algebra has received from his sister geometry. On the other hand, algebra has a very brilliant record of services rendered to his ambitious sister.

The importance of the adoption of the complex numbers (T) cannot be fully appreciated if we confine our attention to the quadratic equation. If the general algebraic operations of each of the following degrees had again required equally great extension in the number system this would soon have become exceedingly difficult and the progress in the solution of the algebraic equations could not have been so rapid. The great importance of the adoption of the totality of numbers T may therefore be said to be due to the fact that it forms a group with respect to the general algebraic operation indicated by the following equation

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots+a_n=0 \qquad (B),$$

where a_0, a_1, a_2, ..., a_n belong to T and n belongs to T_1. In other words if the coefficients of (B) belong to the domain of the indeterminate parameter

$$R=\sqrt{a} \qquad (a \text{ belonging to } T_6)$$

x will also belong to this domain. Or again, B can be resolved into factors without using numbers that lie outside of T.

While we cannot fully appreciate the importance of the adoption of the complex numbers when we confine our attention to the quadratic equation, yet in this equation we see the source of T and with it we naturally associate the wonderful progress achieved by means of T. It is however not our object to enter upon the discussion of the important position which the quadratic equation occupies in the development of mathematics even if an idea of this position naturally increases the interest in this equation and hence also in its discussion as an algebraic operation.

To convey an idea of the importance of the concept domain of rationality we shall consider an application. Suppose that we have an equation of the form (B) and that its coefficients belong to a certain domain of rationality T' while none of its factors belong to this domain.* Suppose that we have any other equation whose coefficients also belong to T' and that these two equations have a common root. We can then say that the first equation is a factor of the second. In other words, we know that all the roots of the first equation are also roots of the second. The truth of this statement follows directly from the fact that the coefficients of the greatest common divisor of the two equations must belong to T'. This greatest common divisor must therefore be the first member of the first equation.

If we have any complex number

$$a+bi$$

the quadratic equation which contains this number and its conjugate for its roots is

$$x^2-2ax+a^2+b^2=0. \qquad (C).$$

The coefficients of this equation belong to T_6. Suppose now that we have any other equation that contains $a+bi$. From the proof just given it follows that it must also contain (C). The same remarks clearly apply to surd roots of the form

$$a\pm\sqrt{b}.$$

Hence we see that the given statement includes the theorems that if an equation with real coefficients contains the root $a+bi$ it also contains the root $a-bi$, and if an equation whose coefficients are rational contains a surd root of the form $a+\sqrt{b}$ it also contains $a-\sqrt{b}$ as a root.

*In this case we say that the equation is irreducible in the domain T'.

THE NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from May Number.]

SCHOLION.

Nevertheless it might be doubted, whether, from whatever point K (assumed indeed in BX before the meeting of this BX with the other AX) erected toward the parts of the straight AX, a perpendicular must meet this (fig. 29) in some point L; provided of course those two, before the aforesaid meeting, are assumed ever more to approach each other mutually [and not to meet at any finite remove].

Fig. 29.

But I say it will follow completely thus.

Proof. Let there be assigned in BX any point whatever K. In AX is taken a certain AM equal to the sum of this BK and of twice AB.

Then from the point M is drawn to BX (according to Eu. I. 12) the perpendicular MN. According to the present supposition, MN will be less than AB. Wherefore AM (made equal to the sum of BK and of double AB) will be greater than the sum of BK, AB, and NM. Now it behooves to show this same AM to be less than the sum of BN, AB, and MN, that thence it may follow this BN is greater than the aforesaid BK, and therefore the point K lies between the points B and N.

Join BM. The side AM will be (from Eu. I. 20) less than the two remaining sides together AB and BM. Again the side BM (from the same Eu. I. 20) will be less than the two sides together BN and MN. Therefore the side AM will be by much less than the three sides together AB, BN, and NM. But this was to be shown, in order to deduce that the point K lies between the points B and N. Thence however it follows, that the perpendicular from the point K erected toward the parts of AX must meet this in some point L stationed between the points A and M; else obviously (against Eu. I. 17) it must cut either AB or MN perpendiculars to BX.

Quod etc.

[To be Continued.]

W AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from December Number.]

XXII. Let ABC be a triangle, right-angled at C. Assume $CB<CA$. plete the figure as indicated, the point E being e middle of AB.

There are three cases: (1) when E is with- he circle, (2) in the circumference, (3) within circle. We treat but one case, when E is with- circle. The other cases may be treated in a manner.

Fig. 17.

$\frac{H\cdot AL}{AB}=\frac{b^2-a^2}{c}$ ....(1).

$\frac{KE\cdot EP}{EB}=\frac{(a-\frac{1}{2}c)(a+\frac{1}{2}c)}{\frac{1}{2}c}$ ....(2).

Add (1) and (2), $\frac{1}{2}c=\frac{a^2+b^2-\frac{1}{2}c^2}{c}$. $\therefore c^2=a^2+b^2$.

NOTE. This method is credited, by Wipper, to Krueger, 1746. Though in the original, the legs are arily assumed as unequal, we may pass, by the theory of limits, to the case when the legs are equal.

XXIII. Let ABC be a triangle, right-angled at Circumscribe the triangle by a circle. Complete the ingle. Then,

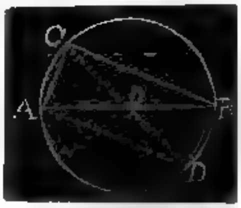


Fig. 18.

$AB\cdot CD=CB\cdot AD+AC\cdot DB$, or $c^2=a^2+b^2$.

NOTE. Though this method must have been known independently ny persons, it seems that its first appearance in print was in 1880, in e's *Mathematical Monthly*.

Fig. 19.

XXIV. Let ABC be a triangle, right-angled at C. Construct circles with AC and BC as diameters, respectively. These circles will intersect in AB, as at D.

Then, $\overline{AC}^2=AD\cdot AB$, and $\overline{BC}^2=BD\cdot AB$.

Add, $\overline{AC}^2+\overline{BC}^2=AB(AD+BD)=\overline{AB}^2$.

NOTE. This is one of Richardson's

XXV. Let ABC be a triangle, right-angled at C. Complete the fig indicated.

Fig. 20.

Then, $\overline{AC}^2=AH\cdot AD$, and $\overline{BC}^2=BL\cdot BE$.

Add, $\overline{AC}^2+BC=AH\cdot AD+BL\cdot BE$

$=(AB-BC)(AB+BC)+(AB-AC)(AB+AC)$.

$\therefore \overline{AB}^2=\overline{AC}^2+\overline{BC}^2$.

Richardson's method is somewhat different. Thus:

$\overline{AC}^2+\overline{BC}^2=AH(AB+BC)+BL(AB+AC)$

$=AB(AH+BL)+AH\cdot BC+BL\cdot AC+BL\cdot AB-BL\cdot AB$

$=AB(AH+BL)+AH\cdot BC+BL(BL+2AC)-BL\cdot AB$

$=AB(AH+BL)+AH\cdot BH+\overline{BH}^2-BL(AH+BH)$

$=AB(AH+BL)+(AH+BH)(BH-BL)$

$=AB(AH+BL)+AB\cdot HL=AB(AH+BL+HL)=\overline{AB}^2$.

XXVI. Fig. 20.

$EH : ED :: HL : DL$. (See Olney, §971). $\therefore EH\cdot DL=ED\cdot HL$,

or $(AC+AB-BC)(BC+AB-AC)=(AC+AB+BC)(AC+BC-AB)$.

$\therefore \overline{AB}^2=\overline{AC}^2+\overline{BC}^2$.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield; Mo. All contributions to this department should be sent to

SOLUTIONS OF PROBLEMS.

70. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pitt Pennsylvania.

A owes me \$100 due in 2 years, and I owe him \$200 due in 4 years; when can I him \$100 to settle the account equitably, money being worth 6%?

I. Solution by the PROPOSER, and P. S. BERG, Larimore, North Dakota.

Let x=the time.

Now the amount of $200 for $(x-4)$ years—the amount of $100 for $(x-2)$ must $100.

$200+12(x-4)=152+12x$=amount of $200 for $(x-4)$ years at 6%.

$100+6(x-2)=88+6x$=amount of $100 for $(x-2)$ years at 6%.

$\therefore (152+12x)-(88+6x)=100. \quad \therefore x=6$ years.

II. Solution by Hon. J. H. DRUMMOND, LL. D., Portland, Maine.

Computing at simple interest,

$\$\frac{5000}{31}-\$\frac{625}{7}=\$72\frac{1}{217}$, amount equitably due now.

Hence, $\$100-\$72\frac{1}{217}=\$27\frac{216}{217}$ to be earned as interest.

$\$27\frac{216}{217}\div(72\frac{1}{217}\times\frac{6}{100})=6\frac{12}{25}$, the number of years required.

III. Solution by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.

In this I assume interest to remain unpaid until the time of settlement, and to *draw no interest*.

A owes me to-day the present worth of $100 due in 2 years at 6%, $89.29.

I owe A to-day the present worth of $200 due in four years at 6%, $161.29.

That is, I owe A $72 more than he owes me. Hence the problem reduces itself to the question, when will the excess of my interest over his plus 72 amount to $100? That is, my interest must exceed his by $28.

My yearly excess is $4.32. Hence to gain $28, 6.481 years will be required.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

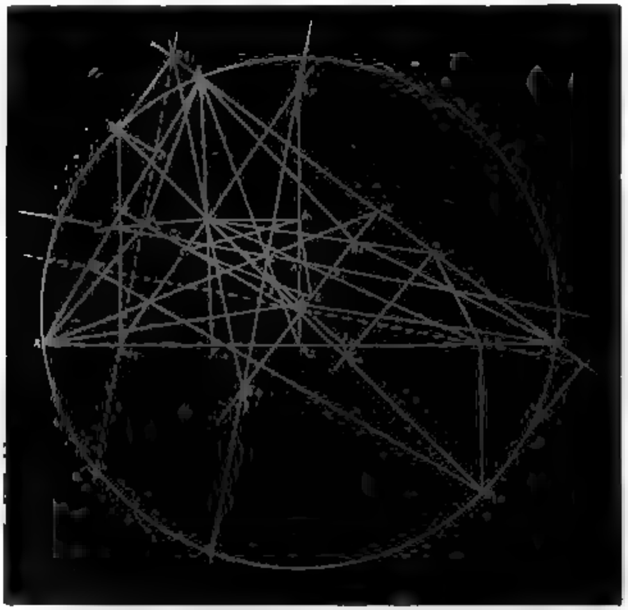
SOLUTIONS OF PROBLEMS.

66. Proposed by I. J. SCHWATT, Ph. D., Professor of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania.

The axes of the ellipse isogonal to Lemoine's line with respect to a triangle (Steiner's Ellipse), are parallel to Simson's lines belonging to the extremities of Brocard's Diameter.

Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bor New Jersey.

Let the triangle be ABC; center of circumcircle, M; $A_1B_1C_1$, the B triangle with vertices on parallels thro' Grebe's point and perpendiculars a points of sides of ABC. It is known that ABC and $A_1B_1C_1$ are similar. be the medio-centre, or centre of gravity, of ABC. It is known that E i medio-centre of $A_1B_1C_1$. Let MK, Brocard's diameter (or the diameter of about Brocard's triangle) be produced to meet circumcircle of ABC in the Q_3 and Q_4.



By the construction, the vertices of Brocard's triangle are also the ve of three similar isosceles triangles; for these isosceles triangles have as alti the perpendiculars from Grebe's point upon the sides of ABC, and it is k that these perpendiculars are as the sides. Hence the triangles have base altitudes proportional, and therefore are similar.

If, now, any three similar isosceles triangles be constructed upon the of ABC, their vertices A_2, B_2, C_2, will be the vertices of a triangle havin same medio-centre as ABC or $A_1B_1C_1$. The proof of this is similar to which is known to establish E the same for ABC, and $A_1B_1C_1$.

Draw KA_2, KB_2, KC_2 to meet sides of triangle ABC in points A_{2a} C_{2y} respectively. Then triangle $A_{2a}B_{2\beta}C_{2y}$ is similar to the triangle $A_2B_2C_2$ centre of similitude is K. This may be proved as follows: Erect a perpe lar at A_{2a} to cut Brocard's diameter (produced) at Q_2, then

$$A_2M_a : KK_a = A_{2a}A_2 : A_{2a}K = Q_2M : Q_2K.$$

Triangles A_2BC, and B_2AC are similar by construction, hence

$$A_2M_a : B_2M_b = M_aC : M_bC = a : b = A_1M_a : B_1M_b,$$

where $a : b$ is ratio of two sides of triangle ABC.

We may write this last

$$A_2M_a : A_1M_a = B_2M_b : B_1M_b = B_{2\beta}B_2 : B_{2\beta}K = Q_2M : Q_2K \qquad (2).$$

From (1) and (2) we have,

$$A_{2a}A_2 : A_{2a}K = B_{2\beta}B_2 : B_{2\beta}K,$$

hence the lines A_2B_2 and $A_{2a}B_{2\beta}$ become parallel, and if the same course of reasoning be pursued with regard to the other sides, the triangles are seen to be similar, with K the center of similitude.

Now, from the equality, $B_{2\beta}B_2 : B_{2\beta}K = Q_2M : Q_2K$, it follows that $B_{2\beta}Q_2$ is parallel to B_2M; and since B_2M is perpendicular to AC, $B_{2\beta}Q_2$ is also perpendicular to AC. Similarly, the perpendicular to AB at $C_{2\gamma}$ passes through Q_2, and we have already that the perpendicular to BC at A_{2a} passes through Q_2. If Q_2 be caused to coincide with either Q_3 or Q_4, then triangle $A_{2a}B_{2\beta}C_{2\gamma}$ will degenerate into the straight lines $Q_{3a}Q_{3b}Q_{3c}$, and $Q_{4a}Q_{4b}Q_{4c}$ which are the Simson lines belonging to Q_3 and Q_4. Also triangle $A_2B_2C_2$ will degenerate into the straight lines $A_3B_3C_3$, and $A_4B_4C_4$, which are parallel to Simson's lines, to Q_3, Q_4. These lines will also pass through the medio-centre, E, since the triangles which degenerate continually have E as the medio-centre. Since the Simson lines are perpendicular to each other (see Geometry of Simson lines), these last mentioned lines through E, are perpendicular to each other. Since we know that the ellipse (Steiner's) has these lines for axes, the proposition is proved. Q. E. D.

NOTE. The above solution I got from Dr. Schwatt. An elegant demonstration of properties of the ellipse is given in Schwatt's Isogonal Curves, (Leach, Shewell & Sanborn, New York). F. M. M.

This problem was also solved by Prof. G. B. M. Zerr. Prof. William Hoover did not solve it, but referred to the proof given in *Casey's Analytical Geometry*, Edition of 1893, Articles 304, 305 (Cor. 1).

66. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The locus of points whose polars with respect to a given parabola touch the circle of curvature at the vertex is an equilateral hyperbola.

I. Solution by the PROPOSER.

By Salmon's *Conic Sections*, Sixth Edition, Ex. 4, page 234, the equation to the circle osculating a parabola $y^2=px$......(1) at (x', y') is

$$(p^2+4px')(y^2-px)=\{2yy'-p(x+x')\}\{2yy'+px-3px'\}\ldots\ldots\ldots(2).$$

At the vertex, $x'=0$, $y'=0$, and (2) becomes

$$x^2+y^2-px=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

If (x_1, y_1) be any point on the required locus, its polar with respect to (1) is

$$px-2y_1y+px_1=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The condition that (4) touches (3) is

$$x_1^2-y_1^2-px_1=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

an equilateral hyperbola.

II. **Solution by J. SCHEFFER, A. M., Hagerstown, Maryland; CHAS. C. PURYEAR, Professor of Mathematics, Agricultural and Mechanical College, College Station, Texas; and G. B. M. ZERR, Texarkana, Ark.**

Let $y^2=4ax$ be the equation to the parabola, (b, c) any point.

Then $cy=2a(x+b)$......(1) is the polar of (b, c). $x^2+y^2=ax$.......(2) is the circle of curvature at the vertex. The value of y from (1) in (2) gives

$$c^2x^2+4a^2x^2+8a^2bx+4a^2b^2=ac^2x\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From (3) we find the condition that (1) should be tangent to (2) to be

$$a^2c^4=8a^3bc^2+16a^2b^2c^2. \quad \therefore c^2=8ab+16b^2.$$

$$\therefore a^2=(4b+a)^2-c^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

(4) represents an equilateral hyperbola.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

55. **Proposed by GEORGE LILLEY, Ph. D., LL. D., Principal of Park School, 394 Hall Street, Portland, Oregon.**

A horse is tethered by a rope, a feet long, fastened to a post in a circular fence enclosing a circular piece of grouhd b feet in diameter. If the horse is tethered outside of the fence over how much ground can he feed? If he is inside the fence over how much ground can he feed? b is greater than a in each case.

I. Solution by the PROPOSER.

Consider area $C'HFMA$. $C'A=C'P=a$, feet. $CC'=CH=\frac{1}{2}b$, feet. Let $\angle HCF=\phi$. $HM=$arc $HF=\frac{1}{2}b\phi$. The area element is $\frac{1}{2}(HM)^2d\phi$.

$$\therefore \text{ area } C'HFMA=\tfrac{1}{8}\int_0^{\frac{2a}{b}}b^2\phi^2d\phi,=\frac{a^3}{3b}.$$

Hence, when the horse is outside, he can graze over

$$\frac{a^2}{6b}(4a+3b\pi), \text{ square feet.}$$

Draw PE at right angles to DC'. Let $x=CE$, and $\angle PC'E=\theta$.

$$\text{Area of sector } PC'D=\int_0^\theta\int_0^a d a d\theta,=\tfrac{1}{2}a^2\cos^{-1}\frac{b+2x}{2a},=\text{sector } NC'D.$$

$$\text{Area of sector } PCC'HF=\tfrac{1}{8}b^2\times\text{angle } PCC',=\tfrac{1}{8}b^2\cos^{-1}\left(-\frac{2x}{b}\right).$$

$$\text{Area of triangle } PCC'=\tfrac{1}{8}b\sqrt{b^2-4x^2}.$$

$$\therefore \text{ area of segment } PFHC'=\tfrac{1}{8}b^2\cos^{-1}\left(-\frac{2x}{b}\right)-\tfrac{1}{8}b\sqrt{b^2-4x^2},=\text{segment } HC'F.$$

$$x=\frac{2a^2-b^2}{2b}.$$

Hence, area of $C'HPNF=a^2\cos^{-1}\frac{a}{b}+\frac{1}{4}b^2\cos^{-1}\frac{b^2-2a^2}{b^2}-\frac{1}{2}a\sqrt{b^2-a^2}$, square feet ; the space grazed over inside the fence.

II. Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

In first part required area equals that of semicircle $ALB+2\times AMFHC$. Let HM be a portion of rope unwound and equal to HF. Let $\angle HCF=\phi$, $\angle MCF=\theta$, $CM=\rho$, and take CF for polar axis. Then

$$\rho^2=\tfrac{1}{4}b^2+\overline{HM}^2,=\tfrac{1}{4}b^2+\tfrac{1}{4}(b^2\phi^2).$$

$$\phi=\frac{2}{b}\sqrt{4\rho^2-b^2}. \qquad \frac{b}{2\rho}=\cos(\phi-\theta),=\cos\left(\frac{2}{b}\sqrt{4\rho^2-b^2}-\theta\right).$$

$$\theta=\frac{2}{b}\sqrt{4\rho^2-b^2}-\cos^{-1}\left(\frac{b}{2\rho}\right).$$

$$\frac{d\theta}{d\rho}=\frac{4\rho}{b\sqrt{4\rho^2-b^2}}-\frac{b}{2\rho^2\sqrt{1-\frac{b^2}{4\rho^2}}},\ =\frac{\sqrt{4\rho^2-b^2}}{b\rho}.$$

$$\frac{dA}{d\rho}=\frac{dA}{d\theta}\frac{d\theta}{d\rho},\ =\frac{\rho^2}{2}\times\frac{\sqrt{4\rho^2-b^2}}{b\rho},\ =\frac{\rho\sqrt{4\rho^2-b^2}}{2b}.$$

Limits of ρ for $CFMA$ are seen to be $\frac{1}{2}b$ and $\sqrt{a^2+\frac{1}{4}b^2}$.

$$\therefore \text{Area } CFMA=\frac{1}{2b}\int_{\frac{1}{2}b}^{\sqrt{a^2+\frac{1}{4}b^2}}\rho\sqrt{4\rho^2-b^2}\,d\rho,\ =\frac{a^3}{3b}.$$

Area $FHCAM=CFMA+C'CA-CFHC'$, $=a^3/3b+ab/4-ab/4=a^3/3b$.

$\therefore$ Area $FBLAFHC'=2a^3/3b+\pi a^2/2$.

Internal area is composed of $2\times$segment PHC' + sector $PDNC'$

$$\sin\tfrac{1}{2}\angle PCC'=(\tfrac{1}{2}a/\tfrac{1}{2}b)=a/b.\quad \angle PCC'=2\sin^{-1}(a/b).$$

$$\angle PCN=2\pi-4\sin^{-1}(a/b).\quad \angle PC'N=\pi-2\sin^{-1}(a/b).$$

$$\text{Sec. } PDNC'=\frac{a^2}{2}\left(\pi-2\sin^{-1}\frac{a}{b}\right),\ \text{sector } CPHC'=\frac{b^2}{8}\left(2\sin^{-1}\frac{a}{b}\right),=\frac{b^2}{4}\sin^{-1}\left(\frac{a}{b}\right)$$

$$\triangle PCC'=\tfrac{1}{2}a\sqrt{\tfrac{1}{4}b^2-\tfrac{1}{4}a^2},=\tfrac{1}{4}a\sqrt{b^2-a^2}.$$

$$\text{Segment } PHC'=\frac{b^2}{4}\sin^{-1}\frac{a}{b}-\tfrac{1}{4}a\sqrt{b^2-a^2}.$$

$$\therefore \text{Internal area }=\frac{a^2}{2}\left(\pi-2\sin^{-1}\frac{a}{b}\right)+\frac{b^2}{2}\sin^{-1}\frac{a}{b}-\tfrac{1}{2}a\sqrt{b^2-a^2},$$

$$=\tfrac{1}{2}\pi a^2-\frac{a}{2}\sqrt{b^2-a^2}+(\tfrac{1}{2}b^2-a^2)\sin^{-1}\left(\frac{a}{b}\right).$$

III. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let A be the point where the horse is tethered. $AF=a$, $AO=b/2$. Area $EADGKFE=2$ area EAF + area of semicircle GKF.

$$\therefore A=\int\rho^2d\theta+\tfrac{1}{2}\pi a^2\ ;\ \text{but } \rho=\tfrac{1}{2}b\theta.$$

$$\therefore\ A=\tfrac{1}{4}b^2\int_0^{2a/b}\theta^2\,d\theta+\tfrac{1}{2}\pi a^2,=\frac{a^2}{6b}(4a+3\pi b).$$

Let $x^2+y^2=\frac{1}{4}b^2$, be the equation to circle center O. $(x-\frac{1}{2}b)^2+y^2=a^2$, be the equation to circle center A.

$$\therefore OH=\frac{b^2-2a^2}{2b}, \quad \therefore BH=\frac{a}{b}\sqrt{b^2-a^2}.$$

A'=area of segment BLC+area of segment BAC,

$$\frac{b^2}{4}\left\{\sin^{-1}\left(\frac{2a}{b^2}\sqrt{b^2-a^2}\right)-\frac{2a(b^2-2a^2)\sqrt{b^2-a^2}}{b^4}\right\}$$

$$+a^2\left\{\sin^{-1}\frac{\sqrt{b^2-a^2}}{b}-\frac{a\sqrt{b^2-a^2}}{b^2}\right\},$$

$$=\tfrac{1}{4}b^2\cos^{-1}\left(\frac{b^2-2a^2}{b^2}\right)+a^2\cos^{-1}\frac{a}{b}-\frac{a}{2}\sqrt{b^2-a^2}.$$

Also solved by *J. SCHEFFER* and *A. H. HOLMES*.

60. Proposed by B. F. BURLESON, Oneida Castle, New York.

Find (1) the length s of the closed curve of the cardioid; (2) its area A; (3) if made to olve about its axis $2a$, find the maximum longitudinal circumference C of the solid gented; (4) find the surface K of the same; (5) its volume V; (6) the distance x_0 of the cenof gravity of the solid from the origin O; and (7) the distance ρ_0 of the center of gravof the plane curve from the origin O.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland, and the PROPOSER.

Let $AB=a$ be the diameter of a circle. From A draw any chord AC. ke CP and $CP'=b$, then will the locus of P P' be the Limaçon. If $AP=r$, $\angle PAB=\theta$, we d at once the polar equation of the Limaçon be $r=a\cos\theta+b$. If $b>a$, the curve consists out one loop; if $b<a$, it has two loops, and if a the curve becomes the Cardioid, the polar ation of which is $r=a(1+\cos\theta)$. It can eas- be shown that the cardioid is an epicycloid, generating circle of which is equal to the fixed ; also, drawing through the center O of the

le a line parallel to AP cutting the circumference of the circle at D, and drawthrough P a line parallel to CD, this line is a tangent to the cardioid at P. different problems proposed are best solved by means of the polar equation curve.

(1). The length $s=2\int_0^\pi d\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2},=2a\int_0^\pi\cos\tfrac{1}{2}\theta d\theta,=8a.$

(2). The area $A=\int_0^\pi r^2 d\theta,=a^2\int_0^\pi(1+\cos\theta)^2 d\theta,=\tfrac{3}{2}\pi a^2.$

(3). To find the maximum ordinate, $r\sin\theta=a(\sin\theta+\frac{1}{2}\sin2\theta$ is to be a maximum. By differentiation we find $\theta=60°$. $\therefore$ maximum ordinate$=\frac{3}{4}a\sqrt{3}$, and circumference $C=\frac{3}{2}\pi a\sqrt{3}$.

(4). Surface $K=2\pi\int_0^\pi r\sin\theta d\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)}$,

$$=4\pi a^2\int_0^\pi(1+\cos\theta)\sin\theta\cos\tfrac{1}{2}\theta d\theta,=-16\pi a^2\int_0^\pi\cos^4\tfrac{1}{2}\theta d\cos\tfrac{1}{2}\theta,=\frac{32\pi a^2}{5}.$$

For the distance x_0 of the center of gravity of this surface we have

$$Kx_0=2\pi\int_0^\pi r^2\sin\theta\cos\theta(ds/d\theta).d\theta,=4\pi a^3\int_0^\pi(1+\cos\theta)^2\sin\theta\cos\theta.\cos\tfrac{1}{2}\theta d\theta,$$

$$=-64\pi a^3\left[2\int_0^\pi\cos^8\tfrac{1}{2}\theta d\cos\tfrac{1}{2}\theta.\int_0^\pi\cos^6\tfrac{1}{2}\theta d\cos\tfrac{1}{2}\theta\right],=\frac{320\pi a^3}{63};$$

$$\therefore x_0=\frac{320\pi a^3}{63}\div\frac{32\pi a^2}{5}=\frac{50}{63}a.$$

(5). Volume, $V=2\pi\int_0^\pi\int_0^{a(1+\cos\theta)}rdrd\theta.r\sin\theta,=\frac{2\pi a^3}{3}\int_0^\pi(1+\cos\theta)^3\sin\theta d\theta$

$$=-\frac{2\pi a^3}{3}\int_0^\pi(1+\cos\theta)^3d(1+\cos\theta),=\frac{8\pi a^3}{3}.$$

(6). The distance x_0 of this volume from the origin we find from

$$Vx_0=2\pi\int_0^\pi\int_0^{a(1-\cos\theta)}r^2drd\theta\sin\theta r\sin\theta,=\frac{\pi a^4}{2}\int_0^\pi(1+\cos\theta)^4\sin^2\theta d\theta,$$

$$=32\pi a^4\int_0^\pi\cos^{10}\tfrac{1}{2}\theta\sin^2\tfrac{1}{2}\theta d\theta,=\frac{64\pi a^4\Gamma(\frac{3}{2})\Gamma(\frac{11}{2})}{2\Gamma(7)},=\frac{21}{32}\pi^2a^4;$$

$$\therefore x_0=\frac{21\pi^2a^4}{32}\div\frac{8\pi a^3}{3}=\frac{63}{256}\pi a.$$

(7). The distance x_0 of the center of gravity of the arc of the curve from the origin is found by

$$sx_0=2\int_0^\pi r\sin\theta.2a\cos\tfrac{1}{2}\theta.d\theta,=16a^2\int_0^\pi\cos^4\tfrac{1}{2}\theta.\sin\tfrac{1}{2}\theta.d\theta,$$

$$=-32a^2\int_0^\pi\cos^4\tfrac{1}{2}\theta d(\cos\tfrac{1}{2}\theta),=\frac{32a^2}{5},\quad\therefore x_0=\frac{32a^2}{5}\div8a=\frac{4}{5}a;$$

...d the distance x'_0 of the area of the curve from the origin is found by

$$x'_0=\tfrac{2}{3}\int_0^\pi r^3\cos\theta d\theta,=\tfrac{2}{3}a^3\int_0^\pi(1+\cos\theta)^3\cos\theta d\theta,=\tfrac{16}{3}a^3\int_0^\pi(2\cos^8\tfrac{1}{2}\theta-\cos^6\tfrac{1}{2}\theta)d\theta,$$

$$=\tfrac{5}{4}\pi a^3. \qquad \therefore x'_0=\tfrac{5}{4}\pi a^3\div\frac{3\pi a^2}{2}=\tfrac{5}{6}a.$$

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.**

Let $r=a(1+\cos\theta)$, be the equation to the Cardioid.

(1). $s/2=2a\int_0^\pi\cos(\tfrac{1}{2}\theta)d\theta,=4a. \quad \therefore s=8a.$

(2). $A=2a^2\int_0^{2\pi}\cos^4(\tfrac{1}{2}\theta)d\theta,=\tfrac{1}{2}(3\pi a^2).$

(3). $C=2\pi\rho$, where $\rho=r\sin\theta,=a\sin\theta(1+\cos\theta)$. $d\rho=2a\cos^2\theta+a\cos\theta-a.$

$\therefore \cos\theta=\tfrac{1}{2}$ or -1. $\therefore$ for a maximum $\theta=60°$.

$\therefore C=2\pi a(1+\cos\tfrac{1}{3}\pi)\sin\tfrac{1}{3}\pi,=\tfrac{1}{2}(3\sqrt{3}\pi a).$

(4). $K=8\pi a^2\int_0^\pi\cos^3(\tfrac{1}{2}\theta)\sin\theta d\theta,=\tfrac{1}{5}(32\pi a^2).$

(5). $V=2\pi\int_0^\pi\int_0^{a(1+\cos\theta)}r^2\sin\theta d\theta dr,=\dfrac{16\pi a^3}{3}\int_0^\pi\cos^6(\tfrac{1}{2}\theta)\sin\theta d\theta.$

$\therefore V=\tfrac{1}{3}(8\pi a^3).$

$$(6).\quad x_0=\frac{\int_0^\pi\int_0^{a(1+\cos\theta)}r^3\sin\theta\cos\theta d\theta dr}{\int_0^\pi\int_0^{a(1+\cos\theta)}r^2\sin\theta d\theta dr},=\tfrac{3}{2}a\frac{\int_0^\pi\cos^8(\tfrac{1}{2}\theta)\cos\theta\sin\theta d\theta}{\int_0^\pi\cos^6(\tfrac{1}{2}\theta)\sin\theta d\theta}.$$

$\therefore x_0=\tfrac{4}{5}a.$

$$(7).\quad g_0=\frac{\int_0^{2\pi}\int_0^{a(1+\cos\theta)}r^2\cos\theta d\theta dr}{\int_0^{2\pi}\int_0^{a(1+\cos\theta)}r d\theta dr},=\tfrac{4}{3}a\frac{\int_0^{2\pi}\cos^6(\tfrac{1}{2}\theta)\cos\theta d\theta}{\int_0^{2\pi}\cos^4(\tfrac{1}{2}\theta)d\theta}$$

$\therefore g_0=\tfrac{5}{6}a.$

Also solved by *C. W. M. BLACK.*

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

40. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the law of the force, in order that the orbit may be a Cassinian Oval.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $r^4+2c^2r^2\cos2\theta=m^4-c^4=a^4$ ......(1) be the equation to the oval. Then $a^4u^4=2c^4u^2\cos2\theta+1$, where $u=1/r$.

$$\frac{du}{d\theta}=\frac{c^2u\sin2\theta}{c^2\cos2\theta-a^4u^2}\text{ ....(2).}$$

$$\frac{d^2u}{d\theta^2}=\left\{\frac{(c^2\sin2\theta\frac{du}{d\theta}+2c^2u\cos2\theta)(c^2\cos2\theta-a^4u^2+(2c^2\sin2\theta+2a^4u\frac{du}{d\theta})c^2u\sin2\theta}{(c^2\cos2\theta-a^4u^2)^2}\right\}$$

$$=\left\{\frac{3c^4r^8+10r^4c^4m^4-3r^4c^8-(m^4-c^4)^3-7r^4m^8+9r^8m^4-r^{12}}{r(a^4+2r^4)^3}\right\}.$$

$$F=\text{force}=h^2u^2(u+\frac{d^2u}{d\theta^2})=\frac{h^2}{r^3}+\frac{h^2}{r^2}\cdot\frac{d^2u}{d\theta^2}.$$

$$\therefore F=\frac{h^2(7r^{12}+21r^8m^4+3r^4c^8-9c^4r^8-2c^4m^4r^4-m^8r^4)}{r^3(m^4-c^4+2r^4)^3}$$

$$=\frac{h^2\{7r^9+21m^4r^5-9c^4r^5+3rc^8-2c^4m^4r-m^8r\}}{(m^4-c^4+2r^4)^3}.$$

41. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

If the earth were a perfect sphere and had a frictionless surface, what would be the motion of a ball placed at a given latitude?

[No solution of this problem has been received. EDITOR].

42. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

Find the time of vibration of a particle slightly displaced from the center of a solid cylinder in direction of the axis, the matter of the cylinder attracting according to the laws of nature.

Solution by the PROPOSER.

Consider the particle displaced by the amount x from the center of the cylinder. The matter attracting it will be a cylinder of length $2x$ at the opposite end of the cylinder. Call y the distance of any particle from the axis of cylinder and z the distance of particle from the end of cylinder with length $2x$ measured from the end towards the center.

The attraction of the cylinder upon the particle displaced from the center is

$$A=2\pi\int_0^R\int_0^{2x}\frac{y(a-x+z)dydz}{[(a-x+z)^2+y^2]^{\frac{3}{2}}}.\qquad A=2\pi\int_0^{2x}\left[dz-\frac{(a-x+z)dz}{[(a-x+z)^2+R^2]^{\frac{1}{2}}}\right].$$

$$=4\pi x-2\pi\sqrt{(a+x)^2+R^2}-2\pi\sqrt{(a-x)^2+R^2}$$

$$=4\pi x-2\pi\sqrt{(a^2+R^2)+(x^2+2ax)}-2\pi\sqrt{(a^2+R^2)+(x^2-2ax)}$$

$$=4\pi x-2\pi\{(a^2+R^2)^{\frac{1}{2}}+\tfrac{1}{2}\frac{x^2+2ax}{(a^2+R^2)^{\frac{1}{2}}}+\ldots\ldots\}$$

$$-2\pi\{(a^2+R^2)^{\frac{1}{2}}+\tfrac{1}{2}\frac{x^2-2ax}{(a^2+R^2)^{\frac{1}{2}}}\ldots\ldots\}$$

$$=4\pi x-4\pi(a^2+R^2)^{\frac{1}{2}}-2\pi\frac{x^2}{(a^2+R^2)^{\frac{1}{2}}}\ldots\ldots$$

Since the displacement is to be slight, we may neglect x^2 and all higher powers.

$\therefore\ (d^2x/dt^2)=4\pi x-4\pi(a^2+R^2)^{\frac{1}{2}}=cx-d$, for brevity.

$$\frac{d^2x}{dt^2}=\frac{dv}{dt}=\frac{dv}{dx}\,\frac{dx}{dt}=v\frac{dv}{dt}.\quad\therefore v\frac{dz}{dx}=cx-d.\quad v^2=cx^2-2dx+k.$$

When the displacement is a maximum the particle is at rest. Call the amplitude α.

$$\text{Then } t=\int_0^{\alpha}\frac{dx}{\sqrt{cx^2-2dx-c\alpha^2+2d\alpha}}=\int_0^{\alpha}\frac{dx}{\sqrt{(2d\alpha-c\alpha^2)+(cx^2-2dx)}}$$

$$\int_0^{\alpha}[(2d\alpha-c\alpha^2)+(cx^2-2dx)]^{-\frac{1}{2}}dx=\frac{\alpha}{\sqrt{2d\alpha-c\alpha^2}},$$

neglecting higher powers of x.

$$=\sqrt{\frac{\alpha}{8\pi\sqrt{a^2+R^2}}},\text{ neglecting the square of }\alpha.$$

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

47. Proposed by M. A. GRUBER, A. M., War Department, Washington D. C.

Find the first six sets of values in which the sum of two consecutive integral squares equals a square.

I. Solution by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Let $\frac{1}{2}(x_n-1)$ and $\frac{1}{2}(x_n+1)$ be two consecutive integers, and y_n^2 the sum of their squares; then we must have

$$\tfrac{1}{4}(x_n-1)^2+\tfrac{1}{4}(x_n+1)^2=y_n^2, \text{ or } x_n^2-2y_n^2=-1\ldots\ldots\ldots\ldots(1),$$

which may be written

$$(x_n-y_n\sqrt{2})(x_n+y_n\sqrt{2})=-1\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

When $n=1$, we have

$$(x_1-y_1\sqrt{2})(x_1+y_1\sqrt{2})=-1\ldots\ldots\ldots\ldots\ldots\ldots(3);$$

also, raising (3) to the $(2n+1)$th power, we have

$$(x_1-y_1\sqrt{2})^{2n+1}(x_1+y_1\sqrt{2})^{2n+1}=-1\ldots\ldots\ldots\ldots\ldots(4),$$

where n may be 0, 1, 2, 3, 4, etc.

Assuming $x_n-y_n\sqrt{2}=(x_1-y_1\sqrt{2})^{2n+1}$,

$x_n+y_n\sqrt{2}=(x_1+y_1\sqrt{2})^{2n+1}$, as we are at liberty to do, we find

$$x_n=[(x_1+y_1\sqrt{2})^{2n+1}+(x_1-y_1\sqrt{2})^{2n+1}]/2,$$

$$y_n=[(x_1+y_1\sqrt{2})^{2n+1}-(x_1-y_1\sqrt{2})^{2n+1}]/2\sqrt{2}.$$

It is easily seen that $x_1=1$ and $y_1=1$; therefore

$$x_n=[(\sqrt{2}+1)^{2n+1}-(\sqrt{2}-1)^{2n+1}]/2,\quad y_n=[(\sqrt{2}+1)^{2n+1}+(\sqrt{2}-1)^{2n+1}]/2\sqrt{2},$$

and the required numbers are

$$\tfrac{1}{4}[(\sqrt{2}+1)^{2n+1}-(\sqrt{2}-1)^{2n+1}-2] \text{ and } \tfrac{1}{4}[(\sqrt{2}+1)^{2n+1}-(\sqrt{2}-1)^{2n+1}+2].$$

The operation of involution is very tedious except when n is a small number. When x_n and y_n are very large numbers we have from (1) very nearly

$/y_n = \sqrt{2}$, and the values of x_n and y_n are the numerators and denominators the odd convergents to $\sqrt{2}$ expanded as a continued fraction, after the first. ıe successive odd convergents are

$$1/1,\ 7/5,\ 41/29,\ 239/169,\ 1393/985,\ 8119/5741, \text{ etc.}$$

The values of x_n and y_n are connected by the relations

$= 6x_{n-1} - x_{n-2}$..................(5), $y_n = 6y_{n-1} - y_{n-2}$..................(6),

hich afford an easy method of computing the successive sets of numbers quired. When $n=1$ we easily find from the general formulas the first set, 3 ıd 4, and then from (5) the successive sets, which are

2nd set	20,	21,
3d "	119,	120,
4th "	696,	697,
5th "	4059,	4060,
6th "	23660,	23661.

See the *Mathematical Visitor*, Vol. I., No. 3, page 56, where the fifth and xth sets are erroneously given as 4058, 4059 and 23657, 23658. The root of ıe sum of the squares of the sixth set should be 33461 instead of 33457. The)0 set is given on the same page; and also on page 122 where the numbers are und by Mr. K. S. Putnam by a different method.

These numbers solve the geometrical problem—"To find rational right-ıgled triangles whose legs are consecutive numbers."

II. Solution by A. H. BELL, Hillsboro, Illinois.

We have $x^2 + (x+1)^2 = \square$, or $2x^2 + 2x + 1 = \square$ ....(1).

Take $Ax^2 \pm Bx + C = \square = y^2$ ....(2).

(2)×A, and add and subtract, etc.

$(Ax \pm B/2)^2 = Ay^2 + B^2/4 - AC = \square = t^2$ ....(3).

$\therefore t^2 - Ay^2 = B^2/4 - AC$ ....(A).

Also $x = (t \mp B/2)/A$ ....(B).

Let (A) reduce to $t^2 - Ay^2 = \pm D$ ....(4);

d a complete quotient$=(\sqrt{A}+M)/D$. Then the preceding convergent will t/y and will answer the $+$ or $-D$ as it is an odd or even number of fraction.

Also $v^2 - Au^2 = 1$ ....(5).

(4)×(5), and add and subtract $2Auvyt$, etc.

$(vt \pm Ayu)^2 - A(ut \pm vy)^2 = \pm D$, or $t_n^2 - Ay_n^2 = \pm D$ ....(C).

But in the problem (A) is $t^2-2y^2=-1$ .... (6).

Then we also have $t_n/y_n=(2vt_{n-1}-t_{n-2})/(2vy_{n-1}-y_{n-2})$ .... (7).

$\sqrt{2}$. Number of complete fractions = integer : 1, 2, etc.

Complete quotients $=(\sqrt{2}+0)/1$: $(\sqrt{2}+1)/1$, $(\sqrt{2}+1)/1$, etc.

Partial quotients = 1 : 2, 2, etc.

Convergents = 1/0, 1/1 : 3/2, 7/5, etc.

$\therefore t/y=1/1$, 7/5, etc. $v/u=1/0$, 3/2. $2v_1=6$, $v_0=1$.

$\therefore$ (C) or (7) $t=1$, 7, 41, 239, 1393, 8119, 47321, etc.

(B) $x=0/2$, 3, 20, 119, 696, 4059, 23660, etc.

$x+1$ will give the six sets of values required.

NOTE. $2v=M$=magic M of Roberts and Robins.

III. Solution by the PROPOSER.

Take the formula for finding the sum of two integral squares equal to a square:

$$(2mp)^2+(m^2-n^2)^2=(m^2+n^2)^2 \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

Then will the difference between $2mn$ and m^2-n^2 be 1. When $m^2-n^2>2mn$, we have $m^2-n^2-2mn=1$; whence $m=n\pm\sqrt{2n^2+1}$. When $2mn>m^2-n^2$, we have $2mn-(m^2-n^2)=1$, whence $m=n\pm\sqrt{2n^2-1}$. Substituting these values of m in (A), we obtain

$$[2n(n\pm\sqrt{2n^2\pm1})]^2+[2n(n\pm\sqrt{2n^2\pm1})\pm1]^2$$

$$=[2n(n\pm\sqrt{2n^2\pm1})+2n^2\pm1]^2 \ldots\ldots\ldots (B).$$

It now remains to make $2n^2+1=\square$, and $2n^2-1=\square$. We find, by inspection, that the first value of n in $2n^2+1=\square$, is 0, and in $2n^2-1=\square$, is 1. Knowing these two values, we find the succeeding values from the formula, $n=2n_1+n_2$, in which n_1 is the last found known value of n, and n_2 the value just preceding. Whence $n=0$, 1, 2, 5, 12, 29, 70, 169, 408, 985, etc. *Zero* and the *even* numbers are the values of n in $2n^2+1=\square$, and the *odd* numbers in $2n^2-1=0$; the first two values of each series being known, the succeeding values can be found by the formula, $n=6n_1-n_2$.

It is also noticeable that the consecutive *odd number* values of n are the consecutive values of the root of the square that equals the sum of two consecutive integral squares. Substituting, now, the values of n in (B), we obtain, respectively, $0^2+1^2=1^2$, $4^2+3^2=5^2$, $20^2+21^2=29^2$, $120^2+119^2=169^2$, $696^2+697^2=985^2$, $4060^2+4059^2=5741^2$, $23660^2+23661^2=33461^2$, etc.

Or, from solution III of Problem 36, Vol. III., No. 3, page 82, we find that when one of the triangular square numbers is taken as $n(n+1)/2$, the next in order, is terms of n is $[2n+1+3\sqrt{n(n+1)/2}]^2$. The difference of the roots of these two successive triangular square numbers is $2n+1+2\sqrt{n(n+1)/2}$. The sum of the roots is $2n+1+4\sqrt{n(n+1)/2}$, which, when $n(n+1)/2=\square$, equals the sum of the two *consecutive integral* numbers, $n+2\sqrt{n(n+1)/2}$ and $n+1+2\sqrt{n(n+1)/2}$.

$$\text{But } [n+2\sqrt{n(n+1)/2}]^2+[n+1+2\sqrt{n(n+1)/2}]^2$$

$$=6n^2+6n+1+(8n+4)\sqrt{n(n+1)/2}=[2n+1+2\sqrt{n(n+1)/2}]^2.$$

We here have a general formula in which the sum of the squares of two consecutive integers equals a square. To obtain integral numerical results, we assign the successive values of n in $n(n+1)/2=\square$, 1, 8, 49, 288, 1681, 9800, etc. Whence we have $3^2+4^2=5^2$; $20^2+21^2=29^2$; $119^2+120^2=169^2$; $696^2+697^2=985^2$; $4059^2+4060^2=5741^2$; $23660^2+23661^2=33461^2$, etc.

IV. Solution by Hon. J. H. DRUMMOND, LL. D., Portland, Maine.

Let x=one number and $x+1$=the other; then x^2+x^2+2x+1 will be the sum of two consecutive squares. Then $2x^2+2x+1=\square=$(say) $(mx-1)^2$, from which we readily obtain $x=2(m+1)/(m^2-2)$. It is readily seen that x is integral when $m=2$. Then we have 2/1, 10/7, 58/41, etc., for values of m which give integral values of x, viz., 3, 119, 4059, etc. The other series which makes x integral is 3/2, 17/12, 99/70, etc., and x=20, 696, 23660, etc. The six values of x, therefore, are 3, 20, 119, 696, 4059, 23660, and of $x+1$, 4, 21, 120, 697, 4060, 23661, and the squares of these values are *probably* the squares required. I say "probably," because it cannot be mathematically determined that some other method of solution will not give other results that show that there are other values less than 23660 besides those I have given.

[The following is my formula for obtaining integral values of a fraction whose denominator is p^2-2, which I assume in this solution. I have never seen the formula in print and do not know how generally it is known.

If r/s is of such a value of p as will give an integral result, then $(3r+4s)/(2r+3s)$ is *another* value of p that gives an integral result, and so on *ad infinitum*. If the numerator is even, there will be two different series of values of p, the initial term in one being 2/1, and in the other 3/2; if the numerator is odd, the series beginning with 3/2 will give integral results. By means of this formula an infinite number (mathematically speaking) of integral values of x may be obtained in the equation $2x^2+ax+b^2=\square$, in terms of a and b; and in the equation $2x^2+2ax+b^2$, two series (infinite) of integral values in terms of a and b may be obtained. In both cases, however, the numbers increase in value very rapidly.]

V. Solution by A. H. HOLMES, Brunswick, Maine.

$x^2+(x+1)^2=\square$ or $2x^2+2x+1=\square$. Let $x=y+p$.

$\therefore 2y^2+(4p+2)y+2p^2+2p+1=\square$, from which we find the law of the series to be: $b=1+3a+2\sqrt{2a^2+2a+1}$. Let $a=3$ and we find $b=20$. Then by the same law, $c=119$, $d=696$, $e=4059$, and $f=23660$. Therefore, we have for the first six sets of values: (3 and 4), (20 and 21), (119, 120), (696 and 697), (4059 and 4060), and (23660 and 23661).

VI. Solution by H. C. WILKES, Skull Run, West Virginia.

We have $x^2+(x+1)^2=y^2=4n+1$, then $x(x+1)/2=n$. Substituting this value for n in $4n+1=y^2$ we have $x^2+x=(y^2-1)/2$. Putting $x+(x+1)=t$ or $x=(t-1)/2$, we obtain $t^2-2y^2=-1$. Since $t=7$, $y=5$ satisfy this equation, the first values of $x+(x+1)$ and y will be 7 and 5.

$\therefore 3^2+4^2=5^2$. From inspection of solution II, Problem 36, Vol. III, page 81, we find a formula for obtaining the succeeding values of $x+(x+1)$ and y.

$x++(x1)$.	y.	
$6\times7-1=41$,	$6\times5-1=29$,	$20^2+21^2=29^2$,
$6\times41-7=239$,	$6\times29-5=169$,	$119^2+120^2=169^2$,
$6\times239-41=1393$,	$6\times169-29=985$,	$696^2+697^2=985^2$,
$6\times1393-239=8119$,	$6\times985-169=5741$,	$4059^2+4060^2=5741^2$,
$6\times8119-2392=47321$.	$6\times5741-985=33461$.	$23660^2+23661^2=33461^2$.

Also solved by *J. SCHEFFER* and *G. B. M. ZERR*.

48. Proposed by B. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, O.

If any positive integral number N be divided by another positive integral number D, leaving a remainder 1, then any positive integral power of N, divided by D, will leave a remainder of 1.

I. Solution by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Let $N=nD+1$, then

$$(nD+1)^m=n^mD^m+mn^{m-1}D^{m-1}+\frac{m(m-1)}{2}n^{m-2}D^{m-2}+$$

$$\ldots\ldots+\frac{m(m-1)}{2}n^2D^2+mnD+1,$$

$$=D[n^mD^{m-1}+mn^{m-1}D^{m-2}+\frac{m(m-1)}{2}n^{m-2}D^{m-3}+\ldots\ldots+\frac{m(m-1)}{2}n^2D+mn]+1,$$

which proves the proposition.

Solved in a similar manner by *M. A. GRUBER* and *G. B. M. ZERR*.

II. Solution by J. C. CORBIN, Pine Bluff, Arkansas; P. S. BERG, Larimore, North Dakota; E. W. MORRELL, Montpelier Seminary, Montpelier, Vermont; A. P. READ, A. M., Clarence, Missouri; and O. S. WESTCOTT, Principal North Chicago High School, Chicago.

Put $N=nD+1$, then it is evident that if $N=nD+1$ be raised to any positive integral power, the last term will be 1 and every other term will contain D as a factor; hence if this power be divided by D the remainder will be 1.

Also solved in a similar way by *A. H. BELL, JOSIAH H. DRUMMOND, ARTEMAS MARTIN* and *J. SCHEFFER*.

III. Solution by J. O. MAHONEY, B. E., M. S., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tennessee.

If $a\equiv a'$, $b\equiv b'$, $c\equiv c'$, $d\equiv d'$, etc., mod(D),

then $abcd......\equiv a'b'c'd'......$mod($D$).

Let $a=b=c=d$, etc.,$=N$, and $a'=b'=c'=d'$, etc.,$=1$, then $N^k\equiv 1$ mod(D).

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

39. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

A man is at the center of a circular desert; he travels at a given rate but in a *perfectly* random manner. What is the probability that he will be off the desert in a given time?

No solution of this problem has been received.

40. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

If every point of an ellipse be joined with every other point, what is the average length of the chords thus drawn?

Solution by the PROPOSER.

Let $a\cos\theta$ and $(b/a)\sin\theta$ be the coördinates of one point, and $a\cos\phi$ and $(b/a)\sin\phi$ those of another.

The length of the chord joining them is

$$K=[a^2(\cos\phi-\cos\theta)^2+\frac{b^2}{a^2}(\sin\phi-\sin\theta)^2]^{\frac{1}{2}}.$$

Let s_1 and s_2=lengths of elliptic arcs from point $(a, 0)$ to points $(a\cos\theta, \frac{a}{b}\sin\theta)$ and $(a\cos\phi, \frac{a}{b}\sin\theta)$ respectively, and let S=whole distance around the ellipse.

Then $\dfrac{ds_1}{d\theta}=a(1-e^2\cos^2\theta)^{\frac{1}{2}}$ and $\dfrac{ds_2}{d\phi}=a(1-e^2\cos^2\phi)^{\frac{1}{2}}$.

Then the required average is

$$A=\frac{H}{S^2}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}Kds_1ds_2=\frac{4a^2}{S^2}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\left[a^2(\cos\phi-\cos\theta)^2+\frac{b^2}{a^2}(\sin\phi-\sin\theta)^2\right]^{\frac{1}{2}}\times$$

$$(1-e^2\cos^2\theta)^{\frac{1}{2}}(1-e^2\cos^2\phi)^{\frac{1}{2}}d\theta d\phi.$$

This equation cannot be integrated in general terms.

Solved in the same manner by *G. B. M. ZERR*,

41. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

A line is drawn at random across the chord and given arc of a circular segment. Find the mean area of the divisions.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let A=area of given segment, A_1, A_2 mean areas of the two divisions.

$\therefore A_1+A_2=A$.

But, since the line is a random line, $A_1=A_2$.

$\therefore A_1=A_2=\frac{1}{2}A$.

Also solved by *HENRY HEATON*.

42. Proposed by CHARLES E. MYERS, Canton, Ohio.

A attends church 4 Sundays out of 5; B, 5 Sundays out of 6; and C, 6 Sundays out of 7. What is the probability of an event that A and B will be at church and C will not?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas, and B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

The chance that A attends church$=\frac{4}{5}$.
The chance that B attends church$=\frac{5}{6}$.
The chance that C attends church$=\frac{6}{7}$.
The chance that A is not at church$=\frac{1}{5}$.
The chance that B is not at church$=\frac{1}{6}$.
The chance that C is not at church$=\frac{1}{7}$.
The chance that A and B attend and C not$=p_1=\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{1}{7}=\frac{2}{21}$.
The chance that A and C attend and B not$=p_2=\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{1}{6}=\frac{4}{35}$.
The chance that B and C attend and A not$=p_3=\frac{1}{5}\cdot\frac{5}{6}\cdot\frac{6}{7}=\frac{1}{7}$.
The chance that A attends and B and C not$=p_4=\frac{4}{5}\cdot\frac{1}{6}\cdot\frac{1}{7}=\frac{2}{105}$.
The chance that B attends and A and C not$=p_5=\frac{1}{5}\cdot\frac{5}{6}\cdot\frac{1}{7}=\frac{1}{42}$.
The chance that C attends and A and B not$=p_6=\frac{1}{5}\cdot\frac{1}{6}\cdot\frac{6}{7}=\frac{1}{35}$.
The chance that A, B and C attend$=p_7=\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7}=\frac{4}{7}$.
The chance that A, B and C do not attend$=p_8=\frac{1}{5}\cdot\frac{1}{6}\cdot\frac{1}{7}=\frac{1}{210}$.
p_1=probability required.
Also $p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8=1$.

Also solved by *HENRY HEATON*.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

40. Proposed by F. M. PRIEST, Mona House, St. Louis, Missouri.

Suppose two cylindrical iron shafts, each 6 inches in diameter and respectively, 20 nd 40 feet in height, are both standing perpendicular at the sea level. They start to fall still air, how long will it require each one to fall to a horizontal position?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Neglecting the atmosphere and supposing the cylinder to revolve about a iameter in the base, we get, if l is the length of an equivalent pendulum, from orks on Mechanics the formula for the time of vibration of a pendulum,

$$t=\sqrt{\frac{l}{g}}\int_0^a \frac{d\theta}{\sqrt{\sin^2\frac{1}{2}a-\sin^2\frac{1}{2}\theta}}.$$

In this problem a is $180^\circ=\pi$, $\theta=90^\circ=\frac{1}{2}\pi$.

$$\therefore t=\sqrt{\frac{l}{g}}\int_{\frac{1}{2}\pi}^{\pi}\frac{d\theta}{\cos\frac{1}{2}\theta}=\left[2\sqrt{\frac{l}{g}}\log.\left\{\frac{\tan(\frac{1}{4}\pi+\frac{1}{4}a)}{\tan(\frac{1}{4}\pi+\frac{1}{4}\theta)}\right\}\right]_{\theta=\frac{1}{2}\pi}^{a=\pi}.$$

$\therefore t=\infty$, which proves that in a perfectly vertical position they will not all unless moved slightly from this position. Let $a=\pi-\delta$ where δ is very small.

$$\therefore t=2\sqrt{\frac{l}{g}}\log.\left\{\frac{\cot\frac{1}{4}\delta}{\tan 3\pi/8}\right\}.$$

Let $\delta=1'$, $l=$length of cylinder, $b=$radius of base.

$\therefore l=(3b^2+4l^2)/6l=13.3349$ feet for first cylinder.

$l=26.66745$ feet for second cylinder.

$\therefore t=2(.644328)(3.153498)=4.0638$ seconds for first cylinder.

$t=2(.911177)(3.153498)=5.7468$ seconds for second cylinder.

41. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific Colge, Santa Rosa, California.

A straight inflexible bar of uniform weight and thickness, length m is suspended at ne two ends by a string without weight, length $l>m$ passing freely over a peg driven in perpendicular wall. Describe and analyze the curve traced on the wall by the ends of ne hanging bar.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let O be the peg, AB the rod. Let $OB=r$, $AO=r'$, $AB=m$, $\angle X$
$\angle XOA=\phi$.

Now in equilibrium OD always passes through the mid-point of AB.

Then $r+r'=l$ ....(1).

$m^2=r^2+r'^2-2rr'\cos(\phi-\theta)$ ....(2).

$r\cos\theta=r'\cos\phi$ ....(3).

(3) is obtained from the two triangles OAC, OBC.
(1) in (2) and (3) gives

$m^2=r^2+(l-r)^2-2r(l-r)\cos(\phi-\theta)$ ....(4).

$r\cos\theta=(l-r)\cos\phi$ ....(5).

(5) in (4) gives $l^2-m^2+2r^2\sin^2\theta-2rl=2r\sin\theta\sqrt{l^2-2rl+r^2\sin^2\theta}$.

$\therefore\ 4r^2(l^2-m^2\sin^2\theta)-4rl(l^2-m^2)+(l^2-m^2)^2=0.$

$$\therefore\ r=\frac{l^2-m^2}{2(l\pm m\sin\theta)}=\frac{l(1-e^2)}{2(1\pm e\sin\theta)},\ \text{where } e=m/l.$$

This equation represents two equal ellipses with eccentricity $=m/l$,
axis $=l$, minor axis $=\sqrt{l^2-m^2}$, and O is one focus of each ellipse.

[The above solves the problem—"An ellipse confined to one vertical plane is suspended fro
point in space, coincident with a movable point on its circumference. Describe the curve mark
foci." EDITOR].

EDITORIALS.

With January, 1897, the Chicago *Open Court* celebrates its decennia
versary and now appears in the form of a monthly instead of a weekly.

Plane and Solid Analytical Geometry, by Frederick H. Bailey, A. M.
Frederick S. Woods, Ph. D., Assistant Professors of Mathematics in Ma
setts Institute of Technology, is announced as ready in March by Gi
Company.

President H. H. Seerley, of the State Normal School of Cedar Falls, Iowa, s just ordered a complete set of the MONTHLY for the library. We only ve a few more complete sets. Who wants them?

We are in correspondence with several excellent mathematicians who are ous of securing better positions for next year. If any of our readers know ch positions which are vacant or likely to become vacant at the end of this ol year, we shall be pleased to refer them to these gentlemen.

With this number begins the fourth volume of the MONTHLY. No pains will pared on the part of the Editors to make this volume better than any of the e previous ones, and in this effort they earnestly solicit the continued aid of ormer contributors and subscribers. This number is sent to all our old subbers, with bill enclosed, and anyone who may wish to discontinue should re this copy with his name written on the wrapper.

BOOKS AND PERIODICALS.

Elements of Analytical Geometry of Two Dimensions. The Fourteenth Edition. By Briot and Boùquet. Translated and Edited by James Harrington Boyd, Instructor in Mathematics in The University of Chicago. 8vo. Cloth, 582 pages. Introduction Price, $2. Chicago: Werner School Book Co.

This celebrated work so long known to mathematicians familiar with the French nguage, is now put in English dress, and is, therefore, at the service of American studts. Comments on the material and the method of this work are unnecessary.

The work is divided into four books. Book I contains four chapters: Chapter I, Conrning Coördinates; Chapter II, Examples—The Circle, the Ellipse, the Hyperbola, the arabola, Cissoid of Diocles, etc.; Chapter III, Concerning Homogenity; Chapter IV, ansformation of Coördinates. Book II contains three chapters: Chapter I, Straight ne; Chapter II, the Circle; Chapter III, the Geometrical Loci. Book III contains twelve apters: Chapter I, Construction of Curves of the Second Degree; Chapter II, Center, iameter, and Axes of Curves of the Second Degree; Chapter III, Reduction of the Equaon of the Second Degree; Chapter IV, the Ellipse; Chapter V, the Hyperbola; Chapter I, Concerning the Parabola; Chapter VII, Foci and Directrices; Chapter VIII, the Conic ections; Chapter IX, the Determination of the Conic Sections; Chapter X, Theory of oles and Polars; Chapter XI, General Properties of Conic Sections; Chapter XII, Secants ommon to Two Conics. Book IV contains seven chapters: Chapter I, the Construction ' Curves in Rectilinear Coördinates; Chapter II, Convexity and Concavity; Chapter III, symptotes; Chapter IV, Construction of Curves in Polar Coördinates; Chapter V, Conrning Similitude; Chapter VI, Graphic Solutions of Equations; Chapter VII, Notions oncerning Unicursal Curves.

From the table of contents it is seen that a leading feature of the work is its scope. . treats all the important methods invented by geometers, and includes some of the most autiful discoveries of ancient and modern times. All subjects are treated in a practical ay and illustrated by the applications of the theories to numerous problems. The book beautiful as well as profound. The typographical and mechanical execution of the work a credit to American text-book making. I very heartily commend this work to the carel consideration of teachers of Analytical Geometry and mathematical students desiring good work on the subject. B. F. F.

The Outlines of Quaternions. By Lieutenant-Colonel H. W. L. Hime. 188 pages. Price, $3. Longmans, Green & Co. 1894. London and New York.

The first chapters deal with the properties of vectors. In the remaining pages we are introduced to quaternions proper,—their various forms and properties. The last chapter treats of the applications of quaternions to trigonometry, the triangle, the circle, conic sections, and other curves, the plane, tetrahedron, sphere and cone. These geometric applications show in some measure the usefulness of quaternions and give freshness and interest to the book. There is no preface. The addition of some exercises for solution would have added to the practical character of the work for class use. J. M. C.

Plane Surveying. By William G. Raymond, C. E., Member American Society of Civil Engineers; Professor of Geodesy, Road Engineering, and Topographical Drawing, in the Rensselaer Polytechnic Institute, Troy, New York. 8vo. Cloth, 486 pages (including tables). Price, $3. Chicago: American Book Co.

Some of the valuable features of this work are the detailed description of the use of instruments, accompanied by excellent illustrations and diagrams of the instruments themselves; the clear and comprehensible presentation of the subject matter of the work; and the fine form in which it appears for public favor. In its pages may be found treated plane table work and the use of the slide rule, planimeter and stadia measurements. Full tables and numerous examples of work in the way, both of underground and general topography are also given. B. F. F.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single numbers, 25 cents. The Review of Reviews Co., 13 Astor Place, New York City.

The *Review of Reviews* for February makes "A Plea for the Protection of Useful Men" from bores and "societies," and all well-meaning people who bother the life out of public men by letters and calls on the pretext of seeking assistance in some worthy undertaking. The editor of the *Review* publishes letters on this subject from the late Gen. Francis A. Walker, written only a few weeks before his death. In one of these letters General Walker wrote, "I am not well, and neither callars nor correspondents have any mercy.
B. F. F.

The Cosmopolitan. An Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1. per year in advance. Single number, 10 cents. The Cosmopolitan Co., Irvington-on-the-Hudson, New York.

The January number of the *Cosmopolitan* not only keeps up the usual literary excellence, artistic merit, and widest interests of that magazine, but also adds new features to its field of usefulness. The February number will contain the second part of Conan Doyle's new story.

During the year 1896, the *Cosmopolitan* reached the largest clientèle of intelligent, thoughtful readers possessed by any periodical in the world. The smallest issue of the year was 300,000 copies. B. F. F.

The Arena. A Monthly Magazine. Price, $3. Single number, 25 cents. Boston: Arena Publishing Co.

The *Arena* is the organ or mouth-piece of no one party, faction, or creed. It is unmortgaged and unbribed—a *free lance*, an *open arena*—wherein all honest and properly expressed and authoritative opinions, having in view the betterment of human conditions and human life, may be expressed. The best writers and authorities on leading questions contribute to its pages. Among the leading articles in the January number are the following: The Religion of Burns' Poems, by Rev. Andrew W. Cross; A Court of Medicine and Surgery, by A. B. Choate; Finance and Currency, by Gen. Heman Haupt; Daniel Webster's School Days, by Forest Prescott Hall; England's Hand in Turkish Massacres; etc.
B. F. F.

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PIRICAL FORMULÆ FOR APPROXIMATE COMPUTATION.

By the Late ANSEL N. KELLOGG, of New York City.

[NOTE. The following paper is here printed in the exact form in which it was left by the author at me of his death, except only a few necessary verbal alterations which are distinguished from the parts of the paper by an enclosure of square brackets.]

The following symbols have the same signification, throughout these form-, the only exception being when the letter "d" is used with a, b, c, etc.

$n=\frac{1}{m}$=index of root.

$m=\frac{1}{n}$=index of power.

q=quantity, or $\frac{q}{p}$, if fractional.

s=sum of $q+p$ (or the semi-sum).*

d=difference of q and p (or the demi-difference).*

$t=\frac{n^2-1}{3}$.

U=the sub-square, or under-square=$(\sqrt{q}-1)^2$.

U_4=the under-fourth=$(\sqrt[4]{q}-1)^4$, etc.

E=Napierian logarithm of number.

K=logarithm of number to base 4.

r=root of number.

*These last values for s and d cannot be used in the same equation with p and q.

$=\!|\!=$ signifies nearly equal.

$=\!\|\!=$ signifies very nearly equal.

Mercator's formula for the extraction of roots of numbers near unity was equivalent to my own formula No. 1, as shown in Hutton's *Tracts on Mathematics*, Vol. I.

$$\frac{ns+d}{ns-d}=\sqrt[n]{q}\text{, nearly} \dots\dots\dots\dots\dots (1).$$

Hutton says he gave a formula for the correction of this result, but I have never been able to find it. Hutton himself gives the derivation of the above formula.

First correction of above:

$$\sqrt[n]{q}=\!|\!=\frac{ns+d-\dfrac{(n^2-1)d^2}{3ns}}{ns-d-\dfrac{(n^2-1)d^2}{3ns}}.$$

Substituting t for $\dfrac{n^2-1}{3}$, we have

$$\sqrt[n]{q}=\!|\!=\frac{ns+d-\dfrac{td^2}{ns}}{ns-d-\dfrac{td^2}{ns}} \dots\dots\dots\dots\dots (2).$$

Second correction of above:

$$\sqrt[n]{\frac{q}{p}}=\!|\!=\frac{ns+d-\dfrac{t(d^2+[\sqrt{q}-\sqrt{p}]^4)}{ns}}{ns-d-\dfrac{t(d^2+[\sqrt{q}-\sqrt{p}]^4)}{ns}}; \dots\dots\dots\dots\dots (3).$$

or by reducing,

$$\sqrt[n]{q}=\!|\!=\frac{(n^2+2)s+(2n^2-2)\sqrt{s^2-d^2}+3nd}{(n^2+2)s+(2n^2-2)\sqrt{s^2-d^2}-3nd} \dots\dots\dots\dots\dots (4).$$

This nearly equals

$$\sqrt[n]{q}=\!|\!=\frac{ns+d-\dfrac{n^2-1}{3}\times\dfrac{d^2}{s^2-\dfrac{d^2}{4s}}}{ns-d-\dfrac{n^2-1}{3}\times\dfrac{d^2}{s^2-\dfrac{d^2}{4s}}} \dots\dots\dots\dots\dots (5).$$

Simpler than these, and nearly related to (3) is

$$\sqrt[n]{q} \doteq \frac{ns+d-\cfrac{td^2}{ns-\cfrac{td^2}{ns}}}{ns-d-\cfrac{td^2}{ns-\cfrac{td^2}{ns}}} \text{........................} (6).$$

Very simple and excellent is

$$\sqrt[n]{q} \doteq \frac{ns+d-\cfrac{2tU}{n}}{ns-d-\cfrac{2tU}{n}} \text{........................} (7).$$

All the foregoing are what I call *equidistant* processes, because for all val-
of n, the difference between the numerator and denominator of the result is
(or some multiple); that is, the subtractive corrections following d are the
e in both terms, whether q/p be a proper or an improper fraction.

Of the same nature, but differently derived, is formula (8) and its equiva-
(8½).

$$\sqrt[n]{q} \doteq \frac{n(2s^2-d^2)+d^2+2ds}{n(2s^2-d^2)+d^2-2ds} \text{....................} (8).$$

$$\sqrt[n]{q} \doteq \frac{n(2s^2-d^2)+d(2s+d)}{n(2s^2-d^2)-d(2s-d)} \text{....................} (8\tfrac{1}{2}).$$

The fact that all these fractional formulae are symmetrical makes their
ation comparatively simple.

In extracting roots of high numbers

$$\sqrt[n]{q} = 2^{\frac{m}{n}} \times \sqrt[n]{\frac{q}{2^m}},$$

we are thus always enabled to use $q/2^m$ between the limits ½ and 2. Hence,
following equation becomes of value:

$$\sqrt[n/m]{2} \doteq \frac{3n+m-\cfrac{n^2-m^2}{9n}}{3n-m-\cfrac{n^2-m^2}{9n}}, \text{....................} (9),$$

$$\text{or } \sqrt[n/m]{2} =|= \frac{3n+m-\dfrac{n^2-m^2}{9n-\dfrac{n^2-m^2}{3n}}}{3n-m-\dfrac{n^2-m^2}{9n-\dfrac{n^2-m^2}{3n}}}, \quad \ldots\ldots\ldots (10),$$

$$\text{or } \sqrt[n/m]{2} =|= \frac{3n+m-\dfrac{29(n^2-m^2)}{507n}}{3n-m-\dfrac{29(n^2-m^2)}{507n}}, \quad \ldots\ldots\ldots (11).$$

In a latent form, the equidistant principle is also present in the following:

$$\sqrt[n]{\frac{q}{p}} =|= \frac{(2n^2+3n+1)q^2+(8n^2-2)qp+(2n^2-3n+1)p^2}{(2n^2-3n+1)q^2+(8n^2-2)qp+(2n^2+3n+1)p^2} \ldots\ldots (13).$$

In some of the following applications p is taken $=1$.
This [viz. (13)] becomes:

for $\sqrt[2]{q}$ .... $\dfrac{5q^2+10q+1}{q^2+10q+5}$ .... (13—2)

for $\sqrt[3]{q}$ .... $\dfrac{14q^2+35q+5}{5q^2+35q+14}$ .... (13—3)

for $\sqrt[4]{\dfrac{q}{p}}$ .... $\dfrac{15q^2+42qp+7p^2}{7q^2+42qp+15p^2}$ .... (13—4)

for $\sqrt[5]{q}$ .... $\dfrac{11q^2+33q+6}{6q^2+33q+11}$ .... (13—5)

for $\sqrt[6]{q}$ .... $\dfrac{91q^2+286q+55}{55q^2+286q+91}$ .... (13—6)

for $\sqrt[7]{\dfrac{q}{p}}$ .... $\dfrac{20q^2+65qp+13}{13q^2+65qp+20}$ .... (13—7)

for $\sqrt[8]{q}$ .... $\dfrac{51q^2+170q+35}{35q^2+170q+51}$ .... (13—8)

for $\sqrt[9]{q}$ .... $\dfrac{95q^2+323q+68}{68q^2+323q+95}$ .... (13—9)

$$\sqrt[10]{\frac{q}{p}} \cdots\cdots\cdots \frac{77q^2+266qp+57p^2}{57q^2+266qp+77p^2} \cdots\cdots\cdots (13\text{—}10).$$

$$\sqrt[100]{q} \cdots\cdots\cdots \frac{6767q^2+26666q+6567}{6567q^2+26666q+6767} \cdots\cdots\cdots (13\text{—}100).$$

For very high indices use, without sensible error,

$$\sqrt[n]{\frac{q}{p}} = \frac{(2n+3)q^2+8npq+(2n-3)p^2}{(2n-3)q^2+8npq+(2n+3)p^2} \cdots\cdots\cdots (14).$$

(14) is the equivalent of (2).

Upon the logarithmic function depend the following formulæ :

$$\sqrt[n]{q} = \frac{n+\frac{E}{2}+\left(\frac{E}{6}\times\frac{r-1}{r+1}\right)}{n-\frac{E}{2}+\left(\frac{E}{6}\times\frac{r-1}{r+1}\right)} \cdots\cdots\cdots (15).$$

$$\sqrt[n]{q} = \frac{3+\frac{2E}{n}+\frac{E^2}{2n^2}}{3-\frac{E}{n}} \cdots\cdots\cdots (16).$$

the logarithm of $q = \dfrac{3(q^2-1)}{(q+1)^2+2q} \cdots\cdots\cdots (17),$

, if q be fractional, $= \dfrac{3(q^2-p^2)}{(q+p)^2+2qp} \cdots\cdots\cdots (18),$

, in terms of d and $s = \dfrac{6ds}{3s^2-d^2} \cdots\cdots\cdots (19).$

This value of E, if q be between .9 and 1.1, is true to the seventh decimal, t may be corrected with very great accuracy, even up to the ninth or tenth dec- al, by adding to the result the $\frac{4d^2}{45s^2}$th part of itself.

If, however q be so great as

1.7 or so small as .6 use 44 in place of 45
1.8 or so small as .56 use 43 in place of 45
1.9 or so small as .53 use 42 in place of 45
2. or so small as .5 use 41 in place of 45.

The accuracy of these formulæ will appear from the natural logarithm in the next [paragraph].

[*Some verifications of formula* (18) :

For $q/p=\frac{11}{10}$:—

By the formula : $\log(q/p)=\frac{63}{661}=.0953101$,

error=1 in last place.

By the tables: 2.3978951

2.3025851

$\log(q/p)=$ .0953100

For $q/p=\frac{16}{15}$:—

By the formula : $\log(q/p)=\frac{93}{1441}=.0645385$,

correct to last place.

By the tables: 2.7725887

2.7080502

.0645385

For $q/p=\frac{101}{100}$:—

By the formula : $\log(q/p)=\frac{603}{60601}=.00995033$.]

Reconverting this by (15) we have

$$\frac{1+.004975165+.000008251}{1-.004975165+.000008251}=\frac{1.004983416}{.995033086}.$$

To this denominator add the 100th part,

.995033086

.009950331

1.004983417

Hence the real number is $\frac{101}{100}$.

From the foregoing we have another value of $\sqrt[n]{q}$ as follows :

$$\sqrt[n]{q}=1=\frac{2n+\left(3+\frac{r-1}{r+1}\right)\left(\frac{q^2-p^2}{q^2+4qp+p^2}\right)}{2n-\left(3-\frac{r-1}{r+1}\right)\left(\frac{q^2-p^2}{q^2+4qp+p^2}\right)} \quad \text{..............(20)}$$

(Of course r can only be taken crudely, but may by successive steps, until the required approximation is reached.)

Another formula akin to (15), for values of q (or r) in terms of E is

$$\sqrt[n]{q}=1=\frac{n+\frac{E}{2}+\frac{E^2}{12}}{n-\frac{E}{2}+\frac{E^2}{12}} \quad \text{..................(15)}$$

Let us call c, which is equal to $\frac{q-1}{E}$, a *root centre*, meaning thereby that, for all values of q, and degrees of n, $\dfrac{c+\frac{q-1}{q+1}}{c-\frac{q-1}{q+1}}$ nearly equals $\sqrt[n]{q}$. Then

$$\frac{nc+\frac{q-1}{2}+\frac{(q-1)(r-1)}{6(r+1)}}{nc-\frac{q-1}{2}+\frac{(q-1)(r-1)}{6(r+1)}}=\sqrt[n]{q} \quad \ldots\ldots\ldots\ldots\ldots (21).$$

c, if q is under 10, is nearly $\dfrac{2\sqrt{q}+\frac{q+1}{2}}{3}$ .... (22).

c, if q is between 50 and 250, is nearly same$-\dfrac{q^{\frac{1}{3}}}{500}$.......... ....(23).

c, if q is between 10 and 50 [correction not given].

FACTOR PROCESS.

Take $\frac{b}{a}\times\frac{c}{b}\times\ldots\ldots\ldots\times\frac{h}{g}=q$; then $\frac{h}{a}=q$ and the series is consecutive.

Take $\frac{b}{a}\times\frac{d}{c}\times\ldots\ldots\ldots\times\frac{h}{g}=q$; then $\frac{bd\ldots.h}{ae\ldots\ g}=q$.

Now if, to take the nth root of q, we assume n terms, consecutive or non-consecutive, and nearly equal in value

$$\sqrt[n]{q}=\frac{\frac{\sqrt{b}}{\sqrt{a}}+\frac{\sqrt{d}}{\sqrt{c}}+\ldots\ldots\ldots+\frac{\sqrt{h}}{\sqrt{g}}}{\frac{\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{c}}{\sqrt{d}}+\ldots\ldots\ldots+\frac{\sqrt{g}}{\sqrt{h}}}, \ldots\ldots\ \ldots\ldots (24),$$

wherein the numerators are the square roots of the terms and the denominators the reciprocals thereof.

If the terms are consecutive and odd in number

$$\sqrt[n]{q}=\frac{ah+2bg+\ldots\ldots\ldots+2de}{2ag+2bf+\ldots\ldots\ldots d^2}; \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

but if consecutive and even in number

$$\sqrt[n]{q}=\frac{ai+2bh+\ldots\ldots\ldots+e^2}{2ah+2bg+\ldots\ldots\ldots+2de}\ \ldots\ldots\ldots\ldots\ldots\ldots \quad (25$$

Also if the series is consecutive,

$$\sqrt[n]{q}=\frac{(g+h)b+(f+g)c+\ldots\ldots\ldots+(b+c)g+(a+b)h}{(g+h)a+(f+g)b+\ldots\ldots\ldots+(b+c)f+(a+b)h}.$$

(25) and (25½) are called the *diagonal* process.

To separate a quantity q/p into consecutive factors which shall be ne oqual, and which shall be as many as there are units in the index of the roo be extracted, and whose differences shall also be in arithmetical progression, for expansor $\frac{n^2 s}{d^2}$, and then arrange terms with differences themselves diffe by unity. This may be done by dividing d', the difference of the expanded frac q'/p', or $q'-p'$, by n, and making the final interval $\frac{n-1}{2}$ less than the quoti —after which ascend accordingly.

Illustrations: 1. Separate $\frac{2}{1}$ into three factors of above nature:—

$$\frac{n^2 s}{d^2}=13.5.\quad \text{Then } \frac{q'}{p'}=\frac{27}{13.5},\quad \frac{13.5}{3}=4.5,\quad \frac{n-1}{2}=1.$$

Then first interval=3.5. Therefore:

$$\frac{17}{13.5},\quad \frac{21.5}{17},\quad \frac{27}{21.5},\ \text{or}\ \frac{34}{27},\quad \frac{43}{34},\quad \frac{54}{43}$$

are the desired terms.

Applying the diagonal process:

$$\frac{1458+1462+1462}{1161+1161+1156}=\frac{4382}{3478}=\frac{2191}{1739}=1.2599195.\quad \text{Error}=.0000015.$$

2. Separate $\frac{5}{3}$ into 4 factors:—
Expansor=16 gives $\frac{5}{3}=\frac{80}{48}$; first interval=6.5.
[The series of consecutive factors is]

$$\frac{54.5}{48},\quad \frac{62}{54.5},\quad \frac{70.5}{62},\quad \frac{80}{70.5},\ \text{or}\ \frac{109}{96},\quad \frac{124}{109},\quad \frac{141}{124},\quad \frac{160}{141}.$$

3. Separate $\frac{3}{1}$ into five factors:—
Expansor=12.5; first interval=3, [and the series of consecutive facto

$\frac{15.5}{12.5}$, $\frac{19.5}{15.5}$, $\frac{24.5}{19.5}$, $\frac{30.5}{24.5}$, $\frac{37.5}{30.5}$, or $\frac{31}{25}$, $\frac{39}{31}$, $\frac{49}{39}$, $\frac{61}{49}$, $\frac{75}{61}$.

If the factors are consecutive and in arithmetical progression, then

$$\sqrt[n]{q}=\frac{n^2q+\frac{n^2-1}{6}(q-1)^2}{n^2q-n(q-1)+\frac{(n-1)(n-2)}{6}(q-1)^2} \quad\ldots\ldots\ldots\ldots (27).$$

All quantities p/q may be represented in n terms, which group in three lasses as follows:

$$\left(\frac{b}{a}\right)^j\times\left(\frac{d}{c}\right)^k\times\left(\frac{b}{e}\right)^l=q/p, \text{ where } j+k+l=n.$$

Then $$\sqrt[n]{q}=\frac{j(fc+de)b+k(fa+be)d+l(ad+bc)f}{j(fc+de)a+k(fa+be)c+l(ad+bc)e}\quad\ldots\ldots\ldots\ldots (28).$$

The above is called the *three-class* process.

Sometimes a quantity will reasonably resolve into n terms, which group ı only two classes. Hence the following *two-class* process:

$$\left(\frac{b}{a}\right)^j\times\left(\frac{d}{c}\right)^k=q;$$

$$\sqrt[n]{q}=\frac{j(ad+bc+2cd)b+k(ad+bc+2ab)d}{j(ad+bc+2cd)a+k(ad+bc+2ab)c}\quad\ldots\ldots\ldots\ldots (29).$$

Or, the following, simpler but not so good:

$$\sqrt[n]{q}=\frac{(c+d)jb+(a+b)kd}{(c+d)ja+(a+b)kc}\quad\ldots\ldots\ldots\ldots (30).$$

Of all these processes the three-class (28) is the most trust-worthy. When ȷ does not naturally resolve in such terms, take q/pv which does, and extract $\sqrt[n]{v}$, and multiply results.

In the consecutive series

$$\frac{b}{a}\times\frac{c}{b}\times\ldots\ldots\ldots\times\frac{e}{d}\times\frac{b}{e}=q,$$

ke a new term, of the first expanded q times, that is qb/qa. Then drop the rm b/a, and call $qb=g$ and $qa=f$, giving the equation

$$\frac{c}{b} \times \frac{d}{c} \times \ldots\ldots\ldots \times \frac{f}{e} \times \frac{g}{f} = q.$$

Now apply the diagonal process, according to the spirit, and not the ter, of (25), and we have,

$$\frac{bg+2cf+2de}{2bf+2ce+d^2} =|= {}^{n}\sqrt{q} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2$$

Now this result will be found no nearer than the result in (25) the mean of the two will give a close result. It is not, however, a formul value, and I think there are cases where the error of (25) and (25½) are both the *same* side. Hence, their mean would be of no value in particular.

PROCESS FOR SPECIAL ROOTS.

For Square Root: $\left[\text{Hutton gives } \frac{ac+b^2}{2ab} = \sqrt{\frac{b}{a} \times \frac{c}{b}}\right].$

$$^{2}\sqrt{q} =|= \frac{(c+d)b+(a+b)d}{(c+d)a+(a+b)c} \ldots\ldots\ldots\ldots\ldots\ldots ($$

$$^{2}\sqrt{q} =|= \frac{3bcd+3abd+b^2c+d^2a}{3abc+3acd+a^2d+b^2c} \ldots\ldots\ldots\ldots\ldots ($$

$$^{2}\sqrt{q} =|= \frac{(\sqrt{cd} \times b)+(\sqrt{ab}.d)}{(\sqrt{cd} \times a)+(\sqrt{ab} \times c)},$$

$$\text{or } =|= \frac{b\sqrt{cd}+d\sqrt{ab}}{a\sqrt{cd}+c\sqrt{ab}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

If we call r an approximate value of the required square root, then

$$\frac{q+r}{p+r} =|= {}^{2}\sqrt{\frac{q}{p}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\text{or } \frac{q^2+6r^2q+r^4}{4r(q+r^2)} =|= {}^{2}\sqrt{q} \ldots\ldots\ldots\ldots\ldots\ldots$$

For Cube Roots:—

$$\frac{7q^3+42q^2+30q+2}{2q^3+30q^2+42q+7} =|= {}^{3}\sqrt{q} \ldots\ldots\ldots\ldots\ldots\ldots$$

Also, if $\frac{b}{a} \times \frac{d}{c} \times \frac{f}{e} = q,$

$$\sqrt[3]{q} =|= \frac{bdc+bcf+adf}{acf+ade+bce} \qquad \text{(37).}$$

This is the best of all cube root processes, and I call it the *interweaving* ss. It has remarkable properties.

For the ⅔ Root :—If $\frac{b}{a}\times\frac{c}{b}\times\frac{d}{c}=q$; then

$$\frac{2bd+c^2}{2ac+b^2} =|= \sqrt[\frac{3}{2}]{q}, \text{ or } \sqrt[3]{q^2} \qquad \text{(38).}$$

For the Fourth Root:—Let

$$\frac{b}{a}\times\frac{c}{b}\times\frac{d}{c}\times\frac{e}{d}=q.$$

Then $\sqrt[4]{q} =|= \dfrac{(be+cd)b+(ae+bd)e+(ad+bc)d+(ac+b^2)e}{(be+cd)a+(ae+bd)b+(ad+bc)c+(ac+b^2)d}$ .... (39).

UNDER-SQUARE FORMULÆ.

$U=(\sqrt{q}-\sqrt{p})^2,\ U_4=(\sqrt[4]{q}-\sqrt[4]{p})^4,\ U_8=(\sqrt[8]{q}-\sqrt[8]{p})^8$........

(7) is an under-square formula and, if q is an even square, is precisely valent to the ½nth root of $\sqrt{q}$, by the first correction of formula 1.

The first correction of (7) is

$$\sqrt[n]{\frac{q}{p}} =|= \frac{ns+d-\dfrac{2(n^2-1)}{3n}U-\dfrac{n^2-4}{6n}U_4}{ns-d-\dfrac{2(n^2-1)}{n}U-\dfrac{n^2-4}{6n}U_4} \qquad \text{(40).}$$

Also, $$\sqrt[n]{\frac{q}{p}} =|= \frac{ns+d-\dfrac{2(n^2-1)}{3n}U-\dfrac{(E-1)U}{72}\times\dfrac{n(n-2)}{n-1}}{ns-d-\dfrac{2(n^2-1)}{3n}U-\dfrac{(E-1)U}{72}\times\dfrac{n(n-2)}{n-1}} \qquad \text{(41).}$$

Still another but complex value of $\sqrt[n]{\frac{q}{p}}$ is :

$$\frac{d-\dfrac{(2n^2-n)}{3n}U-\dfrac{(n^2-4)\,U_4}{6n}-\dfrac{(n^2-16)(\sqrt[8]{q}-\sqrt[8]{p})^4\times(\sqrt{q}+1)\sqrt{p}}{{}^*12n}}{d-\dfrac{(2n^2-n)}{3n}U-\dfrac{(n^2-4)\,U_4}{6n}-\dfrac{(n^2-16)(\sqrt[8]{q}-\sqrt[8]{p})^4\times(\sqrt{q}+1)\sqrt{p}}{{}^*12n}} \qquad \text{(42).}$$

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*It seems that 11n is better in actual practice.

For cube root use for coefficients $\frac{16}{9}$ and $\frac{5}{18}$ instead of $\frac{16}{9}$ and $\frac{5}{8}$.

Illustrations: All the equidistant formulæ, except these, approach accuracy only when q is near unity. No such restriction binds these [the undersquare] formulæ, especially those which are most developed. And here, let me say, is undoubtedly the beginning of a series which I think the Calculi would unfold, and I trust some friend of science will take the burden of solving it. So developed, I am satisfied that all roots of all numbers would be extractable to any required degree of accuracy.

1. Taking (40) extract $\sqrt[3]{4096}$.

$$\text{Root}=\frac{3\times 4097+4095-\frac{16}{9}\times 3969-\frac{5}{18}\times 2401}{3\times 4097-4095-\frac{16}{9}\times 3969-\frac{5}{18}\times 2401}$$

$$=\frac{12291+4095-7056-667 \text{ nearly}}{12291-4095-7056-667 \text{ nearly}}=\frac{8663}{473}=18+;$$

but taking $\frac{5}{19}$ instead of $\frac{5}{18}=\frac{8698}{508}=17+$.

2. Taking $\sqrt[4]{4096}$, we have

$$\text{Root}=\frac{4\times 4097+4095-\frac{8}{9}\times 3969-\frac{1}{2}\times 2401}{4\times 4097-4095-\frac{8}{9}\times 3969-\frac{1}{2}\times 2401}=\frac{9360}{1170}=8.$$

3. Take $\sqrt[6]{4096}$. By (42),

$$\text{Root}=\frac{6\times 4097+4095-\frac{35}{9}\times 3969-\frac{8}{9}\times 2401-\frac{5}{18}\times\frac{100}{9}\times 65}{6\times 4097-4095-\frac{35}{9}+3969-\frac{8}{9}\times 2401-\frac{5}{18}\times\frac{100}{9}\times 65}=\frac{11907.2}{3717}=3.2,$$

but should=4.

To Sum a Series: $S=1, r, r^2, \ldots\ldots. r^{n-1}$, n terms.

Let $s=\text{ratio}+1=r+1$, $d=r-1$.

$$S+=\frac{2n}{s-nd+\dfrac{(n^2-1)d^2}{3s}} \ldots\ldots\ldots\ldots (43).$$

If s exceeds 2, this is not accurate enough to be of value.

Cube Root by Difference Method.

Take $a^3<q$, $b^3>q$. Call $q-a^3=A$, $b^3-q=B$. Then

$$\frac{a^2B+b^2A}{aB+bA}+=\sqrt[3]{q} \ldots\ldots\ldots\ldots (44).$$

Convenient Formula for Roots of 2.

$$\sqrt[n]{2} \mathrel{=\!|\!=} \frac{101n+35+\dfrac{35(r-1)}{6(r+1)}}{101n-35+\dfrac{35(r-1)}{6(r+1)}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (45).$$

A Rough Value of q in Terms of E.

$$q \mathrel{=\!|\!=} \frac{\sqrt{9+3E^2}+2E}{3-E} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (46).$$

Logarithms.

A curious fact, but scarcely a useful one, is: The logarithm of any num- approaches

$$K_B \mathrel{=\!|\!=} \frac{\sqrt{B}\times(q-B^m)}{\sqrt{B}\times(\sqrt{B}-1)B^m+(\sqrt{B}-1)q}+m, \quad\ldots\ldots\ldots\ldots\quad (47),$$

rein K is the logarithm, B the base of the system, m the characteristic of the rithm of q, the quantity. Now if $B=4$, the formula becomes

$$K_4 \mathrel{=\!|\!=} \frac{2(q-B^m)}{2B^m+q}+m, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (48).$$

If $q=q/p$, then $K_4 \mathrel{=\!|\!=} \dfrac{2q-2B^m p}{2B^m p+q}+m \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (49).$

If $K_4=\dfrac{m+c}{d}$, then $q \mathrel{=\!|\!=} B^m\left(1+\dfrac{3c}{2d-c}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (50)$

$$\mathrel{=\!|\!=} B^m\frac{2d+2c}{2d-c} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (51).$$

Since Napierian $\log E \mathrel{=\!|\!=} \frac{44}{61}K$, and also, since

$$\sqrt[n]{q} \mathrel{=\!|\!=} \frac{\dfrac{44}{61}K+\dfrac{1}{2n}}{\dfrac{44}{61}K-\dfrac{1}{2n}}$$

roughly, then

$$\sqrt[n]{q} =|= \frac{\frac{44}{61}K+\frac{1}{2n}}{\frac{44}{61}K-\frac{1}{2n}} \text{ roughly} \quad\ldots\ldots\ldots\ldots\ldots\ldots (52).$$

The term K also may be understood to mean the logarithm of q to base 4.

K of $10=\frac{5}{3}$; then $q=4(1+\frac{6}{4})=10$.

The error by this method is easily represented by a curve.

In the fractional processes continually occur the coaddition of fractions, or the adding of numerators together, and of denominators also, when the latter are not the same. When the fractions, two or more, are "embracing," the results are close, and if "harmoniously embracing" then positively accurate geometric averages. They are harmonious when the product of the two terms of each fraction are the same.

Thus, $\frac{8}{5}$ and $\frac{7}{6}$ are embracing, and their co-sum doubled is $2\times\frac{15}{11}=2.727+$. The real sum is $\frac{83}{30}=2.767+$. $\frac{12}{7}$, $\frac{11}{8}$ and $\frac{10}{9}$ are embracing, and their co-sum tripled is $3\times\frac{33}{24}=4.125$. The real sum is $\frac{2117}{504}=4.200$. $\frac{3}{4}$ and $\frac{6}{2}$ are harmoniously embracing. Their co-sum is $\frac{9}{6}$ or $\frac{3}{2}$ which is their square root and precise geometric average.

Rough addition may be performed by co-addition. Thus, $\frac{11}{16}+\frac{10}{17}$ =|= $\frac{21}{33}$, or $\frac{7}{11}$, which doubled differs from $\frac{347}{272}$ as 3817 differs from 3808, or 1 in 423 parts.

In the foregoing processes for extraction of roots, calculation of logarithms, etc., excepting the factor-process, the accuracy increases rapidly as q approaches unity. In general they have value only when q lies between .5 and 2. Their value should be tested by logarithms. When q exceeds 2, it may generally be divided by the nth power of some simple quantity, which will bring it near unity. If this be difficult, use the nearest power of 2 (say m) as previously explained.

Many apparently absurd problems are readily answered by these formulæ. Thus: *If the cube root of* $2.5=\frac{19}{14}$ *what is the 4th root of* $3.333+$?

Using the 2-class process, formula (29),—

$$\left(\frac{19}{14}\right)^3\times\left(\frac{4}{3}\right) = \text{(say to } 3\tfrac{1}{3}\text{)}\ \frac{(125\times3\times19)+(645\times4)}{(375\times14)+(645\times3)}$$

$$=\frac{9705}{7185}=\frac{1941}{1437}=1.35545, \text{ [error here]}$$

or by (30),

$$\frac{399+132}{294+99}=\frac{531}{393}=\frac{177}{131}=1.35114.$$

True answer=1.35120.

Comparison of Fraction.

It is frequently necessary, or desirable to compare the values of two unwieldly, yet not very unequal fractions, or to ascertain approximately the comparative value of two ordinary fractions quickly, even if only approximately. Hence the following:

Compare b/a with d/c. Suppose them nearly equal to z/y, and divide each fraction by it, giving by/az and dy/cz. Subtract unity from each, and we have $\frac{by \pm az}{az}$ which make equal to $\frac{1}{uaz}$, and $\frac{dy-cz}{cz}$ which make equal to $\frac{1}{vcz}$. Then

$$\frac{uaz \mp vcz}{(uaz)\times(vcz)}=\frac{(ua \mp vc)z}{uavcz}$$

$=$the relative excess of b/a over d/c. If u and v are each unity, the process is very simple.

Illustrations: 1. Compare $\frac{5}{8}$ and $\frac{7}{11}$ with $\frac{2}{3}$ as measure.

$$(\tfrac{5}{8}\div\tfrac{2}{3})-1=-\tfrac{1}{16} \text{ and } (\tfrac{7}{11}\div\tfrac{2}{3})-1=-\tfrac{1}{22}.$$

$$\frac{22-16}{22\times16}=\frac{1}{58\frac{2}{3}}, \text{ or } =\frac{11-8}{11\times8\times2}=\frac{1}{58\frac{2}{3}}.$$

True answer$=\frac{1}{56}$.

2. Compare $\frac{29}{39}$ with $\frac{49}{65}$. Take base$=\frac{3}{4}$.

$$\frac{by}{az}=\frac{116}{117}, \quad \frac{dy}{cz}=\frac{196}{195}, \quad \frac{195+117}{195\times117}=\frac{312}{22815}=\frac{1}{73.1+}.$$

By the formula $\frac{49+29}{49\times29\times4}=\frac{76}{5684}=\frac{1}{74.8}$.

True answer$=\frac{1911-1885}{1911}=\frac{1}{73.5}$.

New York, 1878—1883.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

71. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

A man owes me \$200 due in 2 years, and I owe him \$100 due in 4 years; when can he pay me \$100 to settle the account equitably, money being worth 6%?.

I. Solution by P. S. BERG, Principal of Schools, Larimore, North Dakota; and the PROPOSER.

Let x=the time.

Now, the present worth of \$200 for $(2-x)$ years—the present worth of \$100 for $(4-x)$ years must=\$100.

$\dfrac{10000}{56-3x}$=present worth of \$200 for $(2-x)$ years at 6%.

$\dfrac{5000}{62-3x}$=present worth of \$100 for $(4-x)$ years at 6%.

$$\therefore \frac{10000}{56-3x}-\frac{5000}{62-3x}=100.$$

$\therefore x$=.358615 years=4 months and 9 days.

II. Solution by FREDERIC R. HONEY, New Haven, Connecticut.

The present value of \$1.12 due 2 years hence is \$1.00. Therefore the present value of \$200.00 due in 2 years is \$200÷1.12=\$178.571. The present value of \$1.24 due 4 years hence is \$1.00. Therefore the present value of \$100.00 due 4 years hence is \$100÷1.24=\$80.645. If we deduct \$80.645 from \$178.571 we have \$97.926, the amount due to me at the present time. This sum placed at interest at 6% would yield \$97.926×.06=\$5.876 in 1 year. The difference between \$100.00 and \$97.926 is \$2.074, the interest which must accumulate in order that the sum may become equal to \$1.00. Therefore since the interest \$5.876 accumulates in 1 year, the interest \$2.074 will accumulate in 2.074÷5.876=0.3529 years. Answer.

72. Proposed by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana.

Though the length of my field is 1-7 longer than my neighbor's, and its quality is 1-9 better, yet as its breadth is 1-4 less, his is worth \$500 more than mine. What is mine worth? *Encyclopedia Britannica.*

Solution by Misses EVA JONES and NEVA CAROTHERS, Senior Pupils of West Point Graded School.

1. $l : l' :: 7 : 8.$ 1st condition.

2. $q : q' :: 9 : 10.$ 2nd condition.

3. $b : b' :: 4 : 3$. 3rd condition.

4. $v : v' :: 21 : 21$, multiplying and reducing, and remembering that
e value $\propto l.b.q$.

5. Also $v-v'=\$500$. Whence,

6. $v=\$10500$. From (4) and (5),

7. $v'=\$10000$.

This problem was also solved by *B. F. SINE, NELSON S. BORAY, P. S. BERG, F. M. McGAW, . C. CORBIN, COOPER D. SCHMITT, FREDERIC R. HONEY, H. C. WILKES*, and *G. B. M. ZERR*.

M. A. Gruber sent in a solution of Problem 70, Department of Arithmetic, too late for credit in last ssue. His answer is 6.48 years.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

68. Proposed by ROBERT JUDSON ALEY, M. A., Professor of Mathematics in Indiana University, Bloomgton, Indiana.

Sum to n terms the series, $n\cos\theta+(n-1)\cos2\theta+(n-2)\cos3\theta$, etc.

[*Chrystal's Algebra.*]

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington, C.

Let $S=n\cos\theta+(n-1)\cos2\theta+(n-2)\cos3\theta........$ Also let $S_s=\sin\theta+$ n$2\theta+\sin3\theta........$, and $S_c=\cos\theta+\cos2\theta+\cos3\theta........$

$$S=n[\cos\theta+\cos2\theta+\cos3\theta+........]-[\cos2\theta+2\cos3\theta+........],$$

$$=(n+1)[\cos\theta+\cos2\theta+\cos3\theta+........]-[\cos\theta+2\cos2\theta+3\cos3\theta........],$$

$$=(n+1)S_c-dS_s/d\theta.$$

ow $S_c=[\cos\{\frac{1}{2}(n+1).\theta\}\sin\frac{1}{2}(n\theta)]/\sin\frac{1}{2}\theta$, and $S_s=[\sin\{\frac{1}{2}(n+1).\theta\}\sin\frac{1}{2}(n\theta)]/\sin\frac{1}{2}\theta$.

$$\therefore S=(n+1)\frac{\cos\{\frac{1}{2}(n+1)\theta\}\sin\frac{1}{2}(n\theta)}{\sin\frac{1}{2}\theta}-\frac{d}{d\theta}\left[\frac{\sin\frac{1}{2}(n+1).\theta\sin\frac{1}{2}(n\theta)}{\sin\frac{1}{2}\theta}\right],$$

obably as compact a form as can be obtained.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let S=sum required,

$$2\sin\tfrac{1}{2}\theta\cos n\theta=\sin\{\theta+\frac{2n-1}{2}\theta\}-\sin\{\theta+\frac{2n-3}{2}\theta\}$$

$$4\sin\tfrac{1}{2}\theta\cos(n-1)\theta=2\sin\{\theta+\frac{2n-3}{2}\theta\}-2\sin\{\theta+\frac{2n-5}{2}\theta\}$$

$$6\sin\tfrac{1}{2}\theta\cos(n-2)\theta=3\sin\{\theta+\frac{2n-5}{2}\theta\}-3\sin\{\theta+\frac{2n-7}{2}\theta\}$$

$$8\sin\tfrac{1}{2}\theta\cos(n-3)\theta=4\sin\{\theta+\frac{2n-7}{2}\theta\}-4\sin\{\theta+\frac{2n-9}{2}\theta\}$$

....

$$2n\sin\tfrac{1}{2}\theta\cos\theta=n\sin(\theta+\tfrac{1}{2}\theta)-n\sin(\theta-\tfrac{1}{2}\theta).$$

Adding we get

$$2S\sin\tfrac{1}{2}\theta=(\sin\tfrac{3}{2}\theta+\sin\tfrac{5}{2}\theta+\sin\tfrac{7}{2}\theta+........+\sin\frac{2n+1}{2}\theta)-n\sin\tfrac{1}{2}\theta,$$

$$=[\sin(\frac{n+2}{2})\sin\tfrac{1}{2}(n\theta)]/\sin\tfrac{1}{2}\theta-n\sin\tfrac{1}{2}\theta.$$

$$\therefore\ S=[\sin(\frac{n+2}{2}\theta)\sin\tfrac{1}{2}(n\theta)-n\sin^2\tfrac{1}{2}\theta]/2\sin^2\tfrac{1}{2}\theta.$$

The series in parenthesis above is summed in all trigonometries in the series, $\sin\alpha+\sin(\alpha+\beta)+$, etc., by making $\alpha=\frac{3}{2}\theta$, $\beta=\theta$.

III. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

The given series may be broken up into:

$$n[\cos\theta+\cos2\theta+\cos3\theta+........\cos n\theta]$$
$$-[\cos2\theta+2\cos3\theta+3\cos4\theta+........(n-1)\cos n\theta].$$

To sum the series $\cos\theta+\cos2\theta+\cos3\theta+........\cos n\theta$, we have

$\sin\frac{1}{2}\theta-\sin\frac{3}{2}\theta=-2\cos\theta\sin\frac{1}{2}\theta.$

$\sin\frac{3}{2}\theta-\sin\frac{5}{2}\theta=-2\cos2\theta\sin\frac{1}{2}\theta.$

$\sin\frac{5}{2}\theta-\sin\frac{7}{2}\theta=-2\cos3\theta\sin\frac{1}{2}\theta.$

....

$\sin\frac{1}{2}(2n-1)\theta-\sin\frac{1}{2}(2n+1)\theta=-2\cos n\theta\sin\frac{1}{2}\theta.$

Adding, we have, $\sin\frac{1}{2}\theta-\sin\frac{1}{2}(2n+1)\theta=-2\sin\frac{1}{2}\theta\Sigma(n\theta)$.

$\therefore\ \Sigma(n\theta)=[\sin\frac{1}{2}(2n+1)\theta-\sin\frac{1}{2}\theta]/2\sin\frac{1}{2}\theta=[\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta]/\sin\frac{1}{2}\theta.$

$\therefore\ n\Sigma(n\theta)=[n\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta]/\sin\frac{1}{2}\theta.$

To sum the second part, we have,

$x^2+2x^3+3x^4+........(n-1)x^n=[x^2-nx^{n+1}+(n-1)x^{n+2}]/(1-x)^2.$

Putting $x=\cos\theta+i\sin\theta$, and employing the formula $(\cos\theta+i\sin\theta)^m=$ $\cos m\theta+i\sin m\theta$, we obtain after putting the real parts of both members equal, and making all necessary reductions, for the sum of the second series

$$=\frac{\cos\theta-n\cos n\theta+(n-1)\cos(n+1)\theta}{4\sin^2\frac{1}{2}\theta};$$

so that the sum of the given series

$$=\frac{n\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}+\frac{\cos\theta-n\cos n\theta+(n-1)\cos(n+1)\theta}{4\sin^2\frac{1}{2}\theta}.$$

To test this formula we must of course, leave the coefficient n of the first expression unchanged, while in all the other factors and terms which involve n, n must be put successively $=1, 2, 3, 4$, etc.

Also solved by *E. W. MORRELL.*

69. Proposed by C. E. WHITE, A. M., Trafalgar, Indiana.

Prove that $x^n\pm x^{n-1}+x^{n-2}\pm\ldots\ldots\ldots+(\pm1)^{n-1}x+(\pm1)^n=(x\mp1)^n\pm$ $A(x\mp1)^{n-1}+B(x\mp1)^{n-2}\pm\ldots\ldots\ldots+(\pm1)^nx$, where A, B, C, ... are the binomial coefficients of the $(n+1)$th order.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington, D. C.

$$x^n\pm x^{n-1}+x^{n-2}+\ldots\ldots\ldots,\ \text{etc.}=(x^{n+1}-1)/(x-1)\ \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

or $(x^{n+1}+1)/(x+1)\ldots\ldots\ldots\ldots\ldots\ \ldots\ldots(2)$, $=\{[(x-1)+1]^{n+1}-1\}/(x-1)$, or

$\{[(x+1)-1]^{n+1}+1\}/(x+1)$, $=(x-1)^n+C^1_{n+1}(x-1)^{n-1}+C^2_{n+1}(x-1)^{n-2}+\ldots\ldots,$

$$\text{or } (x+1)^n-C^1_{n+1}(x+1)^{n-1}+C^2_{n+1}(x-1)^{n-2}-\ldots\ldots,$$

$$=(x\mp1)^n\pm A(x\mp1)^{n-1}+B(x\mp1)^{n-2}+\ldots\ldots$$

II. Solution by E. W. MORRELL, Professor of Mathematics in Montpelier Seminary, Montpelier, Vermont.

Let $K=x^n\pm x^{n-1}+x^{n-2}\pm\ldots\ldots\ldots+(\pm1)^{n-1}x+(\pm1)^n$.

Put $x=y\pm1$, expanding and observing that the sign of the last term of each expression is $\pm$ if n is odd but $+$ if n is even, we may write:

$x^n=(y\pm1)^n=y^n\pm ny^{n-1}+\frac{1}{2}[n(n-2)]y^{n-2}\pm\ \ldots\ldots\ldots\ldots\ldots\ +(\pm1)^{n-1}ny+(\pm1)^n$

$\pm x^{n-1}=\pm(y\pm1)^{n-1}=\pm y^{n-1}+(n-1)y^{n-2}\pm\ \ldots\ldots\ldots\ +(\pm1)^{n-1}(n-1)y+(\pm1)^n$

$x^{n-2}=(y\pm1)^{n-2}=\ldots\ldots\ldots\ldots\ldots\ldots\ldots y^{n-2}\pm\ldots\ldots\ldots\ +(\pm1)^{n-1}(n-2)y+(\pm1)^n$

etc .... etc.

$(\pm1)^{n-1}z=(\pm1)^{n-1}(y\pm1)=$ .... $(\pm1)^{n-1}y+(\pm1)^n$

$(\pm1)^n=$...... .... $(\pm1)^n$.

By adding, and simplifying the coefficients of y, we have

$$K=y^n\pm(n+1)y^{n-1}+\tfrac{1}{2}[(n+1)n]y^{n-2}\pm\ \ldots\ +(\pm1)^{n-1}\tfrac{1}{2}[(n+1)n]y+(\pm1)^n(n+1),$$

which has binomial coefficients of the $(n+1)$th order. Substituting A, B, C, ... for the coefficients and restoring the values of y,

$$K=(x\mp1)^n\pm A(x\mp1)^{n-1}+B(x\mp1)^{n-2}\pm\ldots\ldots\ldots+(\pm1)^{n-1}B(x\mp1)+(\pm1)^nA.$$

[Expanding and combining the terms of the second member, we get the first member for a result. Zerr.]

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

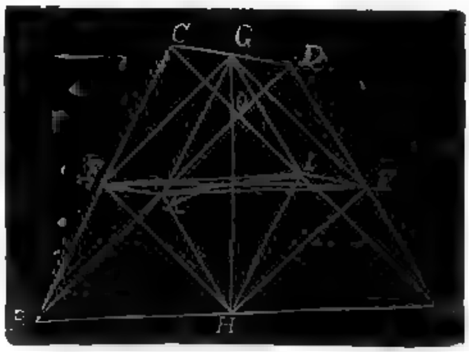
SOLUTIONS OF PROBLEMS.

43. Proposed by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

The consecutive sides of a quadrilateral are a, b, c, d. Supposing its diagonals to be equal, find them and also the area of the quadrilateral.

II. Solution by A. H. BELL, Hillsboro, Illinois.

The solution as published simply demonstrates this theorem, $a^2+b^2+c^2+d^2=2x^2+4\overline{IK}^2$, with two unknowns, and is then solved for a particular case.



Let the sides of the quadrilateral AB, BC, CD, and AD, be a, b, c, and d; and the diagonals each $=2x$; $x+y$, $x-y=$the segments AO and OC; and BO and $OD=x+z$ and $x-z$. In the triangles AOB, BOC, and COD we have $(x+y)\cos A+(x+z)\cos B=a$, $\cos A=(4x^2+a^2-b^2)/(4ax)$, $\cos B=(4x^2+a^2-d^2)/(4ax)$, making

$z(4x^2+a^2-d^2)=(2a^2+b^2+d^2)x-8x^3-(4x^2+a^2-b^2)y$ ....(1).

Similarly BOC, $z(4x^2+b^2-c^2)=(2b^2+a^2+c^2)x-8x^3-(a^2-b^2-4x^2)y$ .....(2),

and COD, $z(-4x^2+b^2-c^2)=(b^2+2c^2+d^2)x-8x^3+(c^2-d^2+4x^2)y$ ....(3).

Subtracting (2) from (1), $(a^2-b^2+c^2-d^2)z=(a^2-b^2-c^2+d^2)x-8x^2y$ .....(4).

Subtracting (3) from (2), $8x^2z=(a^2+b^2-c^2-d^2)x-(a^2-b^2+c^2-d^2)y$.....(5).

Equating the values of z in (4) and (5) and solving for y, after letting

$(a^2-b^2+c^2-d^2)=e$, $a^2-b^2-c^2+d^2=f$, $a^2+b^2-c^2-d^2=g$, we have

$$y=(eg-8fx^2)/(e^2-64x^4).x \text{ ....(6).}$$

Equating the values of z and solving for y in (1) and (4) after letting

$a^2-b^2=m$, $a^2-d^2=n$, $2a^2+b^2+d^2=p$, $2e+f=q$,

and noting that $2n-e=g$, we have

$$y=(4qx^2-ep+fn)/(32x^4+4gx^2-em).x \text{ ....(7).}$$

Equating the values of y in (6) and (7),

$$512x^6-64(a^2+b^2+c^2+d^2)x^4+16[a^2(b^2-2c^2+d^2)+b^2(c^2-2d^2)+c^2d^2]x^2$$

$$+4(ac-bd)(ac+bd)(a^2-b^2+c^2-d^2)=0\text{.......(8).}$$

Let $x^2=\frac{1}{8}y$, $4x^2=\frac{1}{2}y$, $2x=\sqrt{\frac{1}{2}y}$, or multiply (8) by the geometrical series, $\frac{1}{512}$, $\frac{1}{64}$, $\frac{1}{8}$, and 1, ratio=8, then (8) becomes

$$y^3-(a^2+b^2+c^2+d^2)y^2+2[a^2(b^2-2c^2+d^2)+b^2(c^2-2d^2)+c^2d^2]y$$

$$+4(ac-bd)(ac+bd)(a^2-b^2+c^2-d^2)=0\text{........(9).}$$

When this is solved by Cardan's formula, then since there are given the sides of the triangles ABC and ACD, we have in the general formula, with the sides a, b, and $2x$, and c, d, and $2x$, the area for the quadrilateral

$$ABCD=\tfrac{1}{4}\sqrt{[(a+b)^2-4x^2]\times[4x^2-(a-b)^2]}$$

$$+\tfrac{1}{4}\sqrt{[(c+d)^2-4x^2]\times[4x^2-(c-d)^2]}.$$

A better transformation for (8) is to put $4x^2=y$, $2x=\sqrt{y}$. We then get $y^3-\frac{1}{2}(a^2+b^2+c^2+d^2)y^2$ $+\frac{1}{2}[a^2(b^2-2c^2+d^2)+b^2(c^2-2d^2)+c^2d^2]y+\frac{1}{2}(ac+bd)(ac-bd)(a^2-b^2+c^2-d^2)=0$.

Example: a, b, c, d=6, 5, 3, 4, respectively, in (9) gives

$$y^3-86y^2+894y-1216=0.$$

By Horner's method, $y=75.7270176+$. Diagonal $2x=6.1533+$ agreeing with a close drawing.

[Mr. Bell sent us this solution March 14, 1895. We have looked it over carefully and believe that it is entirely correct. The solution published in the July-August number of Vol. II is of a particular case. EDITOR.]

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

57. Proposed by F. M. McGAW, A. M., Professor of Mathematics in Bordentown Military Institute, Bordentown, New Jersey.

Solve the following equation: $(1+x^2)\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+2y=0.$

I. Solution by WILLIAM E. HEAL, A. M., Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

Let $y=bx\left[\int zdx+c\right]$ and the equation becomes

$$x(1+x^2)\frac{dz}{dx}+2z=0, \text{ or } \frac{dz}{z}+\frac{2dx}{x(1+x^2)}=0.$$

$$\therefore z=c'(1+[1/x^2])\,;\; y=bx\{c'(x-[1/x])+c\},\; =Bx+A(1-x^2).$$

II. Solution by H. C. WHITAKER, M. E., Ph. D., Professor of Mathematics in Manual Training School, Philadelphia, Pennsylvania.

Proceeding to obtain a solution in series, both values of y are found to terminate immediately. The complete primitive is $y=Ax+B(x^2-1)$.

III. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.

It is shown (*Forsyth's Differential Equations*, Article 58) that

$$d^2y/dx^2+P(dy/dx)+Qy=R \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

gives, when $y=vw$ ....(2).

$$w\frac{d^2v}{dx^2}+(2\frac{dw}{dx}+Pw)\frac{dv}{dx}+(\frac{d^2w}{dx^2}+P\frac{dw}{dx}+Qw)v=R \quad\text{(3)},$$

with the conditional equations:

$$\frac{d^2w}{dx^2}+P\frac{dw}{dx}+Qw=0 \quad\text{(4)},$$

$$\frac{d^2v}{dx^2}+[(2/w)\frac{dw}{dx}+P]dv/dx=R/w \quad\text{(5)}.$$

w being supposed known from (4) gives

$$w^2\frac{dv}{dx}\varepsilon^{-\int Pdx}=A+\int wR\varepsilon^{\int Pdx}dx \quad\text{(6)},$$

and $v=B+A\int\frac{dx}{w^2}\varepsilon^{-\int Pdx}+\int\frac{dx}{w^2}\varepsilon^{-\int Pdx}\int wR\varepsilon^{\int Pdx}dx \quad\text{(7)}.$

Now (4) is of the same form as (1) excepting that the right member is 0; so that if we have a solution of (4) we have that of (1) when $R=0$.

The given equation is

$$\frac{d^2y}{dx^2}-\frac{2x}{1+x^2}\frac{dy}{dx}+\frac{2}{1+x^2}y=0 \quad\text{(8)};$$

then $P=-2x/(1+x^2)$, and a particular solution is

$y=x$ ..... (9), or $w=x$ ..... (10).

Then (7) gives $v=B-A\int\left(\frac{dx}{x^2}+1\right)=B+A(x-[x/1])$,

and $y=vx=Bx+A(x^2-1)$, the required solution.

[As will be seen from the last solution both forms are correct. The first form is given as the answer, on page 336 of *Byerly's Integral Calculus*. EDITOR.]

Also solved by *O. W. ANTHONY*, *W. C. M. BLACK*, *J. SCHEFFER*, *G. B. M. ZERR*, and *P. S. BERG*.

58. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington, D. C.

A line passes through a fixed point and rotates uniformly about this point. Another line passes through a point which moves uniformly along the arc of a given curve and rotates uniformly about this point. Develop a method for finding the locus of intersection of these two lines. Apply to case of circle and straight line.

Solution by the PROPOSER.

Let $y=f(x)$ be the equation of the curve, taking the fixed point as origin. Let (x_1, y_1) be the position of the moving point at time t. Call θ_1 the angle which the line through the fixed point originally makes with axis of x; also, let ω_1 be the rate of angular rotation. Then the equation of line is

$$y=\tan(\omega_1 t+\theta_1)x;$$

whence $t=(\omega_1/1)[\tan^{-1}(y/x)-\theta_1]$ ....(1).

Also the equation of other line is

$$y-y_1=\tan(\omega_2 t+\theta_2)(x-x_1);$$

whence $t=(\omega_2/1)\left(\tan^{-1}\dfrac{y-y_1}{x-x_1}-\theta_2\right)$ ....(2).

$$ds_1{}^2=dx_1{}^2+dy_1{}^2. \quad ds_1/dt=v_1. \quad \therefore v_1{}^2=[1+\overline{f'(x_1)}^2]^{\frac{1}{2}}\frac{dx_1{}^2}{dt^2}.$$

$t=\dfrac{1}{v_1}\displaystyle\int[1+\overline{f'(x_1)}^2]^{\frac{1}{2}}dx$ ....(3). $y_1=f(x_1)$..........(4).

To solve the problem, then, we integrate (3), solve the resulting equation for x_1, substitute this value in (4), and then substitute the values of x_1 and y_1 in (2), after which t is to be eliminated between the resulting equation and (1). To apply this method to the case of the straight line

$$y_1=f(x_1)=0. \quad x_1=v_1 t.$$

$\therefore$ From (2), $t=\dfrac{1}{\omega_1}\tan^{-1}\left(\dfrac{y}{x-v_1 t}\right)-\theta_1$.

Then $\dfrac{1}{\omega_1}\left(\tan^{-1}\dfrac{y}{x}-\theta_1\right)=\dfrac{1}{\omega_1}\left(\tan^{-1}\dfrac{y}{x-(v_1/\omega_1)(\tan^{-1}[y/x]-\theta_1)}-\theta_2\right)$.

Let $y=\rho\sin\phi$; $x=\rho\cos\phi$.

Then $\dfrac{1}{\omega_1}(\phi-\theta_1)=\dfrac{1}{\omega_2}\left[\tan^{-1}\left(\dfrac{\rho\sin\phi}{\rho\cos\phi-(v_1/\omega_1)(\phi-\theta_1)}\right)-\theta_2\right]$.

$$\therefore \frac{\rho\sin\phi}{\rho\cos\phi-(v_1/\omega_1)(\phi-\theta_1)}=\tan[\theta_2+\frac{\omega_2}{\omega_1}(\phi-\theta_1)].$$

$$\therefore \rho=\frac{(v_1/\omega_1)(\phi-\theta_1)\tan[\theta_2+(\omega_2/\omega_1)(\phi-\theta_1)]}{\cos\phi\tan[\theta_2+(\omega_2/\omega_1)(\phi-\theta_1)]-\sin\phi},$$

the polar equation of the curve.

The case of the circle leads to complicated results. The case of two fixed points is interesting. The equations of the intersecting straight lines may be written

$y=\tan(\omega_1 t+\theta_1)x$............(1), and $y=\tan(\omega_2 t+\theta_2)(x-a)$............(2).

From (1) $\tan\phi=\tan(\omega_1 t+\theta_2)$.

$\therefore\ \phi=\omega_1 t+\theta_1.\quad t=(1/\omega_1)(\phi-\theta_1)$.

$\therefore\ \rho\sin\phi=\tan(\omega_2 t+\theta_2)(\rho\cos\phi-a)$;

whence $\rho=[a\tan(\omega_2 t+\theta_2)]/[\cos\phi\tan(\omega_2 t+\theta_2)-\sin\phi]$, or

$$\rho=\{a\tan\,[\frac{\omega_2}{\omega_1}(\phi-\theta_1)+\theta_2]\}/\{\cos\phi\tan[\frac{\omega_2}{\omega_1}(\phi-\theta_1)+\theta_2]-\sin\phi\},$$

$$=\{\,a\sin\,[\frac{\omega_2}{\omega_1}(\phi-\theta_1)+\theta_2]\}/\{\sin[\,(\frac{\omega_2}{\omega_1}-1)\phi+\theta_2-\frac{\omega_2}{\omega_1}\theta_1]\}\ \ldots\ldots\ldots\ldots\ldots(m).$$

If they both start from the initial horizontal position at the same time,

$$\theta_1=\theta_2=0,\ \text{and}\ \rho=a\frac{\sin[(\omega_2/\omega_1)\phi]}{\sin\{[(\omega_2/\omega_1)-1]\phi\}}.$$

A large number of curves is included in equation (m).

This problem was also solved by O. W. M. Black. His solution will appear in the next issue.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

41. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

If the earth were a perfect sphere and had a frictionless surface, what would be the motion of a particle placed at a given latitude?

Solution by the PROPOSER.

Adopt as coördinates the latitude of the particle and the distance measured in miles along a circle of latitude. Call the latitude λ, and the distance measured along the small circle x. Also let λ_1 be the initial latitude.

As the particle moves towards the equator under the resolved component of centrifugal force there will be no acceleration parallel to the equator.

Then clearly,

$$x=2\pi R(\cos\lambda-\cos\lambda_1)\ldots\ldots\ldots\ldots(1).$$

Also for the acceleration towards the equator,

$$\frac{d^2(R\lambda)}{dt^2}=-\frac{2\pi^2}{T^2}R\sin2\lambda,\ \text{or}\ \frac{d^2\lambda}{dt^2}=-\frac{2\pi^2}{T^2}\sin\lambda.$$

Let $d\lambda/dt=\mu$.

Then $\mu=\sqrt{2}\,\frac{\pi}{T}\sqrt{\cos2\lambda-\cos2\lambda_1}$.

Whence $t=\frac{1}{2}\sqrt{2}\frac{T}{\pi}\int_{\lambda_1}^{\lambda}\frac{d\lambda}{\sqrt{\cos2\lambda-\cos2\lambda_1}}=\frac{1}{2}\frac{T}{\pi}\int_{\lambda_1}^{\lambda}\frac{d\lambda}{\sqrt{\sin^2\lambda_1-\sin^2\lambda}}$

Let $\sin\lambda=\sin\lambda_1\sin\phi$. Then

$$t=\tfrac{1}{2}\int_{\phi_1}^{\phi}\frac{d\phi}{\sqrt{1-\sin^2\lambda_1\sin^2\phi}}.\quad \phi_1=\tfrac{1}{2}\pi,\quad \phi=\sin^{-1}\left(\frac{\sin\lambda}{\sin\lambda_1}\right).$$

$$t=\tfrac{1}{2}\frac{T}{\pi}\left[\int_0^{\phi_1}\frac{d\phi}{\sqrt{1-\sin^2\lambda_1\sin^2\phi}}-\int_0^{\phi}\frac{d\phi}{\sqrt{1-\sin^2\lambda_1\sin^2\phi}}\right]$$

$$=\tfrac{1}{2}(T/\pi)[F(\sin\lambda_1,\ \tfrac{1}{2}\pi)-F(\sin\lambda_1,\ \phi)]\ldots\ldots\ldots\ldots(2).$$

Equation (1) gives the relation between the coördinates at any time, and (2) gives the time of motion.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

30. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington, D. C.

A man is at the center of a circular desert; he travels at a given rate but in a *perfectly* random manner. What is the probability that he will be off the desert in a given time?

—

Solution by the PROPOSER.

Let R=the radius of desert, T=time, v=rate. Let P be the position of e man at any instant. Draw about P an infinitesimal circle, MSK. Call the

angle MPN, θ. Then $\theta=\cos^{-1}\dfrac{PN}{PM}$.

Now the rate at which the man must approach the circumference in order to be off in a given time is R/T. In an infinitely small time the distance will be $(R/T)dt$.

Also $PN=vdt$.

$$\therefore\ \theta=\cos^{-1}\left(\frac{R}{Tv}\right)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now R, T, and v are positive. Therefore the value of θ defined by equa-on (1) has to do with an angle less than 90°.

Now if the man at each instant goes within the angle MPN, he will get off ıe desert in the given time. The chance that he will do this is

$$C=\frac{2\cos^{-1}[(R/Tv)]}{\pi}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Hence the required probability in given by (2).

If $R=0$, or T, $=\infty$, or $v=\infty$, $C=1$. If $R=Tv$, $C=0$.

If $R>Tv$, C is impossible.

Let $R=1$, and $v=1$. To find the time which he must have at his disposal ı order that he may have half a chance to get off the desert.

Clearly $T=\sqrt{2}$.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

75. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, ennsylvania.

If 24 men, in 15 days of 12 hours each, dig a trench 300 yards long, 5 yards wide, 6 et deep for 540 five-cent loaves when flour is \$8 a barrel; what is flour worth a barrel hen 45 men, working 5½ days of 10 hours each, dig a trench 125 yards long, 5 yards wide, feet deep for 320 four-cent loaves? *Solve by proportion.*

76. Proposed by E. W. MORRELL, Professor of Mathematics in Montpelier Seminary, Montpelier, Vermont.

An eastern nobleman willed his entire estate to his three sons on the condition that ıe oldest should have one-half, the next one-third, and the youngest one-ninth. His es-te, on inventory, was found to consist of 17 elephants. What should be the share f each?

ALGEBRA.

78. Proposed by J. MARCUS BOORMAN, Consultative Mechanician, Counselor at Law, Inventor, Etc., Hewlett, Long Island, New York.

Solve $x^2+xy=10$......(1); $y^2+xy=15$......(2), for all the roots, and prove that they are the roots.

[Former solutions in print are defective. See *Analyst*, Vol. VIII, page 111; Vol. IX, page 58. J. M. B.]

79. Proposed by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

Of n persons A, B, C, etc., A first gives to the others as much as each of them already has; then B gives to the others as much as each then has; and so on for each in turn. Finally, A, B, C, etc., have respectively a, b, c, etc., dollars. How much had each at first?

80. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Solve $1+x^4=a(1+x)^4$.

GEOMETRY.

74. Proposed by ROBERT JUDSON ALEY, M. A., Professor of Mathematics in Indiana University, Fellow in Mathematics, University of Pennsylvania.

Let O be the center of the inscribed circle. AO produced meets the circumcircle in A'. Find the ratio of AO to OA'.

75. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

A plane passes through $(0,0,c)$ and touches the circle $x^2+y^2=a^2$, $z=0$; determine the locus of the ultimate intersections of the plane.

76. Proposed by L. B. FRAKER, Bowling Green, Ohio.

Lines run from a point, P, within a triangular piece of land to the angles A, B, and C are 91, 102, and 80 rods, respectively; and a line 78 rods in length passing through the point, P, and terminating in the sides AC and BC cuts off 3024 square rods adjacent to angle C. Required the dimensions of the land.

CALCULUS.

65. Proposed by GEORGE LILLEY, Ph. D., LL. D., Portland, Oregon.

A string is wound spirally 100 times around a cone 100 feet high and 2 feet in diameter at the base. Through what distance will a duck swim in unwinding the string keeping it taut at all times, the cone standing on its base and at right angles to the surface of the water?

66. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

Around the top of a conical frustum,—base 5 feet, top 1 foot, altitude 100 feet,—is wound a rope 100 feet long and 1 inch thick. It is unwound by a hawk flying in one plane. How far does Mr. Hawk fly?

67. Proposed by BENJ. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

A man starts to walk at a uniform rate across a draw-bridge just as it begins to move. He walks the full length of the bridge and back, in the same time that it takes the bridge to make a half revolution. How far does he ride, the length of the bridge being 250 feet, and its velocity uniform about a center axis?

MECHANICS.

52. Proposed by S. ELMER SLOCUM, Union College, Schenectady, New York.

A chain 16 feet long is hung over a smooth pin with one end 2 feet higher than the ther end and then let go. Show that the chain will run off the pin in about 7-5 second. *Wright's Mechanics*, page 92.]

53. Proposed by J. C. NAGLE, M. A., C. E., Professor of Civil Engineering, Agricultural and Mechanil College of Texas.

Find the locus of the center of gravity of an arc of constant length for a parabola.

54. Proposed by C. H. WILSON, Poughkeepsie, New York.

A body slides from rest down a series of smooth inclined planes, whose total heights re h feet. Show that the velocity at the bottom is $\sqrt{2gh}$ feet per second. [From *Wright's lechanics*.]

AVERAGE AND PROBABILITY.

51. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Three points are taken at random in a sphere and a plane passed through them. ind the average volume of the segment cut off from the sphere.

52. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, pringfield, Missouri.

A straight line of length a is divided into three parts by two points taken at random; nd the chance that no part is greater than b. [From *Hall and Knight's Higher Algebra*.]

53. Proposed by Samuel B. Harwood, Professor of Mathematics, Southern Illinois State Normal Univerty, Carbondale, Illinois.

Four Latin sentences are given. Number one has 12 words, two has 18 words, three nd four have 6 each. What are the chances that two pupils will have them in the same rder? Will the result vary with the number of pupils in the class?

EDITORIALS.

President George H. Harter, of Delaware College, Delaware, has just orered a complete set of the MONTHLY.

We shall be pleased to pay 25 cents each for a limited number of copies of lo. 6, Vol. I, and No. 11, Vol. II, of the MONTHLY. Any of our readers wishag to dispose of these numbers should write to us.

We are greatly pleased to note that the Board of City Trusts, Philadelphia, 'ennsylvania, has recognized the long and faithful service of Professor Warren Iolden in the following resolution: Resolved, That in consideration of fortyve years continued and faithful service, Warren Holden, A. M., Professor of Iathematics at Girard College, be retired January 31, 1897, at a salary of $2,500 er annum.

So far, we have received only a few letters respecting the matter of publishing the portraits of our contributors. We shall be pleased to hear still furthur, and those who favor the plan may send their photos to us at once.

The paper by the late Ansel N. Kellogg, of Chicago, published in this issue was sent to the MONTHLY at the suggestion of Professor Irving Stringham, of the University of California. Professor Stringham says, "They [the formulæ] take us back to methods that were in vogue at the beginning of the century. But they are much superior in accuracy and rapidity of convergence to any I have found in the older books. They will be of some interest, I think, to mathematical readers.

Their author, the late Ansel N. Kellogg, of Chicago, was for a number of years prominent in newspaper and business circles throughout the country. Though a very busy man, he found time for mathematical meditation, and that he could think efficiently in this domain the paper presented sufficiently attests."

As we are very anxious to increase the subscription to the MONTHLY we make the following liberal offers :

1. To any person sending us 75 new subscribers at our regular price, we will make a present of a handsome set of the *Century Dictionary and Encyclopedia.*

2. To any person sending us 50 new subscribers at our regular price, we will make a present of a $100 *Acme* or *Monarch Bicycle.*

3. To any person sending us 20 new subscribers at our regular price, we will make a present of the *Standard American Encyclopedia* [see advertisement on cover.]

4. To any person sending us 15 new subscribers at our regular price, we will make a present of a copy, in one volume, of the *Standard Dictionary* (Funk and Wagnalls').

In all cases the money must accompany the list of names sent in.

BOOKS AND PERIODICALS.

Determinants. Designed for High Schools, and Lower Classes of Colleges and Universities. By J. M. Taylor, M. S., Professor of Mathematics and Astronomy in the University of Washington and Director of the Observatory. 8vo. Cloth, 48 pages. Chicago : Werner School Book Company.

In this little book, Professor Taylor has set forth in a very clear and concise manner the fundamental principles of Determinants. We feel sure that this little work will go far towards popularizing the subject and bringing it within the easy comprehension of the students of our best High Schools. B. F. F.

Elements of Theoretical Physics. By Dr. C. Christiansen, Professor of ıysics in the University of Copenhagen. Translated into English by W. F. agie, Ph. D., Professor of Physics in Princeton University. Large 8vo. Cloth, 8 pages. Price, $3.25. New York: The Macmillan Co.

This work, at first sight, presents a formidable appearance in mathematical notation l formulæ, but by beginning with the introduction and carefully reading through it, the ıder is led on to overcome difficulties by a force which can only be accounted for by the mirable, clear, and interesting treatment of the subjects. It presents the fundamental nciples of Theoretical Physics and developes them so far as to bring the reader in touch th much of the new work that is now being done in that subject. It is not exhaustive in ery respect, but is stimulating and informing and furnishes a view of the whole field, ıich will facilitate the reader's subsequent progress in special parts of it. The book is inted on good paper and is well bound. Its appearance could have been somewhat imoved by not printing it so compactly. B. F. F.

Principles of Mechanism. A treatise on the Modification of Motion by eans of the Elementary Combinations of Mechanism or of the Parts of achines. For use in College Classes, by Mechanical Engineers, etc., etc. By illman W. Robinson, C. E., D. Sc., till recently Professor of Mechanical Enıeering in the Ohio State University. First Edition, first thousand. Large o. Cloth, 309 pages. Price, $3.00. New York: John Wiley & Sons.

In this volume we have a thoroughly scientific treatise on mechanical movements. ey are treated from the standpoint of both theory and practice. The work embodies the ɔstance of lectures given by the author during the past twenty-seven years.

The work is largely addressed to those who are more conversant with the drawing ard than with mathematics, so that the subject has been treated more from the standint of graphics than of pure analysis. This feature will popularize the work. The drawgs, which are very suggestive, beautiful, and accurate, are very numerous. There are merous reproductions from actual models. B. F. F.

(1) *Macaulay's Essay on Milton;* (2) *Shakespeare's Midsummer Night's ·eam;* (3) *Scott's Woodstock;* (4) *Milton's L'Allegro, Il Penseroso, Comus, Lycis;* (5) "*George Eliot's*" *Silas Marner.* Price of (1), (2), and (4) 20 cents, of) 60 cents, and of (5) 30 cents. American Book Company, New York, Cincinti, and Chicago.

We notice collectively this group of texts from the "Eclectic English Classics" ser, published by the American Book Company. The texts are well and carefully edited, th introductions and explanatory notes. (1), (3), and (5) have frontispiece portraits of hn Milton, Oliver Cromwell, and "George Eliot", respectively. These books are clearly nted, the notes are concise and sufficient, and the introductions interesting and valua. There is so much advantage in extending the use of these gems of English Literature our schools, that a debt of gratitude is due the publishers for providing them in such viceable shape and at a minimum cost. J. M. C.

An Elementary Treatise on Plane Trigonometry. By E. W. Hobson, Sc. , and C. M. Jessop, M. A. 299 pages. Price, $1.25. Cambridge University ess. New York: Macmillan & Co.

This treatise on trigonometry is a work of recognized merit. The chapter on solun of trigonometrical equations is particularly full and valuable. The plan of the work good and the execution thorough and satisfactory. J. M. C.

The Arena. An Illustrated Monthly Magazine. Edited by John Clark Redpath and Helen H. Gardner. Price, $3.00 per year in advance. Single numbers, 25 cents. Boston: The Arena Company.

The March number of *The Arena* is the initial issue of the magazine under the new management and editorship. The Company has been reorganized on a solid financial basis, and the current number of the magazine comes in a form and substance well calculated to win public favor, and following its well established policy of liberalism and reform.

The number opens with the first of a series of important contributions on the development and reform of city government in the United States. This first article is by the Hon. Josiah Quincy, Mayor of Boston, who therein expresses himself as in favor of the municipal ownership, though not necessarily the municipal operation, of public services, such as gas and electric lighting and street railways. An excellent portrait of Mayor Quincy forms the frontispiece to the number. The article by Professor LeConte, of the University of California, on "The Relation of Biology to Philosophy," is a searching adverse criticism of the seventh chapter of Professor Watson's recent work on "Comte, Mill, and Spencer;" but it is also very much more, being a thoroughly up-to-date exposition of the general theory of organic evolution, and its relation to religion as well as philosophy.

B. F. F.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single numbers, 25 cents. The Review of Reviews Co., 13 Astor Place, New York City.

The editor of the *Review of Reviews* comments in the March number on the Spanish program of reforms in Cuba, the United States Senate's attitude toward the arbitration treaty with England, the immigration bill, the proposed international monetary conference, President-elect McKinley's cabinet selections, the recent Senatorial elections, the New York Trust investigation, the famine situation in India, the affair of the Greeks in Crete, the foreign policy of Russia, the position of England, France, and the other great powers, and many other matters of current interest. B. F. F.

ERRATA IN JANUARY NUMBER.

On page 16, 2nd line of problem 55, for "grouhd" read *ground*.

On page 17, in the figure, join *CF* and *CP*.

On page 17, 1st line of solution II, for "*AMFHC*" read *AMFHC'*.

On page 20, line 1, complete the parenthesis after last term of equation.

On page 20, line 8, place — between two terms enclosed by brackets.

On page 20, line 14, for "$\frac{3}{8}\frac{1}{3}\pi^2a^4$" read $\frac{3}{8}\frac{1}{3}\pi^2a^4$.

On page 21, line 2, for "$\frac{16}{3}a^2$" read $\frac{16}{3}a^3$.

On page 21, line 8, read $d\rho=2a\cos^2\theta+a\cos\theta-a=0$.

On page 25, line 18, for "100" read 100th.

On page 25, line 25, read "add and subtract $B^2/4$, etc."

On page 26, line 15, for "$(2mp)^2$" read $(2mn)^2$.

On page 27, line 3, for "is" read in.

On page 28, line 15, for "$x++(x1)$" read $x+(x+1)$.

On page 28, line 20, for "2392" read 1393.

On page 32, lines 6, 11, and 12, read *l* where 1 occurs.

HUBERT ANSON NEWTON.

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BIOGRAPHY.

HUBERT ANSON NEWTON.

BY PROFESSOR ANDREW W. PHILLIPS.

HUBERT ANSON NEWTON was born in Sherburne New York, March 19, 1830, and died at New Haven on the 12th day of August, 1896.* He graduated from Yale, taking the degree of A. B., in 1850, and spent the next two and one-half years in mathematical study.. He became tutor at Yale in 1853 and on account of the sickness and subsequent death of Professor Stanley, the whole work of the department of mathematics devolved upon him from the first. In 1855 his great ability was recognized in his election, at the early age of twenty-five, to a full professorship of mathematics at Yale, the duties of which he assumed after spending a year of study in Europe, where, under the inspiration of Chasles, he became especially interested in the subject of Modern Higher Geometry. He carried on most vigorously work and studies in various lines in addition to the duties of his professorship. Sometimes it was a profound study in pure Mathematics, sometimes a rich contribution to the education of the public, and sometimes an original investigation in the field of Astronomy.

He published in 1857 a paper on the Gyroscope in the *American Journal of Science*, and soon after, a paper in the *Mathematical Monthly*, in which he seems to have been the first to apply the principle of inversion in the solution of the problem of constructing circles tangent to three given circles. He showed how deeply rooted in his mind were the ideas of the Modern Geometry in his

*Professor Newton was Vice President of the American Mathematical Society at the time of his death. EDITOR.

elaborate papers published in the same journal in 1861 on the geometrica struction of certain curves by points, where he extended the ideas of Chasl de Jonquières, and of Poncelet. The subject of transcendental curves he st for a long time with great interest, and constructed a myriad of interestin terns, but contented himself with publishing, in the joint name of himsel his pupil, the discussion of the single group of equations which he found w give the most beautiful and symmetric forms, and which he had set for his to investigate.

Professor Newton was very active in securing the prompt adoption o Metric System of Weights and Measures, both by the Connecticut Legislatur by Congress after the Conference of Nations on the subject, held in Berl 1863. He wrote a popular tract in 1864, giving an explanation of the sys He contributed in 1865 to the Report of the House Committee on Weights Measures at Washington, and also to the Report of the Smithsonian Institutio this subject. He prepared an appendix consisting of these tables in for school instruction for one of the leading arithmetics, and interested the ers of scales rulers in graduating their devices for weighing and measu according to the Metric System. He gave his ideas to the public freely in r ence to the graphical representation of all sorts of statistical information, and tributed lavishly his ideas to the authors of mathematical books used in sc and college class-rooms, although he published no text-books in his own n He was the joint author with Professor Loomis of a most elaborate paper o climate of New Haven, which was published in the Transactions of the Conn cut Academy of Arts and Sciences. He prepared articles on the subje Meteors for two leading cyclopædias and contributed the mathematical astronomical definitions to Webster's International Dictionary. Professor ton was one of the highest authorities on the subject of Life Insurance and sides the important actuarial work which he did, computed valuable tables lished by the New York Insurance Department in 1868, and later in the Englander, a paper on the Law of Mortality that prevailed among former bers of the Yale Divinity School, and still later, in Professor Dexter's A and Biographies, on the Length of Life of the Early Yale Graduates.

But the contributions to human knowledge, which most entitle hi fame, are those which he made on the subject of meteors, shooting and comets. The facts of the great star shower of 1833 had given t New Haven men—Professors Twining and Olmstead—a clue to the true t of the shooting stars, and this, together with the interest which the men of sc at Yale had kept up in the subject of meteors, influenced Professor Newt direct his studies towards these bodies as the time drew near for a possible r rence of the great November shower of 1833. In 1860 he published his paper on this subject in the *Journal of Science*, entitled "The Fireball of No ber 15, 1859," and this was followed by two other papers in the same jou one on the great fireball of August 10, 1861, in which also the August gro meteors was discussed; and the other on the two fireballs of August 2 and A

1860. Professor Newton had gathered a large number of observations made persons in the localities where these bodies had attracted attention, and treat- the subject with special reference to determining their nature and their veloc-. Early in 1863, at the request of the Connecticut Academy, he prepared a llar chart suited to observations at all times, which was distributed to persons various stations for observing the August meteors. A vast amount of material s thus collected for computing the altitudes of the meteors and for obtaining ne idea of their velocities. In June, 1863, Professor Newton published in the *urnal* a paper on the "Evidence of the Cosmical Origin of Shooting Stars de- ed from the Dates of early Star Showers," which not only established beyond estion the fact that the star showers are caused by the entrance into the earth's nosphere of bodies revolving about the sun, but gave the key to the complete ution of the problem of the November meteors. In May, 1864, he published original accounts of thirteen remarkable displays of the November shooting rs, ranging from A. D. 902 to 1833, and in July of the same year he published second paper in which he derived from these accounts the length of the annual riod, the length of the cycle, the mean motion along the ecliptic of the node of e orbit of the group, and the length of the part of the cycle during which show- s may be expected. He also showed that there were only five possible period- times which could satisfy the observed conditions, and of these the true orbit s probably either one with a period of 354.6 days or one with a period of 33.25 ars. The first of these two he thought the more likely, and computed e other elements of that orbit, but he pointed out at the same time a criterion r determining which was the true orbit when the position of the radiant should more accurately established.

In August, 1864, Professor Newton presented to the National Academy of iences a comprehensive memoir on the Sporadic Shooting Stars. He had ortly before this compiled a table of computed altitudes of certain shooting rs which included substantially all that had ever been published. Using this ble as a basis, he deduced the distribution of meteor paths over the sky in alti- de and in azimuth, the number of shooting stars that come into our atmosphere ch day, the mean length of the visible part of the meteor paths, and the num- r of meteoroids in the space which the earth traverses. He also deduced the markable fact that the mean velocity could be determined from the number of ooting stars in the different hours of the night.

These papers of Professor Newton aroused the greatest interest among athematicians and astronomers in the subject of meteors, and especially in the ır showers predicted for November, 1865 and 1866. The facts of these showers nfirmed to a remarkable degree Professor Newton's theories. Leverrier and hiaparelli, however, by independent methods showed that the period of e group was most probably 33.25 years, and Professor Adams, in 1867, applying Professor Newton's criterion added the last link in establishing this the true orbit of the November meteoroids.

Professor Newton, by his papers of 1863 and 1864, laid the foundation of the Science of Meteoric Astronomy. His subsequent papers, nearly thir in number, cover almost every topic connected with the subject. Wheth in his reviews of the facts concerning the November shooting stars in the suc- sive years from 1864 to 1869, or in the discussion of the Biela meteors of 18 and of 1885, or in his treatment of such topics as the origin of comets, or the di- rect motion of comets of short period, the capture of comets by Jupiter, the eff upon the earth's velocity produced by small bodies entering the asmosphere, th relation to the earth's orbit of the former orbits of those meteorites in our col- tions, which were seen to fall, one prominent characteristic of his investigati was always its exhaustive character. For, whatever Professor Newton, did was not worth the while of any one else to cover the same field.

Besides the papers which he published, his scientific activities outside th duties of his professorship were numerous and important. He organized a math- ematical society in the early '60s to which he was the principal contributor, a to the successor of this society, the Yale Mathematical Club, organized in 18 he contributed more than a score of papers. He was for many years a memb of the Publishing Committee of the Connecticut Academy of Arts and Sciences. He was an associate editor of the *American Journal of Science* for thirty years. He was one of the principal founders of the Yale Observatory and practically its director till near the time of his death.

The appreciation in which his scientific ability and his labors were held is shown in the honors which he received. In 1862 he was made a member of the American Academy of Arts and Science. He was one of the original charter members of the National Academy of Sciences, founded in 1863. In 1867 he was made a member of the American Philosophical Society of Philadelphia. The degree of LL. D. was conferred upon him by Michigan University in 1868. He was made an Associate of the Royal Astronomical Society in 1872. He was Vice President of the American Association for the Advancement of Science, pre- siding over the section of Mathematics and Astronomy in 1875, and was President of the Association in 1885. He was made a Foreign Honorary Fellow of the Royal Society of Edinburgh in 1886, and a Foreign Member of the Royal So- ciety of London in 1892.

At the April meeting of the National Academy in 1888 the value of Pro- fessor Newton's scientific work was publicly recognized by that body, in award- ing to him the J. Lawrence Smith gold medal for his contributions to Meteoric Astronomy. His reply to the address of presentation reveals at once his modes- ty and his own true scientific spirit.

"Sir: I beg to express to the Academy my high appreciation of the hon- or you have conferred upon me. To discover some new truth in nature, even though it concerns the small things in the world, gives one of the purest pleas- ures in human experience. It gives joy to tell others of the treasure found. When, therefore, those best able to judge of the value of this addition to human knowledge say that it is worthy of their special public commendation, that joy is greatly increased.

I shall cherish this memorial also for that it bears the likeness of ie whose true scientific spirit we all learned to admire, and whom, for his gen- l character, we all learned to love."

The achievements of Professor Newton, great as they were from a scienti- standpoint, give no adequate idea, taken in themselves, of his power and in- ence. These, in a larger sense have become a part of the organic life of the iversity where his work was done. He built up, during a leadership of forty ars, a strong and symmetrical department of Mathematics, by his comprehen- e grasp of the trend of Mathematical thought, and by his wonderful power of ining the paths which lead out to fruitful fields of research, both within the main of pure mathematics and in its applications to other sciences. Nor was best part of his academic activities merely in his own department of studies. moulding the general policy of the institution his counsel was invaluable; in ablishing and maintaining the moral and intellectual standards, his influence preëminent; the University bears the indelible impress of a life consecrated the development of the noblest ideals.

Yale University.

ON THE SOLUTION OF THE QUADRATIC EQUATION.

By G. A. MILLER, Ph. D., Paris, France.

[Continued from January Number.]

The solution of the quadratic equation

$$a_0x^2+a_1x+a_2=0 \qquad A$$

clearly equivalent to finding the two factors which are linear in x of the quantic

$$a_0x^2+a_1x+a_2.$$

When we ask whether this quantic has linear factors it is necessary to con- ler the domain of rationality to which we confine our attention. For illustra- n, we may consider the special quantic

$$x^2-4x+1.$$

If we confine ourselves to the simplest domain of rationality, viz: the do- in which consists of all the rational numbers, we have to say that this quantic no linear factors. In other words, it is irreducible in this domain. However, if we enlarge this domain by adding to it the irrational number $\sqrt{3}$* we obtain a domain in which the quantic is clearly reducible. This domain is composed of all the numbers whose form is

$$\alpha+\beta\sqrt{3} \qquad (\alpha \text{ and } \beta \text{ being any rational numbers}).$$

According to the fundamental theorem of algebra a quantic which involves only a single variable can always be resolved into its linear factors in the domain obtained by enlarging the domain of its coefficients, if necessary, so as to include suitable new numbers. If the coefficients lie in the domain of the complex numbers the added numbers must also lie in this domain. If a quantic involves several variables it may remain irreducible even when the domain is enlarged in every possible manner.

Let x_1 and x_2 be the two roots of A. Since every rational symmetric function of the roots of an algebraic equation can be expressed rationallly in terms of its coefficients we know the value of any rational symmetric function of x_1 and x_2. This value must lie in the domain of the coefficients. In particular, we know the value of any even power of x_1-x_2. The value of the square is given by the equation

$$(x_1-x_2)^2=x_1{}^2+x_2{}^2-2x_1x_2=(x_1+x_2)^2-4x_1x_2=a_1{}^2-4a_0a_2/a_0{}^2.$$

To find the difference of the roots from the last equation we have to extract the square root of the last member. This may be impossible in the domain of the coefficients. If this domain forms a group with respect to the extraction of the square root it is clearly possible in this domain. We know that the system of ordinary complex numbers forms a group with respect to the extraction of any root. Hence we see that, if a_0, a_1, a_2 lie in the domain formed by the ordinary complex numbers, the difference of the roots of A as well as the sum of these roots must lie in the same domain.

The roots themselves may be found from these two functions by means of addition and subtraction. As any domain includes all the quantities resulting by applying these operations to any of its quantities the roots of A must also lie in the given domain of rationality. The roots may also be found by observing that their general linear function

$$\alpha x_1+\beta x_2$$

is rationally expressible as follows :†

$$\alpha x_1+\beta x_2\equiv\tfrac{1}{2}(\alpha+\beta)(x_1+x_2)+\tfrac{1}{2}(\alpha-\beta)(x_1-x_2)$$

---

*By enlarging a domain of rationality by the addition of a quantity is meant the forming of the smallest domain that contains the given domain and the added quantity.

†This is an illustration of the general theorem that any rational function of the roots of an algebraic equation of degree n is rationally expressible in terms of a $n!$ valued function of the n roots.

$$\equiv\tfrac{1}{2}-(\alpha+\beta)(a_1/a_0)+(\alpha+\beta)/2a_0\sqrt{a_1^2-4a_0a_2}.$$

By letting $\alpha=1$, $\beta=0$, and $\alpha=0$, $\beta=1$ in this identity we obtain the values of x_1 and x_2 respectively.

As the ordinary complex numbers do not only form a domain of rationality but also a group* with respect to what is frequently called the most general algebraic operation, viz: that represented by

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots a_n=0$$

(a_0, a_1, a_2, ..., a_n being ordinary complex numbers and n any positive integer), and as they obey the commutative, distributive and associative laws of operation just like real numbers and also the law that a product cannot be zero unless one of the factors is zero, it is clear that we can reason quite generally in regard to symbols representing such numbers. It is probably largely due to this fact that other number systems are not more generally employed. In fact, no really different number system was developed until 1843. In this year Sir William Hamilton discovered and communicated to the Royal Irish Academy the system known as *Quaternions*, which is perhaps still the most important system besides that of the ordinary complex numbers. In the following year Grassmann published his *Ausdehnungslehren* in which he used a number system of a somewhat different form.

Among the investigations of later years those of Weierstrass have probably received the most attention† although important developments have been made in other directions. The fact that the ordinary complex numbers correspond to the points of a plane very naturally led to the thought that a system of higher complex numbers of the form

$$\alpha+\beta i+\gamma j \qquad (\alpha,\ \beta,\ \gamma \text{ being any real numbers})$$

might correspond to the points of space. It was easy to show that the product of two such numbers, multiplied according to the rules of ordinary numbers, may be zero when neither of the factors is zero.‡ This result naturally led to the study of numbers which do not obey all the laws of operations which the ordinary numbers obey.

The main purpose of the preceding remarks was to obtain a fairly clear view of number and of the domain of rationality as these two concepts are fundamental in the study of the solution of algebraic equations. Incidentally we indicated several methods of solving the quadratic equation A. We proceed now to consider some of the other methods of solving this equation. We shall not aim at a complete enumeration of the methods by which A may be solved. In

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*It seems that Poincara was the first who considered the general number systems directly as groups. Cf. Comptes Rendus, t. 99, p. 740.

†*Gottinger Nachrichten*, 1884, page 395.

‡Cf. Harkness and Morley, *Theory of Functions*, page 8.

fact, if we would consider each modification of the operations of finding the roots of A as a new method the number of these methods would clearly be infinite. We may, for instance, form an infinite number of quantics of the form of a quadratic each of which contains the first member of A as a factor. For A may be written in the form

$$a_0x^2+a^2=a_1x.$$

Squaring both members and combining we have

$$ax^4+bx^2+c=0,$$

(a, b, c belonging to the same domain as a_0, a_1, a_2). Since the result is of the same form as A we may repeat the operation any number of times. Hence A is a factor of the quantic

$$A_0x^{2^a}+A_1x^{2^{a-1}}+A_2,$$

(A_0, A_1, A_2 belonging the same domain as a_0, a_1, a_2 and a being any positive integer). The roots of any one of the equations obtained by making these quantics equal zero include the roots of A. As the roots of

$$A_0y^2+A_1y+A_2=0$$

are the 2^{a-1} powers of the roots of A it is clear that none of these transformations can simplify the solution of A. By elimination we may clearly obtain an indefinite number of additional equations containing the roots of A from the given system. In particular, if we eliminate the constant from the biquadratic equation by means of A we obtain a biquadratic equation which has the roots of A and two zero roots. Upon this elimination depends a solution recently publish- in this journal. The same result might be obtained by multiplying both members of A by x^2. It is, in general, not well to raise the degree of A in the process of solution since this introduces additional roots and therefore makes the operation more complex.

Perhaps the best known method of solving A is that by which its first member is made a perfect square by the addition of the same quantity to each member. To make the quantic

$$a_0x^2+a_1x+a_2$$

a perfect square without altering its degree we may add to it the quantic

$$ax^2+bx+c$$

where two of the three numbers a, b, c are entirely arbitrary since it is only necessary that the discriminant vanishes. This idea is frequently expressed by saying that the quantic to be added can be chosen in a doubly infinite number of ways. Since this quantic must also be a perfect square its own discriminant must also vanish. As this imposes another condition on its coefficients we can select the trinomial to be added to both members of A in only a simply infinite number of ways.

This number of choices might at first appear too small since in the ordinary method by which we add a constant to both members of A we apparently select both a and b arbitrarily since we let both equal zero. This would imply a doubly infinite number of choices. This apparent contradiction is explained by the fact that the vanishing of the discriminant of the added trinomial, i. e., the equation

$$b^2=4ac$$

indicates that at least two of the coefficients, including b, must be zero when one is zero. Hence the ordinary method implies that one of the coefficients of the added trinomial is selected arbitrarily and the other in accord with this equation.

To illustrate we inquire what quantics may be added to both members of the special equation

$$x^2-4x+1=0$$

so as to make both members perfect squares. Adding the given general quantic we have the equations

$$(a+1)x^2+(b-4)x+c+1=ax^2+bx+c.$$

Since the discriminants of both members must vanish we have

$$(b-4)^2=4(a+1)(c+1) \text{ and } b^2=4ac.$$

If we assign to b the arbitrary number 2 and eliminate c we have

$$a^2+a+1=0.$$

Hence a and c are the imaginary cube roots of unity, ω_1 and ω_2, and the given equation becomes*

$$-\omega_1{}^2x^2-2x-\omega_2{}^2=\omega_1x^2+2x+\omega_2$$

or

$$-1(\omega_1{}^2x^2+2x+\omega_2{}^2)=\omega_2{}^2x^2+2x+\omega_1{}^2.$$

Extracting the square root from both members we have

$$\pm i(\omega_1x+\omega_2)=\omega_2x+\omega_1$$

*It should be observed that the product of the two imaginary cube roots of unity is unity and that the square of one is equal to the other.

or
$$x=\frac{\omega_1\mp i\omega_2}{\pm i\omega_1-\omega_2}=2\pm\sqrt{3}.$$

If we let $b=4$ the first discriminant shows that one of the two factors $a+1$, $c+1$ must vanish. If we suppose that the former vanishes the given equation becomes

$$-3=-x^2+4x-4 \text{ or } x^2-4x+4=3.$$

If we suppose that the latter of the given factors vanish we obtain the equation
$$4x^2-4x+1=3x^2.$$

Instead of assigning an arbitrary value to b we might clearly assign an arbitrary value to either of the other coefficients. The simplest method is probably that in which a is made equal to zero. By making a and b equal to the corresponding coefficients with the signs changed of the equation which is to be solved and selecting c so as to satisfy the equation

$$b^2=4ac$$

we have another simple rule for completing the square. A number of other fairly convenient rules can easily be derived from what precedes.

That we can assign the given values to a and b follows from the first of the given discriminants. If we assign this value to a we determine the value of b at the same time but if we commence by assigning the given value to b neither a nor c are fully determined. We still say that the number of choices is simply infinite since a finite number multiplied into a simply infinite number is said to give a simply infinite product. The preceding remarks apply evidently also to the slight modification of the given method which consists in writing A in the form
$$a^2-b^2=0 \text{ instead of } a^2=b^2$$

and factoring the first member according to the well known formula

$$a^2-b^2=(a+b)(a-b)=(-b-a)(b-a)$$

instead of extracting the square root of the two members.

Another simple method of solving A may be described as follows: The equation A is satisfied by the affixes of two points and gives the elementary symmetric functions of these affixes. As all rational symmetric functions can be expressed rationally in terms of the elementary symmetric functions we know the affix of the middle point of the join of the roots. If the points of the plane are so transformed that this point becomes the origin the roots are the affixes of the extremities of a diameter of a circle whose center is the origin. Hence the equa-

in the new variable must be a pure quadratic and the solution is readily
pleted. If we do not assume that the coefficients are real, one root may be
while the other is imaginary. In fact the roots may be the affixes of any
points.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

ORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from January Number.]

PROPOSITION XXV. *If two straights (Fig. 30.) AX, BX existing in the plane (standing upon AB, one indeed at an acute angle in the point A, and ther perpendicular at the point B) so always approach more to each other mu-ly, toward the parts of the point X, that nevertheless their distance is always ter than a certain assigned length, the hypothesis of acute e is destroyed.*

Fig. 30.

PROOF. Let R be the assigned length. If therefore X is assumed a certain BK any chosen multiple of the osed length R; it follows (from the preceding Scholion) the perpendicular erected from the point K toward the s of AX will meet it at some point L; and again (from present hypothesis) it follows that this KL will be ter than the aforesaid length R. Furthermore BK is erstood divided into portions KK, each equal to R, even KB is itself equal to the length R. Finally from the ts K are erected to BX perpendiculars meeting AX in ts L, H, D, M, even to the point N nearest the point A. Now I proceed thus.

The four angles together of the quadrilateral $KHLK$, more remote from base AB, will be (from the preceding Proposition) greater than the four an-together of the quadrilateral $KDHK$, nearer to this base; of which quadri-al in the same way the four angles together will be greater than the four an-together of the quadrilateral $KMDK$ subsequent toward this base. And so ys even to the last quadrilateral $KNAB$, whose four angles together assured-ill be the least, in reference to the four angles together of each of the quadri-als ascending toward the points X.

But since are present as many quadrilaterals described in the aforesaid ner, as are, except the base AB, perpendiculars let fall from points of AX to the straight BX; the sum of all the angles together, which are comprehended in these quadrilaterals can be reckoned. We assume that there are nine such perpendiculars let fall, and therefore so nine quadrilaterals.

We get (from Eu. I. 13) as equal to four rights the angles comprehended hither and yon at the two points of those eight perpendiculars, which lie in the middle between the base AB and the more remote perpendicular LK. So the sum of all these angles will be 32 rights.

There remain two angles at the perpendicular LK, and two at the base AB. But the angles one indeed at the point K and the other at the point B are supposed right; but the angle at the point L (from the Cor. after P. XXIII.) is obtuse. Wherefore (even neglecting the acute angle at the point A) the sum of all the angles which are comprehended by these nine quadrilaterals exceeds 35 rights. But hence follows, that the four angles together of the quadrilateral $KHLK$, more remote from the base lack less from four rights than the ninth part of one right; and that indeed even if an equal portion of the aforesaid sum of all the angles pertained to each of those quadrilaterals.

Therefore less yet will be the entered defect, since the sum of the four angles together of this quadrilateral $KHLK$ was shown the greatest of all, in relation to the four angles together of the remaining quadrilaterals.

But again; in consequence of the supposition upon which this proposition proceeds, so great a length of BK can be assumed, that as many quadrilaterals as we choose may be made on bases KK, each equal to the assigned length R.

Wherefore the defect of the four angles together of this more remote quadrilateral $KHLK$ from four rights is shown ever less both than a hundredth and than a thousandth, and thus under any assignable part of a right. Further however, LK and HK will be always (in accordance with the aforesaid supposition) greater than the designated length R. Therefore if in KL and KH are assumed KS and KT equal to KK or the length R; ST being joined, the two angles together KST, KTS will be greater, in hypothesis of acute angle, than the two angles together (from Cor. after P. XVI.) at the points H and L in the quadrilateral $THLS$, or the quadrilateral $KHLK$; and therefore (the common right angles at the points K, K being added) the four angles together of the quadrilateral $KTSK$ will be greater than the four angles together of that quadrilateral $KHLK$.

But now, since on one hand is stable and given the quadrilateral $KTSK$, in as much as constant in the given base KK, which indeed is taken equal to the assigned length R, and again constant in the two perpendiculars TK, SK equal to this base, and finally in the joining TS, which comes out completely determinate; and on the other hand the four angles together of this stable and given quadrilateral have now been shown greater than the four angles together of the quadrilateral $KHLK$ distant as far as we choose from the base AB; assuredly it follows, that the four angles together of this stable and given quadrilateral $KTSK$ are greater than any sum of angles, which lacks however you choose of being four right angles; since already it has been shown that a quadrilateral $KHLK$ can always be designated such that its four angles together shall fall short of four rights less than any assignable part of a right. Therefore the four angles together of this stable and given quadrilateral either are equal to four rights or greater.

But then (from P. XVI.) is established the hypothesis either of right angle or of obtuse angle; and therefore (from P. V. and P. VI.) the hypothesis of acute angle is destroyed.

So is established that the hypothesis of acute angle will be destroyed, if two straights existing in the same plane so approach each other mutually ever more, that nevertheless their distance is always greater than any assigned length.

Hoc antem erat demonstrandum.

Corollary I. But (the hypothesis of acute angle destroyed) the controverted Pronunciatum Euclidaeum is manifest from P. 13 of this; just as that by me in this place it would be disclosed I promised in Scholion III after P. XXI of this, where we spoke of the attempt of the Arab Nassaradin.

Corollary II. On the other hand from this proposition, and from the preceding XXIII is manifestly gathered that not sufficient for establishing Euclidean geometry are two following points. One is: that we designate by the name of parallels those straights, which existing in the same plane possess a common perpendicular. The second indeed; that all straights existing in the same plane, of which there is no common perpendicular, and therefore which according to the assumed definition are not parallel, must, being produced toward either part ever more, somewhere meet each other, if not at a finite, at least at an infinite distance.

For again it would be requisite to demonstrate, that any two straights existing in the same plane, upon which a certain straight cutting makes two internal angles toward the same parts less than two right angles, nowhere else can receive a common perpendicular.

But that, this demonstrated, Euclidean geometry is most exactly established, will be shown below.

[To be Continued.]

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from January Number.]

XXVII. Let ABC be a triangle, right-angled at C. With O, the middle of AB, as a center, describe a circle to which either of the other sides, as BC, shall be tangent. Then,

Fig. 21.

$$BD \cdot BE = \overline{BP}^2;$$

or $(\tfrac{1}{2}c-\tfrac{1}{2}b)(\tfrac{1}{2}c+\tfrac{1}{2}b)=\tfrac{1}{4}a^2$. $\therefore c^2=a^2+b^2$.

This and XVI are special cases of a more general form. For O may be any point in AB, such that the ratio of OB to AB shall be n. Our equation would then become $(nc-nb)(nc+nb)=n^2a^2$; whence, $c^2=a^2+b^2$.

XXVIII. Fig. 21.

Suppose $BC<AC$. Then $HC \cdot FC=\overline{PC}^2$.

But $HC \cdot FC=AF \cdot AH=AE \cdot AD=(\tfrac{1}{2}c-\tfrac{1}{2}b)(\tfrac{1}{2}c+\tfrac{1}{2}b)$, and $\overline{PC}^2=\tfrac{1}{4}a^2$.

$\therefore c^2=a^2+b^2$.

From $BC<AC$ pass to $BC=AC$ by the theory of limits.

XXIX. Let ABC be a triangle, right-angled at C. Describe a circle, such that its center O shall be in AB, and to which the other sides shall be tangent.

Fig. 22.

Draw OD perpendicular to AB. Then,

$$\overline{AT}^2=AE \cdot AF=\overline{AO}^2-\overline{EO}^2=\overline{AO}^2-\overline{TC}^2 \text{ .... } (1).$$

$$\overline{BP}^2=BF \cdot BE=\overline{BO}^2-\overline{FO}^2=\overline{BO}^2-\overline{CP}^2 \text{ .... } (2).$$

Now, $AO : OT :: AD : OD$;

$\therefore AO \cdot OD=OT \cdot AD$.

And, since $OD=OB$, $OT=TC=CP$, and $AD=AT+TD=AT+BP$,

$$\therefore AT \cdot TC+CP \cdot BP=AO \cdot OB \text{ .... } (3).$$

Adding (1), (2), and $2\times(3)$,

$$\overline{AT}^2+\overline{BP}^2+2AT \cdot TC+2CP \cdot BP=\overline{AO}^2-\overline{TC}^2+\overline{BO}^2-\overline{CP}^2+2AO \cdot OB;$$

$$\therefore \overline{AT}^2+2AT \cdot TC+\overline{TC}^2+\overline{BP}^2+2BP \cdot CP+\overline{CP}^2=\overline{AO}^2+2AO \cdot OB+\overline{BO}^2;$$

$$\therefore \overline{AC}^2+\overline{BC}^2=\overline{AB}^2.$$

XXX. Let ABC be a triangle, right-angled at C. Describe a circle, such that its center O shall be in one of the legs, as AC, and to which the other leg l hypotenuse shall be tangent.

Then $\overline{AD}^2=AE\cdot AC=\overline{AC}^2-2AC\cdot OE$;

$\overline{BD}^2=\overline{BC}^2$;

Adding, $\overline{AD}^2+\overline{BD}^2=\overline{AC}^2+\overline{BC}^2-2AC\cdot OE$.

$\therefore\ \overline{AD}^2+2AC\cdot OE+\overline{BD}^2=\overline{AC}^2+\overline{BC}^2$.

Fig. 23.

Now $AC:AC::OD(=OE):BC(=BD)$;

$\therefore\ AD\cdot BD=AC\cdot OE$; $\therefore\ \overline{AD}^2+2AD\cdot BD+\overline{BD}^2=\overline{AB}^2=\overline{AC}^2+\overline{BC}^2$.

XXXI. Fig. 23.

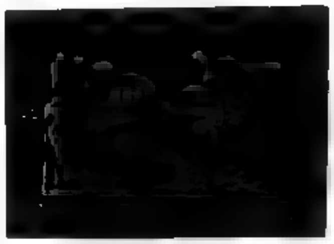
$\overline{AD}^2(=(AB-BD)^2)=\overline{AC}^2-2AC\cdot OE$,

$\therefore\ \overline{AB}^2-2AB\cdot BD+\overline{BD}^2=\overline{AC}^2-2AC\cdot OE$.

Adding $\overline{BD}^2=\overline{BC}^2$, $\overline{AB}^2-2BD\cdot AD=\overline{AC}^2+\overline{BC}^2-2AC\cdot OE$.

$\therefore\ \overline{AB}^2=\overline{AC}^2+\overline{BC}^2$.

XXXII. Let ABC be a triangle right-angled at C. Draw AE parallel to and $=AC$. With O, the middle of AE, as enter, describe a circle, to which both AC l BC shall be tangent. Then,



$(=(a-b)^2)=BD\cdot BA=c(c-AD)$........(1).

Also, $AD:a::2b:c$.

$\therefore\ AD=2ab/c$ ....(2).

in (1), $(a-b)^2=c^2-2ab$. $\therefore\ c^2=a^2+b^2$.

Fig. 24.

From unequal sides about the right angle pass to equal sides by the theory imits.

[To be Continued.]

TWO DEVELOPMENTS.

By E. D. ROE, JR., Associate Professor of Mathematics in Oberlin College, Oberlin, Ohio.

[A paper read at the January meeting of the American Mathematical Society.]

It is desired to call attention here to two developments, whose statement and discussion the writer has no where met, except for $n=2$, and then not from the point of view to be suggested here. Yet it is quite possible that they may be found elsewhere.*

I. Formulas.

If $u=f(x_1, x_2, \ldots\ldots x_n)$ is a function of n variables, Δu will be used to denote the total increment in the function due to a change in all the variables, while $\Delta_{x_1, x_2 \ldots\ldots x_r} u$ will denote an increment due to a change in r variables. The two developments are as follows:

1. $$u+\Delta u=u+\Delta_{x_1}u+\Delta_{x_2}u+\ldots\ldots\Delta_{x_n}u$$
$$+\Delta_{x_1}\Delta_{x_2}u+\Delta_{x_1}\Delta_{x_3}u+\ldots\ldots\Delta_{x_1}\Delta_{x_2}\ldots\ldots\Delta_{x_n}u$$

$$=u+\sum_{r=1}^{r=n}{}^{r}\sum_{\alpha_1=1}^{\alpha_1=n-r+1}{}^{\alpha_1}\sum_{\alpha_2=\alpha_1+1}^{\alpha_2=n-r+2}{}^{\alpha_2}\ldots\ldots$$

$$\cdot\sum_{\alpha_{r-1}=\alpha_{r-2}+1}^{\alpha_{r-1}=n-1}{}^{\alpha_{r-1}}\sum_{\alpha_r=\alpha_{r-1}+1}^{\alpha_r=n}{}^{\alpha_r}\Delta_{x_{\alpha_1}}\Delta_{x_{\alpha_2}}\ldots\ldots\Delta_{x_{\alpha_r}}u.$$

Or in determinant notation,

$$\begin{vmatrix}\Delta & 1\\ -1 & 1\end{vmatrix}u=\begin{vmatrix}\Delta_{x_1} & 1 & 1\ldots\ldots\ldots 1 & 1\\ -1 & \Delta_{x_2} & 1\ldots\ldots\ldots 1 & 1\\ -1 & -1 & \cdot\;. \qquad \vdots & \vdots\\ \ldots\ldots\ldots & & \cdot\;.\Delta_{x_n} & 1\\ -1 & -1 & \ldots\ldots -1 & 1\end{vmatrix}u$$

*Since the above was read Professor Fiske of Columbia, has informed the writer that "Dr. McClintock called attention to the fact, that the first result is contained in a general formula which he gave in Vol. II. page 116 of the *American Journal of Mathematics*. His formula (77) reduces in a special case to

$$\phi(E)=\phi(E_{x_1}E_{x_2}\ldots\ldots E_{x_n})\text{ where }E=1+\Delta.$$

His operation S reduces to E when

$$\Psi=1."$$

$$=(1+\Delta_{x_1})(1+\Delta_{x_2})\ldots\ldots(1+\Delta_{x_n}u)=\overset{r=n}{\underset{r=1}{\pi}}r(1+\Delta_{x_r})u,$$

so that as operators $(1+\Delta)=\overset{r=n}{\underset{r=1}{\pi}}r(1+\Delta_{x_r})$, or the operator $1+\Delta$ is developable into the product of n operators.

2. $$u-\int du=u-\int d_{x_1}u-\int d_{x_2}u\ldots\ldots-\int d_{x_n}u+\int\int d_{x_1}d_{x_2}u+\int\int d_{x_1}d_{x_3}u+$$

$$\ldots\ldots+(-1)^n\int\int\ldots\ldots\int d_{x_1}d_{x_2}\ldots\ldots d_{x_n}u$$

$$=u+\sum_{r=1}^{r=n}{}^{r}\sum_{\alpha_1=1}^{\alpha_1=n-r+1}{}^{\alpha_1}\sum_{\alpha_2=\alpha_1+1}^{\alpha_2=n-r+2}{}^{\alpha_2}\ldots\ldots$$

$$\sum_{\alpha_r=\alpha_{r-1}+1}^{\alpha_r=n}{}^{\alpha_r}(-1)^r\int\int\ldots\ldots\int d_{x_{\alpha_1}}d_{x_{\alpha_2}}\ldots\ldots d_{x_{\alpha_r}}u.$$

Or in determinant notation,

$$\begin{vmatrix}-\int d & 1\\ -1 & 1\end{vmatrix}u=\begin{vmatrix}-\int d_{x_1} & 1 & 1\ldots\ldots 1 & 1\\ -1 & -\int d_{x_2} & 1\ldots\ldots 1 & 1\\ -1 & -1 & \vdots & \vdots\\ \ldots\ldots & \ldots\ldots & -\int d_{x_n} & 1\\ -1 & -1\ldots\ldots & -1 & 1\end{vmatrix}u$$

$$=(1-\int d_{x_1})(1-\int d_{x_2})\ldots\ldots(1-\int d_{x_n})u=\overset{r=n}{\underset{r=1}{\pi}}r(1-\int d_{x_r})u=0,$$

so that as operators $(1-\int d)=\overset{r=n}{\underset{r=1}{\pi}}r(1-\int d_{x_r})$, or the operator $1-\int d$ is developable into the product of n operators.

II. Proofs.

1. Let $u=f(x_1)$, then $\Delta u=\Delta_{x_1}u$, and $u+\Delta u=u+\Delta_{x_1}u=(1+\Delta_{x_1})u$.

Let $u=f(x_1x_2)$, then by the preceding, $u+\Delta_{x_2}u=(1+\Delta_{x_2})u$.

Apply the operator $1+\Delta_{x_1}$ to this equation, and

$$u+\Delta_{x_2}u+\Delta_{x_1}u+\Delta_{x_1}\Delta_{x_2}u=(1+\Delta_{x_1})(1+\Delta_{x_2})u.$$

By writing out the left member,

$$f(x_1,x_2)+f(x_1,x_2+\Delta_{x_2})-f(x_1,x_2)+f(x_1+\Delta x_1,x_2)-f(x_1x_2)+f(x_1+\Delta x_1,x_2+\Delta x_2)$$
$$-f(x_1,x_2+\Delta x_2)-f(x_1+\Delta x_1,x_2)+f(x_1,x_2)$$
$$=f(x_1x_2)+f(x_1+\Delta x_1,x_2+\Delta x_2)-f(x_1,x_2)=u+\Delta u.$$

The same process would show that by applying $(1+\Delta_{x_2})$ first, $(1+\Delta_{x_1})$ second, we would also get $u+\Delta u$. Hence would follow commutation of the two operators, so that

$$(1+\Delta_{x_1})(1+\Delta_{x_2})=(1+\Delta_{x_2})(1+\Delta_{x_1})=1+\Delta,$$

or if $u=f(x_1, x_2, \ldots\ldots x_n)$ we have proved that

$$(1+\Delta_{x_i})(1+\Delta_{x_k})=(1+\Delta_{x_k})(1+\Delta_{x_i})=1+\Delta_{x_i x_k},$$

i. e., any two of these operators are commutative. It follows $(n=2)$ that

$$u+\Delta_{x_2}u+\Delta_{x_1}u+\Delta_{x_1}\Delta_{x_2}u=u+\Delta_{x_1}u+\Delta_{x_2}u+\Delta_{x_2}\Delta_{x_1}u,$$

and since the ordinary addition is commutative, $\Delta_{x_1}\Delta_{x_2}u=\Delta_{x_2}\Delta_{x_1}u$, a familiar result.

Assume for $u=f(x_1,x_2\ldots\ldots x_n)$, that $u+\Delta u=(1+\Delta_{x_1})\ldots\ldots(1+\Delta_{x_n})u$.

Let $U=F(x_1, x_2\ldots\ldots x_{n+1})$.

Then by the assumption

$$U+\Delta_{x_1\ldots\ldots x_n}U=(1+\Delta_{x_1})(1+\Delta_{x_2}\ldots\ldots(1+\Delta_{x_n})U.$$

Apply $(1+\Delta_{x_{n+1}})$ to both sides of this equation remembering that we have proved commutation of operators. We get,

$$U+\Delta_{x_1\ldots\ldots x_n}U+\Delta_{x_{n+1}}U+\Delta_{x_{n+1}}\Delta_{x_1\ldots\ldots x_n}U=(1+\Delta_{x_1})\ldots\ldots(1+\Delta_{x_{n+1}})U.$$

By working out the left member, it becomes $U+\Delta U$, hence the next case takes the same form with respect to $n+1$, that the assumption had with respect to n, and since the assumption is true for $n=2$, it is true universally. The commutation of any two operators brings with it the proof that $\Delta_{x_1}\Delta_{x_2}\ldots\ldots\Delta_{x_r}u$ is equal to any other one of $r!$ orders in which the r operations might be brought about. The determinant form was suggested by the formula for the development of a determinant in terms of the elements of its principal diagonal, a formula which has the same limits and number of operators in the summation. It is easily shown by adding to each column the elements of the last, when it reduces to one term, its principal diagonal term.

2. The second formula is easily shown after it has been proved for two variables. Let $u=f(x_1, x_2)$. Now u may be composed linearly of a constant, a function of x_1 alone, a function of x_2 alone, and a function of (x_1, x_2). The last must always be present, though the others may be wanting. About the constant we are not here concerned. Neglecting it,

$$u=f_1(x_1)+f_2(x_2)+\phi(x_1, x_2).$$

$$d_{x_1}u=[f'(x_1)+D_{x_1}\phi]d_{x_1},\quad d_{x_2}u=[f_2'(x_2)+D_{x_2}\phi]d_{x_2},$$

$$d_{x_1}d_{x_2}u=D_{x_1}D_{x_2}\phi dx_1dx_2.\quad \int d_{x_1}u=f_1(x_1)+\phi(x_1, x_2),$$

$$\int d_{x_2}u=f_2(x_2)+\phi(x_1, x_2).\quad \int\int d_{x_1}d_{x_2}u=\int\int D_{x_1}D_{x_2}\phi d_{x_1}d_{x_2}=\phi(x_1, x_2).$$

Hence without a constant,

$$u=\int d_{x_1}u+\int d_{x_2}u-\int\int d_{x_1}d_{x_2}u.$$

i. e., we have here complete indefinite integral of du, and

$$(1-\int d)=(1-\int d_{x_1})(1-\int d_{x_2})u=0,$$

where it is evident that there is a commutation of operators, since the ordinary additions are commutative, and $d_{x_1}d_{x_2}u=d_{x_2}d_{x_1}u$.

Assume for $u=f(x_1, x_2\ldots\ldots x_n)$, that

$$(1-\int d_{x_1})(1-\int d_{x_2})\ldots\ldots(1-\int d_{x_n})u=0.\quad \text{Let } U=F(x_1, x_2\ldots\ldots x_{n+1}).$$

Then $U=\int d_{x_1}U+\phi(x_2, x_3\ldots\ldots x_{n+1})$, where ϕ is an arbitrary function of all the variables but x_1. By the assumption

$$(1-\int d_{x_2})(1-\int d_{x_3})\ldots(1-\int d_{x_{n+1}})\phi=0,\ \text{also}\ \phi=U-\int d_{x_1}U=(1-\int d_{x_1})U.$$

Apply the operator $(1-\int d_{x_2})\ldots\ldots(1-\int d_{x_{n+1}})$ to both sides of this equation, remembering that commutation of operators has been proved. We get

$$(1-\int d_{x_1})\ldots\ldots(1-\int d_{x_{n+1}})\phi=(1-\int d_{x_1})\ldots\ldots(1-\int d_{x_{n+1}})U=0,$$

since the left member is zero; also the last has the same form with respect to $n+1$, that the assumption had with respect to n, and since the assumption is true when $n=2$, it is universally true.

III. Applications.

1. An elegant application of the first development is its use in demonstrating that the total differential of a function is equal to the sum of its partial differentials. Let $u=f(x_1, x_2 \ldots\ldots x_n)$. Then

$$\Delta u=\Delta_{x_1}u+\Delta_{x_2}u+\ldots\ldots\Delta_{x_n}u+\Sigma\Delta_{x_i}\Delta_{x_k}+\ldots\ldots\Sigma\Delta_{x_1}\Delta_{x_2}\ldots\ldots\Delta_{x_n}u.$$

Multiply through by n, where n is a number which becomes indefinitely great as the principal increment becomes indefinitely small, but so that $\lim_{\Delta x_r \doteq 0}\left(n\Delta_{x_r}u\right)=$ a finite number, which is called by Hamilton, Serret, and J. M. Pierce, differential of u with respect to x_r, and may be denoted by $d_{x_r}u$. Consider $\lim_{\substack{\Delta x_r \doteq 0\\ \Delta x_k}}\left(n\Delta_{x_r}\Delta_{x_k}u\right)$. This equals $\lim_{\substack{\Delta x_r \doteq 0\\ \Delta x_k}}\left(\Delta_{x_r}n\Delta_{x_k}u\right)=\lim_{\Delta x_r \doteq 0}\Delta_{x_r}\left(d_{x_k}\right)=0.$

A *fortiori*, zero will be the limit of any term of higher order. We have then at once by taking the limits of both members,

$$du=d_{x_1}+d_{x_2}u+\ldots\ldots d_{x_n}u.$$

2. The second formula gives us the complete solution of a differential equation, when it is a perfect differential. The solution is,

$$u=\int d_{x_1}u+\int d_{x_2}u+\ldots\ldots\int d_{x_n}u-\sum\int\int d_{x_i}d_{x_k}u+\sum\int\int\int d_{x_i}d_{x_k}d_{x_j}u\ldots$$

$$+(-1)^{n-1}\int\int\ldots\ldots\int d_{x_1}d_{x_2}\ldots\ldots d_{x_n}u.$$

To this a constant may be added which is determined as usual by corresponding values of the variables and function. It may or may not be zero. When the function is such that farther differentiation of the first differentials will cut them down rapidly, this formula ought to be practically useful. When $n=2$, we can by this formula solve the problem of finding the orthogonal and isothermal curves to a given system, $u=c$, when u satisfies the equation $D_x{}^2u+D_y{}^2u=0$.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

73. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.

A man owes me \$100 due in 2 years, and I owe him \$200 due in 4 years. When can I pay him \$100 to settle the account equitably, money being worth 6%, and the interest to draw interest until the time of settlement?

Solution by FREDERIC R. HONEY, New Haven, Connecticut.

One dollar placed at 6% compound interest, in two years will amount to 1.06^2=\$1.1236. Therefore the present value of \$100.00 due in 2 years is \$100.00÷1.1236=\$89.00 *very nearly*.

One dollar placed at 6% compound interest in four years will amount to 1.06^4=\$1.2625. Therefore the present value of \$200.00 due in 4 years is \$200.00÷1.2625=\$158.416.

Therefore the difference between \$158.416 and \$89.00=\$69.416, is the amount of my debt to A at the present time.

Since \$1.00 placed at 6% compound interest in 6 years will amount to 1.06^6=\$1.4185, \$69.416 at the same rate will, in 6 years, amount to 69.416×1.4185=\$98.4666.

And since the simple interest on one dollar for 1 year is \$0.06, the simple interest on \$98.4666 is 98.4666×0.06=\$5.908 for one year. Therefore the interest \$100.00−\$98.4666=\$1.5334 will accrue in 1.5334÷5.908=0.2594 years.

And 6.+0.2594=6.2594 the number of years hence when \$100.00 should be paid, in order to settle the account equitably.

J. M. Bandy sent solutions of Nos. 71 and 72 too late for credit in February number.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

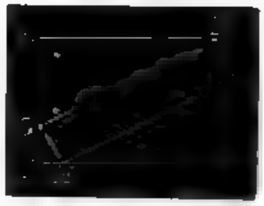
SOLUTIONS OF PROBLEMS.

67. Proposed by F. M. PRIEST, St. Louis, Mo.

Required: The length of a piece of carpet that is a yard wide with square ends, that can be placed diagonally in a room 40 feet long and 30 feet wide, the corners of the carpet just touching the walls of the room.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas; P. S. BERG, Larimore, North Dakota; J. SCHEFFER, A. M., Hagerstown, Maryland; J. M. COLAW, A. M., Monterey, Virginia; R. H. WAGONER, Westerville, Ohio; J. F. YOTHERS, Westerville, Ohio; J. T. FAIRCHILD, Crawfis College, Ohio; CHAS. C. CROSS, Laytonsville, Maryland; and O. S. WESTCOTT, Chicago, Illinois.

Let $AB=40=a$, $BC=30=b$, $EF=3=c$, $BF=x$, $BE=y$.



$\therefore\ x^2+y^2=c^2$ ....(1).

From the triangles CHF and BEF we get $HC : CF=BF : BE$ or $a-y : b-x=x : y$.

$\therefore\ ay-y^2=bx-x^2$ ....(2).

(1) in (2) gives $bx-x^2=a\sqrt{c^2-x^2}-c^2+x^2$.

$\therefore\ 4x^4-4bx^3+(a^2+b^2-4c^2)x^2+2bc^2x+c^4-a^2c^2=0$.

$\therefore\ 4x^4-120x^3+2464x^2+540x-14319=0$.

$\therefore\ x=2.43372+,\ y=1.75414+$.

$HG=\{(a-y)^2+(b-x)^2\}^{\frac{1}{2}}=47.14494+$.

II. Solution by COOPER D. SCHMITT, Professor of Mathematics, University of Tennessee, Knoxville, Tennessee; W. H. HARVEY, Portland, Maine; and B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

In the figure used above, let $AB=DC=a=40$ feet, the length of the room.

$AD=BC=b=30$ feet, the width of the room.

$EF=GH=c=3$ feet, the width of the carpet.

Let $x=HF=GE$, the length of the carpet, and the angle $CFH=$the angle $HGD=\theta$.

Then $x\sin\theta=CF$, $c\sin\theta=DH$, $x\cos\theta=HC$, and $c\cos\theta=DG$.

$\therefore\ c\sin\theta+x\cos\theta=DC=a$ ....(1),

and $x\sin\theta+c\cos\theta=BC=b$ ....(2).

Multiplying (1) by (2), and collecting, we get

$cx(\sin^2\theta+\cos^2\theta)+(x^2+c^2)\sin\theta\cos\theta=ab$, or

$$cx+(x^2+c^2)\sin\theta\cos\theta=ab \quad\text{.............(3).}$$

Squaring (1) and (2) and adding the results, we get

$$c^2+x^2+4cx\sin\theta\cos\theta=a^2+b^2 \quad\text{....................(4).}$$

From (3), $\sin\theta\cos\theta=(ab-cx)/(x^2+c^2)$. Substituting this value of $\sin\theta\cos\theta$ in (4) and reducing, we get,

$$x^4-(a^2+b^2+2c^2)x^2+4abcx-c^2(a^2+b^2-c^2)=0 \quad\ldots\ldots\ldots\ldots(5).$$

Restoring numbers in (5), we have

$$x^4-2518x^2+14400x-22419=0.$$

Solving this equation by Horner's Method, we find $x=47.145$ *feet*, nearly.

CALCULUS.

[Co]nducted by J. M. COLAW, Monterey, Virginia. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

58. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington,

A line passes through a fixed point and rotates uniformly about this point. Another passes through a point which moves uniformly along the arc of a given curve and ro[tate]s uniformly about this point. Develop a method for finding the locus of intersection [of t]hese two lines. Apply to case of circle and straight line.

II. Solution by G. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, [Mass]achusetts.

Let O be the origin, P_3 the fixed point, its coördinates being (r_3, θ_3), and AB be a given position of line through P_3. Let $P_1(r_1, \theta_1)$ be position of [poi]nt on curve and CD the line through it, both cor[res]ponding to the position AB of other line. Also let [KH] be position of AB revolved through an $\angle\phi$, and $P_2(r_2, \theta_2)$ and EF be the corresponding position [of P]$_1$ and CD.

Let $r=f(\theta)$ be equation to curve P_1P_2. Let [η be] the angle made by AB, and η_1 the one made by CD with a polar axis. Let [a be] angular rate of revolution of AB, and na of CD.

$\therefore\ \angle$ between CD and $EF=n\phi$.

Let b=linear rate of movement of P_1. Then $\phi/a=P_1P_2/b$ .... (1).

Equation to KH is $r=[r_3\sin(\eta+\phi-\theta_3)]/\sin(\eta+\phi-\theta)$ ....(2).

Equation to EF is $r=(r_2\sin(\eta_1+n\phi-\theta_2)]/\sin(\eta_1+n\phi-\theta)$..........(3).

By integration,

$$P_1P_2=\int_{\theta_1}^{\theta_2}\sqrt{r^2+[(dr/d\theta)]^2}\,d\theta=\int_{\theta_1}^{\theta_2}\sqrt{[f(\theta)]^2+[f'(\theta)]^2}\,d\theta \ldots\ldots\ldots\ldots\ldots (4),$$

which gives P_1P_2 in terms of θ_2, θ_1 being known. Then substitute from (4) in (1) to get ϕ in terms of θ_2. Substitute this value, and also $f(\theta_2)$ for r_2 in (2) and (3). Then by eliminating (θ_2) we have resulting the equation to the locus of the intersecting of the lines. The solution depends on our ability to integrate (4). Now if the given lines are not straight, it is evident that the only changes are in equations (2) and (3). These may be derived from the equations to the lines in original position by a method of transformation of coördinates. For example, the equation to HK may be derived from that to AB by revolving the pole and polar axis about P_2 through an angle equal to ϕ and in the opposite direction. If the given curve is a circle and the lines straight, the problem can be definitely solved as follows:

Transform coördinates so that center of circle shall be pole and OP_1 the the polar axis. Then $r=f(\theta)$ becomes $r=c$.

Let $(r_1, \theta_1)(r_2, \theta_2)(r_3, \theta_3)$ represent the new coördinates of points P_1, P_2, P_3, respectively.

Then $r_1=r_2=c$. $\theta_1=0$. $P_1P_2=c\theta_2$. From (1) $\phi=ac\theta_2/b$.

(2) becomes $r=[r_3\sin(\eta+ac\theta_2/b-\theta_3)]/\sin(\eta+ac\theta_2/b-\theta)$ ....(5).

(3) becomes $r=[c\sin(\eta_1+nac\theta_2/b-\theta_2)]/\sin(\eta_1+nac\theta_2/b-\theta)$ ....(6).

(5) can be solved for θ_2 and the result can be substituted in (6), giving the equation required. Then if desired the coördinates can be again transformed to the original form.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

43. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

Two weights P and Q rest on the concave side of a parabola whose axis is horizontal, and are connected by a string, length l, which passes over a smooth peg at the focus, F. [*Bowser's Analytical Mechanics*, page 54.]

I. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi; COLMAN BANCROFT, M. Sc., Professor of Mathematics, Hiram College, Hiram, Ohio; GEORGE LILLEY, LL. D., Portland, Oregon; G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, and the PROPOSER.

Let AX be the horizontal axis of the parabola, F the focus, P' and Q' the positions of the weights P and Q, T the tention of the string $P'FQ'$, N and N' the normal reactions at P' and Q' respectively, θ and θ' the angles between the axis AX and the focal radii to P' and Q' respectively, A the intersection of the axis and the tangent at P'. Denote the latus rectum by $4m$.

$$\angle\theta=\angle FAP'+\angle FP'A=2\angle FAP',$$

by property of parabola.

$\therefore\ \angle FAP'=\frac{1}{2}\theta.$

Since N is inclined to the vertical at the same angle that the tangent is inclined to the horizontal, we have for equilibrium of the forces at P', resolving vertically and horizontally,

$N\cos\frac{1}{2}\theta+T\sin\theta=P$, and $N\sin\frac{1}{2}\theta=T\cos\theta$,

from which

$T(\cot\frac{1}{2}\theta\cos\theta+\sin\theta)=P$, or, $T\cot\frac{1}{2}\theta=P$.

Similarly, $T\cot\frac{1}{2}\theta'=Q$. $\therefore\ \cot\frac{1}{2}\theta=P/Q\cot\frac{1}{2}\theta'$.

From the polar equation of a parabola,

$$FP'=\frac{2m}{1-\cos\theta},\quad FQ'=\frac{2m}{1-\cos\theta'}.$$

But $FP'+FQ'=l$. $\therefore\ \dfrac{2m}{1-\cos\theta'}=l-\dfrac{2m}{1-\cos\theta'}$, $\cos\theta'=1-\dfrac{2m(1-\cos\theta)}{l(1-\cos\theta)-2m}$,

$$\sin^2\tfrac{1}{2}\theta'=\frac{m(1-\cos\theta)}{l(1-\cos\theta)-2m},\quad \cot\tfrac{1}{2}\theta'=\sqrt{\frac{l(1-\cos\theta)+m\cos\theta-3m}{m(1-\cos\theta)}}.$$

Then, $\cot\frac{1}{2}\theta=\dfrac{P}{Q}\sqrt{\dfrac{l(1-\cos\theta)+m\cos\theta-3m}{m(1-\cos\theta)}}$. Since $1-\cos\theta=\dfrac{2}{\cot^2\frac{1}{2}\theta+1}$, the preceding equation gives $\cot\frac{1}{2}\theta=\dfrac{P\sqrt{l-2m}}{\sqrt{m(P^2+Q^2)}}$.

II. Solution by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

If, in the figure above, θ represents the angle XFP, and θ_1, XFQ, then

$$FP=r=\frac{2m}{1-\cos\theta}=\frac{m}{\sin^2\frac{1}{2}\theta}.\quad \therefore\ \sin^2\tfrac{1}{2}\theta=\frac{m}{r},\ \text{and}\ \cos^2\tfrac{1}{2}\theta=\frac{r-m}{r}.$$

Since $FQ=l-r$, $\sin^2\frac{1}{2}\theta_1=\frac{m}{l-r}$, and $\cos^2\frac{1}{2}\theta_1=\frac{l-r-m}{l-r}$.

Since the tension is the same in all parts of the string and the angle between the radius vector and tangent is half the angle between the radius vector and the X axis, $T=P\tan\frac{1}{2}\theta=Q\tan\frac{1}{2}\theta_1$.

$$\therefore \frac{P}{Q}=\frac{\text{ctn}\frac{1}{2}\theta}{\text{ctn}\frac{1}{2}\theta_1}, \quad \therefore \frac{P^2}{P^2+Q^2}=\frac{r-m}{l-2m}=\frac{m\,\text{ctn}^2\frac{1}{2}\theta}{l-2m},$$

$$\therefore \text{ctn}\tfrac{1}{2}\theta=\frac{P\sqrt{l-2m}}{\sqrt{m(P^2+Q^2)}}.$$

III. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

Let r', r'' be the parts of the string l joining the focus and the weights P and Q; θ and θ' the angles which r' and r'' make with the axis of X.

For the equilibrium of P and Q, resolving along the tangents through P and Q, T being the tension in the string,

$T\sin\frac{1}{2}\theta=P\cos\frac{1}{2}\theta$...............(1). $T\sin\frac{1}{2}\theta'=Q\cos\frac{1}{2}\theta'$..... .... (2).

These give $P/Q\cot\frac{1}{2}\theta=\cot\frac{1}{2}\theta'$........... ....(3).

The equations to the curve are

$$r'=\frac{2m}{1+\cos\theta}, \quad r''=\frac{2m}{1+\cos\theta'};$$

then $r'+r''=2m\left(\frac{1}{1+\cos\theta}+\frac{1}{1+\cos\theta'}\right)=l$..... ....(4).

Now $\cos\theta=\frac{\cot^2\frac{1}{2}\theta-1}{\cot^2\frac{1}{2}\theta+1}$, $\cos\theta'=\frac{\cot^2\frac{1}{2}\theta'-1}{\cot^2\frac{1}{2}\theta'+1}$ ....(5).

(3) and (5) and the resulting values of $\cos\theta$ and $\cos\theta'$ in (4) and reducing gives

$$\frac{2P^2\cot^2\frac{1}{2}\theta+(P^2+Q^2)}{2P^2\cot^2\frac{1}{2}\theta}=\frac{l}{2m}\text{.................. .(6).}$$

Subtracting unity from both members of (6) and taking the square root of the result,

$$\tan\tfrac{1}{2}\theta=\frac{P}{\sqrt{P^2+Q^2}}\sqrt{\frac{l-2m}{m}}\text{................(7).}$$

In which put $\pi-\theta$ for θ for Bowser's result.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

43. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

In a circle whose radius is a, chords are drawn through a point distant b from the center. What is the average length of such chords, (1), if a chord is drawn from every point of the circumference, and (2), if they are drawn through the point at equal angular intervals?

Solution by the PROPOSER.

In the figure, let BC represent the chord passing through the point A whose distance from O is $OA=b$. Put $BC=x$, $\angle BOA=\theta$, $\angle BAO=\phi$, A_1=first average required, and A_2=second average required.

Then $x=2(a^2-b^2\sin^2\phi)^{\frac{1}{2}}$.

Hence, $A_1=1/\pi\int_0^\pi xd\theta$. From triangle AOB,

$a\sin(\theta+\phi)=b\sin\phi$. $\therefore\ \theta+\phi=\sin^{-1}(b/a\sin\phi)$.

$$\therefore\ d\theta=-d\phi+\frac{b\cos\phi d\phi}{(a^2-b^2\sin^2\phi)}.$$

$$\therefore\ A_1=2/\pi\int_0^\pi[(a^2-b^2\sin^2\phi)^{\frac{1}{2}}-b\cos\phi]d\phi=\frac{4a}{\pi}\ E\left(\frac{b}{a},\ \tfrac{1}{2}\pi\right).$$

$$A_2=1/\pi\int_0^\pi xd\phi=\frac{4a}{\pi}\int_0^{\frac{1}{2}\pi}[1-(b^2/a^2)\sin^2\phi]^{\frac{1}{2}}\,d\phi=\frac{4a}{\pi}\ E\left(\frac{b}{a},\ \tfrac{1}{2}\pi\right).$$

$\therefore\ A_1=A_2$. If $b=0$, $A_1=A_2=2a$. If $b=a$, $A_1=A_2=4a/\pi$.

This problem was also solved by G. B. M. Zerr. His solution will appear in the next issue.

44. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

What is the average length of all the chords that may be drawn from one extremity of the major axis of an ellipse if they are drawn at equal angular intervals?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas; J. SCHEFFER, A. M., Hagerstown, Maryland, and the PROPOSER.

Let r=length, e=eccentricity of ellipse, θ=angle the chord makes with major axis. Then

$$r=\frac{2a(1-e^2)\cos\theta}{1-e^2\cos^2\theta}.\quad \Delta=\text{average length}=\frac{\int_0^{\frac{1}{2}\pi}rd\theta}{\int_0^{\frac{1}{2}\pi}d\theta}.$$

$$\therefore\ \Delta=\frac{4a(1-e^2)}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\cos\theta d\theta}{1-e^2\cos^2\theta}.\quad \therefore\ \Delta\frac{4a\sqrt{1-e^2}}{\pi e}\tan^{-1}\frac{e}{\sqrt{1-e^2}}.$$

When $e=\theta$, $\Delta=\frac{4a}{\pi}$, since $\frac{1}{e}\tan^{-1}\frac{e}{\sqrt{1-e^2}}=1$.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Virginia. All contributions to this department should be sent t

SOLUTIONS OF PROBLEMS.

42. Proposed by E. B. ESCOTT, 6123 Ellis Avenue, Chicago, Illinois.

To find a triangle whose sides and median lines are commensurable.

Solution by J. W. TESCH, in "L'Intermediare des Mathematiciens" for October, 1896. Translat adapted by J. M. COLAW, A. M., Monterey, Virginia.

Suppose the sides to be $2a$, $1+2b-b^2+\frac{1}{4}a^2$, $1-2b-b^2+\frac{1}{4}a^2$; then we have $m_1=\pm(1+b^2-\frac{1}{4}a^2)$,

$m_2{}^2=\frac{1}{2}[4a^2+(1-2b-b^2+\frac{1}{4}a^2)^2]-\frac{1}{4}(1+2b-b^2+\frac{1}{4}a^2)^2$,

or $m_2{}^2=\frac{1}{64}a^4+\frac{1}{8}(17-6b-b^2)a^2+\frac{1}{4}(1-12b+2b^2+12b^3+$

In order that the second member may be a perfect square, it is nece that $\frac{1}{64}(17-6b-b^2)^2=4\times\frac{1}{64}\times\frac{1}{4}(1-12b+2b^2+12b^3+b^4)$, whence $2b=3$.

Thus the sides become $2a$, $\frac{7}{4}+\frac{1}{4}a^2$, $-\frac{17}{4}+\frac{1}{4}a^2$, or

(1) $16a$, $2(a^2+7)$, $\pm2(a^2-17)$; $m_1=\pm2(a^2-13)$, $m_2=a^2+23$.

We will have $m_3{}^2=a^4+190a^2-191$. The values of a, which rende second number a perfect square, are 1, 3, 5, ... ; $m_3=0$, 40, 72, ... none of these values satisfy (1); therefore, after the method of Euler, (*Voll dige Anleitung zur Algebra*, or the French translation by J.-G. Garnier, 2 with the additions of Lagrange, Paris, 1807), it is necessary to proceed as foll

By supposing $a=3+h$, we may write

$$(3+h)^4+190(3+h)^2-191=(40+ph+h^2)^2,$$

where p is a coefficient to be determined. Developing, we have

$$1248+244h+12h^2=80p+(80+p^2)h+2ph^2.$$

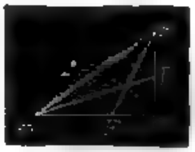
If we take $80p=1248$ or $p=15+3/5$, we find $h=-(4+2/15)$, which gives $=(17/15)^2$. We also have for the three sides, after some easy reductions: 0, 466, 884, and for the medians 659, 683, 208. This is perhaps the simplest e in whole numbers.

43. Proposed by H. C. WILKES, Skull Run, West Virginia.

To find, if possible, a right angled triangle, the bisectors of the acute angles hich, can be expressed by integral whole numbers.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let ABC be a right triangle, right angled at C, AD the bisector of $\angle A$, BE the bisector of $\angle B$.



Put $BC=a$, $AC=b$, $AB=c$, $DC=a_1$, $EC=b_1$, EB , and $AD=c_2$. Then $BD=a-a_1$, and $AE=b-b_1$. n geometrical relations we obtain $a^2+b^2=c^2$......(1); $=ac-b_1(b-b_1)$......(2); $b-b_1 : b_1=c : a$......(3).

From (3) we get $b : b_1=c+a : a$; whence $b_1=ab/$ $a)$, and $b-b_1=bc/(c+a)$.

$\therefore c_1^2=ac-ab^2c/(c+a)^2=ac-ac(c^2-a^2)/(c+a)^2=2a^2c/(c+a)$.

By a similar process, we find $c_2^2=2b^2c/(c+b)$.

From (1), $c^2-b^2=a^2$, or $(c+b)(c-b)=a^2$. Put $c+b=tp^2$ and $c-b=tq^2$. and q being any values. Then $a=tpq$, $b=t(p^2-q^2)/2$, and $c=t(p^2+q^2)/2$. ence $c_1^2=2t^2p^2q^2(p^2+q^2)/(p+q)^2$, and $c_2^2=t^2(p^2-q^2)^2(p^2+q^2)/4p^2$.

When $p^2+q^2=\square$, $c_2^2=\square$, and $c_1^2=2\times\square$. When $p^2+q^2=2\times\square$, $=2\times\square$, and $c_1^2=\square$.

$\therefore$ *Both* bisectors cannot be rational; *one* of them will be $\sqrt{2}$ times a num- when the *other* is a rational whole number.

II. Solution by the PROPOSER.

Let bx, by, and $x+y$ be, respectively, the sides and base of a right angled ngle, and let x and y be the greater and less segments of the base cut by the ctor. Then the bisector will be $\sqrt{y^2(b^2+1)}$ and if the bisector be integral, 1 must$=\square$. b must therefore be an improper fraction, and will always be quotient of the sum of the other two sides divided by the bisected side.

Now let CAB be a triangle, and let $AB=x^2+y^2$, $CA=x^2-y^2$ and $=2xy$, $CA+AB/CB=x/y$. $(x/y)^2+1$ may be a square, but $AB+CB/CA$ $(x+y)/(x-y)$. $[(x+y)/(x-y)]^2+1$ will be a multiple of the $\sqrt{2}$ and cannot be uare.

$\therefore$ If a rational right angled triangle have an integral bisector of one of its e angles, the bisector of the other acute angle must be a multiple of $\sqrt{2}$ and cannot be integral.

[Remark.—On page 155, Vol. II. of the MONTHLY, we have, when the s are 59.4107, 47.4072, 35.8067, the bisectors 40 and 50. It is doubtful ther the sides and bisectors both can be integral. ZERR.]

44. Proposed by P. S. BERG, Principal of Schools, Larimore, North Dakota.

Two trees whose heights are 40 and 80 feet, respectively, stand on opposite sides of a stream 30 feet wide. What path does a squirrel take in leaping from the top of the higher to the top of the lower? What is the length of the path?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

The path is a parabola. Let the top of the higher tree be origin, "across the river" positive, v=velocity, β=angle of projection, then the equation is $y=\tan\beta x-gx^2/(2v^2\cos^2\beta)$, in which we must know either v, or β. Substituting $x=30$, $y=-40$ we get

$$v^2=45g/(4\cos^2\beta+3\sin\beta\cos\beta)\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$S=\frac{1}{v^2\cos^2\beta}\int_0^{30}\sqrt{v^4\cos^4\beta+(gx-v^2\sin\beta\cos\beta)^2}\,dx.$$

Let $v^2\cos^2\beta=a$, $gx-v^2\sin\beta\cos\beta=y$, $30g-v^2\sin\beta\cos\beta=y_1$, $-v^2\sin\beta\cos\beta=-y_2$.

$$\therefore\ S=\frac{1}{ag}\int_{-y_2}^{y_1}\sqrt{a^2+y^2}\,dy$$

$$=\frac{1}{2ag}\{y_1\sqrt{a^2+y_1^2}+y_2\sqrt{a^2+y_2^2}\}+\frac{a}{2g}\log[(\sqrt{a^2+y_1^2}+y_1)/(\sqrt{a^2+y_2^2}-y_2)]$$

Let $\beta=45°$; then $v^2=90g/7$, $y_1=165g/7$, $y_2=45g/7=a$.

$$\therefore\ S=\tfrac{5}{14}(11\sqrt{130}+9\sqrt{2})+\tfrac{45}{14}\log[(\sqrt{130}+11)/(3\sqrt{2}-3)].$$

Let $\beta=0$; then $v^2=45g/4$, $y_1=30g$, $y_2=0$, $a=45g/4$.

$$\therefore\ S=5\sqrt{73}+\tfrac{45}{8}\log[(\sqrt{73}+8)/3].$$

NOTES.

INTERNATIONAL CONGRESS OF MATHEMATICIANS AT ZURICH IN 1897.

"It is known that the idea of an international congress of mathematicians has been, above all in these latter days, the object of numerous deliberations on the part of scientists interested in its realization. It has appeared to them, by reason of the excellent results obtained in other scientific domains by an international 'entente', that assuring the execution of this project would have very weighty advantages.

As outcome of a very active exchange of views, accord was reached on a e point. Switzerland, by its central geographic situation, by its traditions its experience of international congresses, appeared designated to invite a attempt at a reunion of mathematicians. In consequence Zurich is chosen ›at of the Congress.

The mathematicians of Zurich do not disguise from themselves the diffiies they will have to surmount. But in the interest of this enterprise, they thought it their duty not to decline the overtures so flattering that have been le them from all sides. They have decided therefore to take all preparatory isures for the future congress and, to the extent of their powers, to contribute ts success. So, with the concurrence of mathematicians of other nations, was ned the undersigned committee of organization, charged *to bring together at* *ich in* 1897 *the mathematicians of the entire world.*

The congress, in which you are cordially invited to take part, will e place at Zurich the 9, 10 and 11 of August, 1897, in the halls of the federal ytechnic school. The committee will not fail to communicate to you, in time ›ortune, the text of the program determined, begging you to inform them of ir adherence. But even at present it may be said that the scientific works l questions of administration will pertain to subjects of general interest recognized importance.

Scientific congresses have also this precious advantage, to favor and keep personal relations. The local committee will not fail to accord all its solici.e to this part of its task, and, with this aim, it will elaborate a modest prommе of fêtes and intimate reunions.

May the hopes reposed in this first congress be fully realized! May numus participants contribute by their presence to create, among colleagues, not ne coherent scientific relations, but also cordial bonds based on personal acuintance! Finally, may our congress serve the advancement and the progress the mathematical sciences!"

The invitation of which the above is a translation is signed by eleven from rich and ten associates, as committee.

Readers of the American Mathematical Monthly already know the pertent efforts of Vasiliev of Kazan and Laisant of Paris to establish this congress.

It is matter for rejoicing that their noble endeavors have been crowned h this definite success. George Bruce Halsted.

THE SAME OLD BLUNDER.

In the *Nation* of November 26, 1896, in a review of Cajori's History of mentary Mathematics, the reviewer himself makes a blunder so appalling that hould not go unnoticed.

He says Cajori "does not name Prof. J. J. Littrow of Vienna, whose demtration is yet worth notice. Littrow proves first that the three angles of a ngle are $=2R$. Thus: When a side a and angles BC are given, angle A is ermined; it is $=F(aBC)$; and as an angle may be viewed as an abstract num- it has no relation to one measure in space: angle $A=F(BC)$ simply," etc.

Now where has the *Nation's* reviewer been buried not to know that this very pseudo-proof was given by Legendre, and its fallacy shown by George Paxton Young in the *Canadian Journal* for November, 1856, forty years ago, and again in the *Canadian Journal* for July, 1860, pages 356-358 ?

GEORGE BRUCE HALSTED.

Austin, Texas.

James Joseph Sylvester, the great mathematician, Sivilian professor of geometry at Oxford, formerly professor of mathematics at the Johns Hopkins University, and in 1841 at the University of Virginia, died in London on March 15th, aged eighty-three years. Also the eminent mathematician Dr. Karl Weierstrass, died at Berlin on February 19th, aged eighty-one years. In the death of these two men, mathematics sustains a great loss. Both did much to broaden and deepen mathematical knowledge. Sylvester has written much on invariants, the theory of equations, theory of partitions, multiple algebra, the theory of numbers, the theory of reciprocants, etc., while Weierstrass has given special attention to the theory of functions of a complex variable. For a biography of Sylvester, by Dr. Halsted, see MONTHLY, Vol. I., No. 9. B. F. F.

BOOKS.

Composite Geometrical Figures. By George A. Andrews, A. M. 63 pages. Price 55 cents. Ginn and Company, Boston, and London. 1896.

The figures in this little book are constructed for the demonstration of more than one proposition. Ten of the figures are designed for review work, while the last general figure is intended for re-review work of all the theorems of plane geometry. Under each figure are illustrative demonstrations, followed by series of easy examples which require the pupil to apply the general principles of geometry to the specific conditions of the figures. It will be seen that the book is not designed to take the place of other text-books, but is intended to be used with them for reviews and for supplemental easy original work. The plan of the work is rather unique, and it will be useful to teachers who feel the need in their classes for the specific application of geometrical principles. J. M. C.

National Geographic Monographs: (1) *Physiographic Processes*, by J. W. Powell ; (2) *Physiographic Regions of the U. S.*, by J. W. Powell ; (3) *Lakes and Sinks of Nevada*, by I. C. Russell; (4) *Mt. Shasta*, by J. S. Diller. Price 20 cents each, or $1.50 for a set of ten. American Book Company, New York, Cincinnati, and Chicago.

These monographs on the physical features of the earth's surface furnish fresh and interesting material with which to supplement the regular text books. They have been written with exceptional care and ability and are not only very serviceable for such use but are very interesting to the general reader as well. J. M. C.

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BIOGRAPHY.

HOÜEL.

BY GEORGE BRUCE HALSTED.

GUILLAUME JULES HOÜEL, of a very old protestant family of Normandy, was born at Thaon (Calvados) on April 7th, 1823, and died at Périers, near Caen, June 14th, 1886.

The key to his whole mental life was this old protestant blood, which ans so much in a Roman catholic country.

After studying at the lyceum of Caen, and the college Rollin, he entered great Normal School of Paris in 1843. On leaving, he taught at Bourges, rdeaux, Pau, Alençon and Caen.

In 1855 he took his Doctor's degree at the Sorbonne, and then, declining overtures of Le Verrier to join the working force at the Observatory, he reed to his home at Thaon to continue his researches. In 1859 he was called to ceed Le Besgue in the chair of pure mathematics of the Faculty of Sciences of rdeaux. Here he found dignity and facilities for work, and considered the sition as final.

The idea of duty was the essence of his character, remarkably sweet and m. Not only in his official position did he forward science, he spread it with fusion all around him.

So precise and rigorous was his mind that he scorned Legendre's and irant's geometries, and the conventional neatness of the French texts, in favor he eternal geometer Euclid.

In 1863 he published at Greifswald an essay on the fundamental principles of geometry. In this he has already reached by himself the idea that a demonstration of the postulatum of Euclid is impossible. He says: "Since long, the scientific researches of mathematicians on the fundamental principles of elementary geometry have concentrated themselves almost exclusively on the theory of parallels; and if, hitherto, the efforts of so many eminent minds have produced no satisfactory result, it is perhaps permitted to conclude thence that in pursuing these researches they have followed a false path and attacked an insoluble problem, of which the importance has been exaggerated in consequence of inexact ideas on the nature and origin of the primordial verities of the science of space."

To the mind so self-prepared came an important communication in 1866 from Dr. R. Baltzer informing Hoüel of the fundamental idea of Lobachévski and Bolyai and announcing that Baltzer would mention it in the forthcoming second edition of his Elements of Geometry. That very year 1866 Hoüel issued his translation of Lobachévski's "Geometrische Untersuchungen zur Theorie der Parallellinien," and in the preface to his translation quotes from W. Bolyai's "Kurzer Grundriss eines Versuchs etc.," and mentions the work of J. Bolyai with date 1832. In this preface he says: "The aim of the author is to prove that there exists *à priori* no reason to affirm that the sum of the three angles of a rectilineal triangle is not less than two right angles, or, what comes to the same thing, that one cannot draw, through a given point more than a single straight not meeting a given straight in the same plane.

In spite of the high value of these researches, they have not hitherto drawn the attention of any geometer. We do not believe however that we exaggerate their philosophic import in saying that they throw a new day on the fundamental principles of geometry, and that they open a path yet unexplored capable of leading to unexpected discoveries. Not to go beyond elementary questions, one cannot deny that they accomplish an immense advance in methods of teaching by relegating among the chimeras the hope still nourished by so many geometers of demonstrating the postulatum of Euclid.

Henceforth these attempts must be ranked with the quadrature of the circle and perpetual motion."

He mentions the assumption, (three points are costraight or concyclic), given by W. Bolyai to replace Euclid's·

A translation of J. Bolyai was delayed until 1868 by Hoüel's inability to procure a copy of the now celebrated Appendix. How this difficulty was fortunately overcome I learned while in Hungary where my friend Franz Schmidt entrusted to me a precious file of Hoüel's own letters. From these letters it appears that a copy of Hoüel's 'Essai' of 1863 having come by chance into the hands of a young architect of Temesvár in Hungary, this youth (Franz Schmidt), desirous of continuing his mathematical studies wrote for counsel to Hoüel. Hoüel had answered helpfully, and later implored the aid of Schmidt to procure Bolyai's work, and besought Schmidt to collect what materials he could for a biography. This Schmidt did, and his article on Grunert's Archiv, 1868, mained, until my own researches and my journey to Hungary, the only source information on these wonderful Magyars. Schmidt succeeded in procuring for oüel two copies of Bolyai's work. One Hoüel proceeded to translate himself; e other he sent to Battaglini, asking him to make known in Italy this wonderl idea. This he did by an Italian translation. Thus to Hoüel belongs a perstly definite and permanent place in the final history of human thought.

Much else he did; so much that I could not attempt to enumerate it in e brief space at my disposal here. Fortunately it has been most sympathetilly done by M. G. Brunel in a book of 78 pages most obligingly furnished me r Hoüel's son-in-law, Monsieur H. Barckhausen.

M. Brunel cites on page 34 my Bibliography of Hyper-Space and Nonuclidean Geometry (1878), and also that published at Kiev in 1880 by ashtchenko-Zaharchenko, but omits to state that this latter was simply a reint of mine with slight additions, as is also that given at the end of the Kazan lition of Lobachévski's Works, 1886. Some grotesque effects are produced by printing or attempting to reprint my English. Thus under P. G. Tait the title the work is given as follows: "Mentions Hyper-Space in his Address Pres. of Math. Sect. of Brit. Assoc. at Edinbvrgh." Under G. P. Young we ad "The relation which can be proved to subsist between the Area of a Plane riangle and the Sum of the Hypothesis that Euclid's twelfth Axiom is false." must take it, from this extraordinary summation of a hypothesis, that English nearly as difficult as Russian, though neither can for one instant compete with e Magyar.

In his personal character Hoüel reached that perfection which he has done much to introduce into the foundations of Geometry.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from March Number.]

Proposition XXVI. *If the aforesaid AX, BX (fig. 31.) must indeed meet ch other, but only at their infinite production toward the parts of the point X: I y there will be no assignable point T in AB, from which a perpendicular erected wards the parts of AX does not at a finite or terminated distance meet this AX in me point F.*

Demonstratur. For (from the preceding hypothesis) there will be in *AX*

some point N, from which the perpendicular NK let fall to BX is less t
assigned length, as suppose this TB.


Fig. 31.

But then is assumed in TB a portion CB equal to NK, and CN is joined. In the hypothesis of acute angle it is known that the angle NCB will be acute. Therefore (from Eu. I. 13) NCT, which is the adjacent angle, will be obtuse.

Therefore the straight which is erected toward the parts of AX perpendicularly from the point T (disposed between the points A and C), does not meet (from Eu. I. 17) CN at any point; and therefore (lest it should enclose a space with AT, or with TC) it stri
terminated AN in some point F.

Therefore even in the hypothesis of acute angle (which we know c
alone hinder) there will be in this AB no assignable point T, from
the perpendicular erected toward the parts of AX does not, at a finite or
ated distance, meet this AX in a certain point F. Quod etc.

Corollary I. But thence follows, that, point M being assumed
produced, from which towards the parts of the point X is erected a perpen
MZ, this cannot, even if infinitely produced, meet the aforesaid AX; b
otherwise that other straight BX must (from the foregoing demonstratio
finite distance meet this AX; which is against the present hypothesis.

Corollary II. From which again follows, that every perpen
erected from any point, but not however infinitely removed, of this AB pr
indefinitely, must at a finite distance meet the aforesaid AX, as soon as in
is assumed that *every* such perpendicular ever more, without any certain
approaches the other ever produced straight AX.

Corollary III. Whence finally follows, that not even at its infini
duction can BX be cut by that AX; because otherwise from any point of t
beyond the aforesaid intersection a certain perpendicular ZM could be su
let fall to AB-produced; whence again would follow, that BX (against th
ent hypothesis) met the aforesaid AX not at an infinite, but wholly at
distance.

But this last dictum is beyond necessity.

[Saccheri here handles a point at infinity, or *figurative* point, as if
a *proper* point. Upon the extent to which he realized this to be unallowa
pends his real mental attitude toward the non-Euclidean geometries he h
covered. Did he intend his work to suggest what he would not have b
lowed to print?]

A METHOD FOR DEVELOPING $\cos^n\theta$ AND $\sin^n\theta$.

By H. C. STEVENS, A. M., Department of Mathematics, Purdue University, Lafayette, Indiana.

De Morgan in his Calculus gives a method for expanding $\cos^n\theta$ and $\sin^n\theta$ n is an integer which I have not noticed in any of our American works on subject. As it leads to an easy method for integrating such expressions as

$$\int\cos^n\theta d\theta, \quad \int\sin^n\theta d\theta,$$

I have thought it might be of interest to some of the readers of the MONTHLY. The method is as follows:

$$\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}. \quad \text{Let } e^{i\theta}=x, \text{ then } e^{-i\theta}=\frac{1}{x}, \text{ and } \cos\theta=\tfrac{1}{2}\left(x+\frac{1}{x}\right)\dots\dots\dots\dots(1)$$

$$e^{ni\theta}=x^n, \text{ then } e^{-ni\theta}=\frac{1}{x^n}, \cos n\theta=\tfrac{1}{2}\left(x^n+\frac{1}{x^n}\right).$$

$$\text{Then from (1) } \cos^n\theta=\frac{1}{2^{n-1}}\left[\tfrac{1}{2}\left(x^n+\frac{1}{x^n}\right)+n\tfrac{1}{2}\left(x^{n-2}+\frac{1}{x^{n-2}}\right)\right.$$

$$\frac{n(n-1)}{2}\tfrac{1}{2}\left(x^{n-4}+\frac{1}{x^{n-4}}\right)\dots\dots$$

$$=\frac{1}{2^{n-1}}\left[\cos n\theta+n\cos(n-2)\theta+\frac{n(n-1)}{2}\cos(n-4)\theta+\dots\dots\right.$$

If n be an even number $=2m$, there will be $2m+1$ terms in the development, which will give m cosines, namely, those of $2m\theta$, $2(m-1)\theta\dots\dots$ down to, and an additional term which will not contain θ, the value of which is $\frac{n(2m-1)\dots\dots m+1}{|m}$. But if n be odd, and $=2m+1$, then there are $+2$ terms giving $m+1$ cosines, namely, those of $(2m+1)\theta$, $(2m-1)\theta\dots\dots\dots$ wn to θ, with no middle term. Thus we have

$$\cos^6\theta=\tfrac{1}{32}(\cos6\theta+6\cos4\theta+15\cos2\theta+10).$$

Whence the integral of $\cos^6\theta d\theta=\frac{1}{192}\sin6\theta+\frac{3}{64}\sin4\theta+\frac{15}{64}\sin2\theta+\frac{5}{16}\theta$.
Also $\cos^7\theta=\frac{1}{64}(\cos7\theta+7\cos5\theta+21\cos3\theta+35\cos\theta)$.

Whence $\int\cos^7\theta d\theta=\frac{1}{448}\sin7\theta+\frac{7}{320}\sin5\theta+\frac{7}{64}\sin3\theta+\frac{35}{64}\sin\theta$.

The advantage of this method will be still more apparent by integra $\cos^3 3\theta\cos\theta d\theta$. Here $\cos^3 3\theta=\frac{1}{8}(x^3+x^{-3})^3=\frac{1}{8}(x^9+x^{-9})+3(x^3+x^{-3})$.

Multiplying this by $\frac{1}{2}(x+x^{-1})$ we at once have

$$\cos^3 3\theta\cos\theta=\tfrac{1}{8}\cos 10\theta+\tfrac{1}{8}\cos 8\theta+\tfrac{3}{8}\cos 4\theta+\tfrac{3}{8}\cos 2\theta.$$

Whence $\int\cos^3 3\theta\cos\theta d\theta=\frac{1}{80}\sin 10\theta+\frac{1}{64}\sin 8\theta+\frac{3}{32}\sin 4\theta+\frac{3}{16}\sin 2\theta$.

It will be noticed that this form is well adapted for substituting value limits of integration. For instance if the inferior limit be 0, and the supe limit $\frac{1}{6}\pi$ then $\frac{1}{80}\sin\frac{10}{6}\pi=-\frac{1}{160}\sqrt{3}$; $\frac{1}{64}\sin\frac{8}{6}\pi=-\frac{1}{128}\sqrt{3}$; $\frac{3}{32}\sin\frac{4}{6}\pi=\frac{3}{64}\sqrt{3}$ $\frac{3}{16}\sin 2\theta=\frac{3}{32}\sqrt{3}$.

$$\therefore\int_0^{\frac{1}{6}\pi}\cos^3 3\theta\cos\theta d\theta=\tfrac{81}{640}\sqrt{3}.$$

The reader will have no difficulty in applying the same method to dev $\sin^n\theta$ and then for integrating $\sin^n\theta d\theta$.

It will be observed that when we put $\cos\theta=\frac{1}{2}(x+\frac{1}{x})$ we do not escap impossible; for this is as much an impossible form as $\cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta})$ $x+\frac{1}{x}$ can never be *less* than 2, and $2\cos\theta$ can never be *greater* than 2.

CONCERNING CONICS THROUGH FOUR POINTS.

By EDGAR H. JOHNSON, Professor of Mathematics, Emory College, Oxford, Georgia.

The equation of the conic through a_1b_1, a_2b_2, a_3b_3, a_4b_4, and a point x_1y_1 is

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1 & x_1 & y_1 & 1 \\ a_1^2 & a_1b_1 & b_1^2 & a_1 & b_1 & 1 \\ a_2^2 & a_2b_2 & b_2^2 & a_2 & b_2 & 1 \\ a_3^2 & a_3b_3 & b_3^2 & a_3 & b_3 & 1 \\ a_4^2 & a_4b_4 & b_4^2 & a_4 & b_4 & 1 \end{vmatrix}=0,$$

or $Ax^2+2Bxy+Cy^2+2Fx+2Gy+H=0$, where the coefficients A, B, C, .. are of the second degree in x_1 and y_1. The conic is an ellipse, parabola, o

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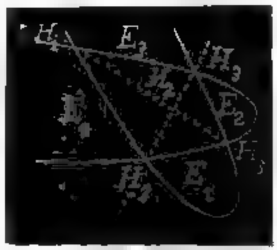
*Professor Waldo first called my attention to this easy method for integrating this par expression.

rbola according as $AC-B^2$ is greater than, equal to, or less than zero. rough every point of the curve $AC-B^2=0$ (x_1 and y_1 being now general co-dinates) may be drawn a parabola also passing through the four given points. ow it is known that through four points two parabolas can be drawn, the para-la being real or imaginary according as one of the four points does not or does in the triangle formed by the other three. (See Salmon's Conic Sections, ge 153, ex. 1; or C. Smith's Conic Sections, pages 233-4).

Since through every point of each of these two parabolas, a parabola pass-g through the four given points is possible, the curve $AC-B^2=0$, of the fourth gree, decomposes into these same two parabolas.

Since $AC-B^2$ changes sign when a point crosses the curve, we have de-rmined the locus of those points which with the four given points determine an lipse (or hyperbola). The curve divides the plane into regions of two kinds, ose for which $AC-B^2$ is positive, and those for which $AC-B^2$ is negative. very point in a region of the first kind determines with the four given points an lipse; every point of the second kind determines likewise an hyperbola. The oints within the region enclosed by the two parabolas determine hyperbolas, nce the four points determine a pair of straight lines, ssing through this region, and for a pair of straight nes $AC-B^2<0$. Points in the regions marked H see figure) determine with the four points of intersec-on of parabolas conics which are hyperbolas; points n the regions marked E determine likewise ellipses. particular case of special interest arises when the ur points become two pairs of coincident points, and he system becomes that of conics tangent to two giv-n lines at given points. It is easy to show that the two parabolas become coin-ident. $AC-B^2$ is then a square and cannot change sign. The two tangents onstitute one conic of the system and for the present purpose a pair of straight ines is a hyperbola. Hence all conics of the system, with the exceptions of the arabola and the pair of tangent lines, are hyperbolas.



In the above we have supposed points and conics to be real. It is easy to ee that the condition for the passing of a real ellipse through four distinct real oints is the same as for a real hyperbola. A real parabola can always be drawn rough four real points not in the same straight line.

INTEGRAL SIDES OF RIGHT TRIANGLES.

By M. A. GRUBER, A. M., War Department, Washington, D. C.

$$a^2+b^2=c^2.$$

Problem I. To find integral sides of right triangles.

Rule 1. Take two integers, both odd or both even. ½ the sum of their squares equals the hypotenuse, or c; ½ the difference of their squares equals one of the legs, or b; and their product equals the other leg, or a.

Rule 2. Take any two integers. The sum of their squares equals the hypotenuse, or c; the difference of their squares equals one of the legs, or b; and twice their product equals the other leg, or a.

Rule 3. If *prime* integral sides are desired, the integers chosen must be prime to each other; in Rule 1, both odd; and in Rule 2, one odd and the other even.

Note. Rules 1 and 2 hold good also for fractional values. These rules are deduced from the two formulas mentioned in Problem II, and, to avoid repetition, are not discussed in this problem.

Problem II. Given one of the legs of a right triangle of integral sides to find the other leg and the hypotenuse.

The sides of a right triangle depend upon the equation $a^2+b^2=c^2$, in which a and b are the legs and c the hypotenuse of the triangle.

In the discussion of this problem, a is taken as the *given leg*.

When integral equations of the form $a^2+b^2=c^2$ are considered, the sets of values for a, b, and c are divided into two classes: (1) Those having no common factor; a, b, and c being prime integral values. (2) Those having a common factor; a, b, and c being found by multiplying a, b, and c of the first class by the highest common factor.

Sets of *prime* integral values are, therefore, the basis of work.

In right triangles of integral sides, any integer from 3 up may be taken as the value of one of the legs.

There are three kinds of integers to be considered: (1) Odd numbers; (2) Even numbers divisible by 4; and (3) Even numbers that are 2 times an odd number.

a may, then, be any one of these three kinds of numbers.

When a is *an odd number*, we have the formula

$$(mn)^2+\left(\frac{m^2-n^2}{2}\right)^2=\left(\frac{m^2+n^2}{2}\right)^2$$

by means of which to find b and c, so that a, b, and c have no common factor.

$$mn=a,\ \frac{m^2-n^2}{2}=b, \text{ and } \frac{m^2+n^2}{2}=c.$$

m and n are odd and are prime to each other, and $m>n$. There are as man sets of prime integral values of a, b, and c as m and n can be made sets of odd, prime, integral factors, the product of each set of which factors equals a.

When a is *an even number divisible by* 4, we have the formula $(2mn)^2+(m^2-n^2)^2=(m^2+n^2)^2$, by means of which to find b and c, so that a, b, and c have no common factor. $2mn=a$, $m^2-n^2=b$, and $m^2+n^2=c$. m and n are prime to each other, one being odd, the other even; and $m>n$. There are as many sets of prime integral values of a, b, and c as m and n can be made sets of prime integral factors, the product of each set of which factors equals $\frac{1}{2}a$.

When a is *an even number that is* 2 *times an odd number*, we first find the set or sets of values for a equal to the odd number, and then multiply them by 2.

When a contains odd factors other than itself and unity, or even factors divisible by 4, there are other sets of values, in which a, b, and c have a common factor. There are as many sets of values of this kind as the sets of prime integral values that can be found for the odd factors and the even factors divisible by 4 contained in a. In this case we first find the sets of prime integral values for each of the factors and then multiply them by the respective numbers that produce a.

In problems relating to the integral sides of right triangles, unity and the number itself are considered factors of a number.

For the purpose of bringing out the foregoing statements more clearly to the mind of the reader, we shall present them by way of illustration.

Put $a=3$, the lowest integer for integral sides of right triangles. Then $mn=3=3\times1$; whence $m=3$, $n=1$. Substituting these values in the formula for $a=$an odd number, we find $b=\frac{1}{2}(3^2-1^2)=4$, and $c=\frac{1}{2}(3^2+1^2)=5$. There is but one set of values; viz., 3, 4, 5.

Put $a=4$. Then $2mn=4=2\times2\times1$; whence $m=2$, $n=1$. Substituting these values in the formula for $a=$an even number divisible by 4, we find $b=2^2-1^2=3$, and $c=2^2+1^2=5$. This set of values, 4, 3, 5, is the same as that for $a=3$, only a and b have interchanged values. There is but one set.

Put $a=12$. Then $2mn=12=2\times6\times1$ and $2\times3\times2$. There are, therefore, two sets of prime integral values. To find first set, $m=6$, $n=1$. To find second set, $m=3$, $n=2$. Whence the sets are 12, 35, 37; and 12, 5, 13. But $12=4\times3$ and 3×4. Hence there are two other sets of values, each set having a common factor. When $a=3$, $b=4$, $c=5$. When $a=4$, $b=3$, $c=5$. Multiplying these sets by the respective numbers that produce $a=12$, we obtain the required sets, 12, 16, 20; and 12, 9, 15, making in all 4 sets.

Put $a=15$. Then $mn=15=15\times1$ and 5×3. There are, therefore, two sets of prime integral values: 15, 112, 113; and 15, 8, 17. But as $15=5\times3$ and 3×5, there are also two sets of values, each set having a common factor. When $a=3$, $b=4$, $c=5$. When $a=5$, $b=12$, $c=13$. Whence the required sets are 15, 20, 25; and 15, 60, 65,—in all 4 sets.

In order to find the number of sets of values that can be formed for $a=$an integer, we shall illustrate by taking $a=60$. Then $2mn=60=2\times30\times1$, $2\times15\times2$, $2\times10\times3$, and $2\times6\times5$. Hence there are 4 sets of prime integral values. But 60 contains also the following factors that are odd numbers: $3=3\times 1$; $5=5\times 1$; and $15=15\times 1$ and 5×3. These give 4 more sets. The factors that are even numbers divisible by 4, are $4=2\times 2\times 1$; $12=2\times 6\times 1$ and $2\times 3\times 2$; and $20=2\times 10\times 1$ and $2\times 5\times 2$. These give 5 additional sets. Hence for $a=60$, there are 13 sets of values for integral sides of right triangles.

A THEOREM ON PRISMOID.

By P. H. PHILBRICK, C. E., Pineville, Louisiana.

THEOREM. *To prove that the error of the "end area volume" of any prismoid or solid to which the prismoidal formula applies, is twice the error of the "middle area volume" and on the opposite side of the true result.*

Let A and B represent the end areas, M the middle area, and l the length of the prismoid.

Then the true volume is, $V=\frac{1}{6}l(A+4M+B)$........................(1),

the end area volume is, $V_e=\frac{1}{2}l(A+B)$..................................(2),

and the middle area volume is, $V_m=lM$....................................(3).

Now (1)−(2) gives error of (2)$=V-V_e=\frac{1}{6}l(4M-2A-2B)$ ....(4),

and (1)−(3) gives error of (3)$=V-V_m=\frac{1}{6}l(A+B-2M)$..................(5).

But (4) is twice (5) with a contrary sign.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

74. Proposed by JOHN T. FAIRCHILD, Principal of Crawfis College, Crawfis College, Ohio.

When U. S. bonds are quoted in London at 108¾ and in Philadelphia at 112¼, exchange $4.89½, gold quoted at 107, how much more was a $1000 U. S. bond worth in London than in Philadelphia?

No solution of this problem has been received.

75. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

If 24 men, in 15 days of 12 hours each, dig a trench 300 yards long, 5 yards wide, 6 feet deep for 540 five-cent loaves when flour is $8 a barrel; what is flour worth a barrel when 45 men, working $5\frac{1}{3}$ days of ten hours each, dig a trench 125 yards long, 5 yards wide, 8 feet deep for 320 four-cent loaves? *Solve by proportion.*

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, and C. A. JONES, Terrence, Miss.

The price of flour is an inverse ratio, hence using the *cause* and *effect* process we get at once

$$\left\{\begin{array}{r}24\\15\\12\\540\\5\end{array}\right\}:\left\{\begin{array}{c}300\\5\\6\\(?)\end{array}\right\}::\left\{\begin{array}{r}45\\5\frac{1}{3}\\10\\320\\4\end{array}\right\}:\left\{\begin{array}{c}125\\5\\8\\8\end{array}\right\}$$

$$\therefore (?)=\frac{24\times15\times12\times540\times5\times125\times5\times8\times8\times3}{300\times5\times6\times45\times16\times10\times320\times4}=16\tfrac{7}{8}.$$

$\therefore$ Flour is worth $16.87½ per barrel.

76. Proposed by E. W. MORRELL, Professor of Mathematics in Montpelier Seminary, Montpelier, Vermont.

An eastern nobleman willed his entire estate to his three sons on the condition that the oldest should have one-half, the next one-third, and the youngest one-ninth. His estate, on inventory, was found to consist of 17 elephants. What should be the share of each?

Solution by FREDERIC R. HONEY, Ph. B., New Haven, Connecticut, and CHAS. C. CROSS, Laytonsville, Maryland.

If the will was obeyed literally the eldest son's share was $\frac{17}{2}$ elephants; the second son's $\frac{17}{3}$; and the youngest's $\frac{17}{9}$, making a total of $\frac{17}{2}+\frac{17}{3}+\frac{17}{9}=16\frac{1}{18}$ elephants. This would leave $\frac{17}{18}$ of an elephant.

The following solution would be satisfactory:

We have $\frac{1}{2}+\frac{1}{3}+\frac{1}{9}=\frac{17}{18}$ as the denominator.

First son receives $\dfrac{\frac{1}{2}}{\frac{17}{18}}=\frac{1}{2}\times\frac{18}{17}=\frac{9}{17}$.

Second son receives $\dfrac{\frac{1}{3}}{\frac{17}{18}}=\frac{1}{3}\times\frac{18}{17}=\frac{6}{17}$.

Third son receives $\dfrac{\frac{1}{9}}{\frac{17}{18}}=\frac{1}{9}\times\frac{18}{17}=\frac{2}{17}$.

Since the estate consisted of 17 elephants, $\therefore$ the first son got $\frac{9}{17}$ of $17=9$ elephants; and the second son got $\frac{6}{17}$ of $17=6$ elephants; and the third son got $\frac{2}{17}$ of $17=2$ elephants.

Remarks by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

As 17 is prime, the elephants should be divided as near the proportion as possible. $\therefore$ oldest should have 9, next, 6, and the youngest, 2.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

68. Proposed by LEONARD E. DICKSON, M. A., Ph. D., Formerly Fellow of Mathematics, University of Chicago, Chicago, Illinois.

Suppose a circle of unit radius divided at the points A, A_1, A_2, A_3, ... into n equal parts. [This division cannot in general be affected by geometry.] Through A draw the diameter OA and join O with A_1, A_2, A_3, ... $A_{\frac{n-1}{2}}$, where n is supposed to be odd.

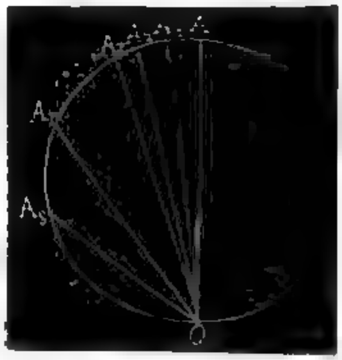
Prove that $OA_1-OA_2+OA_3-OA_4+\ldots\ldots\pm OA_{\frac{n-1}{2}}$, every other chord being affected with the minus sign.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, and C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

Let OA_1, OA_2, OA_3, etc. $=a_1$, a_2, a_3, etc.

Now $OA=2$, $\angle AOA_1=\angle A_1OA_2=$etc.$=\pi/n$.

$\therefore OA_r=a_r=2\cos(r\pi/n)$.



(1) When $\frac{n-1}{2}$ is even,

$\therefore a_1+a_3+a_5+\ldots\ldots+a_{\frac{n-3}{2}}$

$=2(\cos\frac{\pi}{n}+\cos\frac{3\pi}{n}+\cos\frac{5\pi}{n}+\ldots\ldots$

$+\cos\frac{(n-3)\pi}{2n})=\sin\frac{(n-1)\pi}{2n}/\sin\frac{\pi}{n}$

$=\sin(\frac{\pi}{2}-\frac{\pi}{2n})/\sin\frac{\pi}{n}=\frac{1}{2}\cos\frac{\pi}{2n}$ ...... (1).

$$a_2+a_4+a_6+\ldots\ldots+a_{\frac{n-1}{2}}=2(\cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}+\cos\frac{6\pi}{n}+\ldots.+\cos\frac{(n-1)\pi}{2n}$$

$$=\frac{2\cos[(n+3)\pi/4n]\sin[(n-1)\pi/4n]}{\sin(\pi/n)}=\frac{2\cos[\frac{1}{4}\pi+(3\pi/4n)]\sin[\frac{1}{4}\pi-(\pi/4n)]}{\sin(\pi/n)}$$

$$=\frac{[\cos(3\pi/4n)-\sin(3\pi/4n)][\cos(\pi/4n)-\sin(\pi/4n)]}{\sin(\pi/n)}=\frac{\cos(\pi/2n)-\sin(\pi/n)}{\sin(\pi/n)}$$

$$=\frac{1}{2}\text{cosec}\frac{\pi}{2n}-1\ldots\ldots\ \ldots\ldots(2).$$

$$\therefore a_1 - a_2 + a_3 - a_4 + \ldots\ldots + a_{\frac{n-3}{2}} - a_{\frac{n-1}{2}} = 1.$$

(2) When $\dfrac{n-1}{2}$ is odd.

$$\therefore a_1 + a_3 + a_5 + \ldots\ldots + a_{\frac{n-1}{2}} = 2\left(\cos\frac{\pi}{n} + \cos\frac{3\pi}{n} + \cos\frac{5\pi}{n} + \ldots\ldots\right.$$

$$\left. + \cos\frac{(n-1)\pi}{2n}\right) = \sin\frac{(n+1)\pi}{2n} \Big/ \sin\frac{\pi}{n}$$

$$= \sin\left(\frac{\pi}{2} + \frac{\pi}{2n}\right) \Big/ \sin\frac{\pi}{n} = \tfrac{1}{2}\text{cosec}\frac{\pi}{2n} \ldots\ldots\ldots\ldots (3).$$

$$a_2 + a_4 + a_6 + \ldots\ldots + a_{\frac{n-3}{2}} = 2\left(\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cos\frac{6\pi}{n} + \ldots\ldots + \cos\frac{(n-3)\pi}{2n}\right)$$

$$= \frac{2\cos[(n+1)\pi/4n]\sin[(n-3)\pi/4n]}{\sin(\pi/n)} = \frac{2\cos[\frac{1}{4}\pi + (\pi/4n)]\sin[\frac{1}{4}\pi - (3\pi/4n)]}{\sin(\pi/n)}$$

$$+ \frac{[\cos(n/4n) - \sin(\pi/4n)][\cos(3\pi/4n) - \sin(3\pi/4n)]}{\sin(\pi/n)} = \frac{\cos(\pi/2n) - \sin(\pi/n)}{\sin(\pi/n)}$$

$$= \tfrac{1}{2}\text{cosec}(\pi/2n) - 1. \ldots\ldots\ldots\ldots (4).$$

$$\therefore a_1 - a_2 + a_3 - a_4 + \ldots\ldots - a_{\frac{n-3}{2}} + a_{\frac{n-1}{2}} = 1.$$

$$\therefore OA_1 - OA_2 + OA_3 - OA_4 + \ldots\ldots \pm OA_{\frac{n-1}{2}} = 1.$$

78. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

Prove that the locus of the center of the circle which passes through the vertex of a parabola and through its intersections with a normal chord is the parabola $2y^2 = ax - a^2$, the equation to the given parabola being $y^2 = 4ax$.

Solution by the PROPOSER.

The circles being $(x-m)^2 + (y-n)^2 = r^2 \ldots\ldots\ldots\ldots\ldots\ldots (1)$,

and passing through the vertex of $y^2 = 4ax \ldots\ldots\ldots\ldots\ldots\ldots (2)$,

becomes $x^2 - 2mx + y^2 - 2ny = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3)$.

Now the extremities of the normal chord being $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, normal at the former point, we have

$$a^2t_1^4 - 2mt_1^2 + 4a^2t_1^2 - 4nat_1 = 0 \ldots\ldots\ldots\ldots (4),$$

$$\text{and } a^2t_2^4 - 2mt_2^2 + 4a^2t_2^2 - 4nat_2 = 0 \ldots\ldots\ldots\ldots (5).$$

Divide these by at_1, at_2 respectively, and take one result from the other and divide by t_1-t_2; then

$$a(t_1^3+t_1^2t_2+t_1t_2^2)=2mt_1-4at_1 \text{.................} (6).$$

Divide (4) by at_1 and take (6) from the result; then

$$at_1t_2(t_1+t_2)=-4n \text{..........................} (7).$$

But it can be shown that $t_1+t_2=-(2/t_1)$ ....(8).

Substituting in (7) and reducing, $t_2=(2n/a)=-t_1-(2/t_1)$ ....(9),

which gives $t_1=\dfrac{-n+\sqrt{n^2-2a^2}}{a}$ ....(10).

(10) in (6) gives $2n^2=am-a^2$ ....(11),

the required locus of center of (1).

Also solved by *F. M. McGAW.*

[NOTE. Solution of Problem 69 will appear in the next issue. Editor.]

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

44. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

There is a triangle whose sides repulse a center of force within the triangle with an intensity that varies inversely as the distance of the center of force from each point of the sides of the triangle. What the position of equilibrium of the center?

Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

Put p=altitude of the triangle upon the side a as base, s=distance of center of force from the side a, x=distance of any point of side a from the vertex B, y=distance of any point of the side b from the vertex C, and z=distance of any point of the side c from the vertex B.

The force exerted by any portion dx of the side a resolved perpendicular to it, is $msdx$ where m is an arbitrary constant depending on the intensity of the force. The forces exerted by portions dy and dz of the sides b and c are respectively $m[s-(b/p)y]dy$ and $m[s-(c/p)z]dz$. For equilibrium we have

$$m\int_0^a s dx + m\int_0^b (s-\frac{p}{b}y)dy + m\int_0^c (s-\frac{p}{c}z)dz = 0.$$

$$\therefore (a+b+c)s = \tfrac{1}{2}p(b+c). \quad \therefore s = \frac{p(b+c)}{2(a+b+c)}.$$

Hence the distance of the center of force from the side a is $\frac{p(b+c)}{2(a+b+c)}$.

In like manner it may be shown that its distance from the side b is $\frac{q(a+c)}{2(a+b+c)}$, and that its distance from the side c is $\frac{r(a+b)}{2(a+b+c)}$, when q and r are altitudes of the triangle upon the sides b and c respectively.

Also solved by *G. B. M. ZERR*. His solution will appear in the next issue.

45. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

A fifty-pound cannon-ball is projected vertically upward with a velocity of 300 feet per second. Find the height to which it will rise and the time of flight, assuming the initial resistance of the air on the ball to be 10 pounds and the resistance to vary as the square of the velocity.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let h=height required, t=time of ascent, t_1=time of descent, $T=t+t_1$= time of flight, v=velocity=300 feet per second. W=50 pounds, g=32.2 feet per second. μv^2=10 pounds=$\frac{1}{5}W$.

$$\therefore \mu=\frac{W}{5v^2}. \quad \frac{1}{k}=\sqrt{\frac{W}{\mu}}=v\sqrt{5}. \quad \therefore k=\frac{1}{v\sqrt{5}}.$$

From Bowser's Analytical Mechanics, pages 306-7, we get,

$$h=\frac{1}{2gk^2}\log(1+k^2v^2), \quad t=\frac{1}{gk}\tan^{-1}vk.$$

$$t_1=\frac{1}{gk}\log\{\sqrt{1+v^2k^2}+vk\}.$$

$$\therefore h=\frac{5v^2}{2g}\log(\tfrac{6}{5})=1273.9827 \text{ feet}.$$

$$t=\frac{v\sqrt{5}}{g}\tan^{-1}\frac{1}{\sqrt{5}}=8.75931 \text{ seconds}.$$

$$t_1=\frac{v\sqrt{5}}{g}\log\left(\frac{\sqrt{6}+1}{\sqrt{5}}\right)=9.03118 \text{ seconds}.$$

$T=t+t_1$=17.79049 seconds.

Also solved by *HENRY HEATON*.

46. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training Philadelphia, Pennsylvania.

"There was an old woman tossed up in a basket
Ninety times as high as the moon." *Mother Goose.*

Neglecting the resistance of the air, how long did it take the old lady to go u

I. Solution by E. L. SHERWOOD, A. M., Superintendent of City Schools, West Point, Mississip

The equation of motion is

$$1.\quad \frac{d^2s}{dt^2}=-\frac{gr^2}{s^2}.\qquad 2.\quad \left(\frac{ds}{dt}\right)^2=\frac{2gr^2}{s}+c$$

where ds/dt or $v=0$, when $S=90\times 60.3R$ or $5427R$.

Whence $C=-\dfrac{2gr^2}{5427R}$.

$$3.\quad \left(\frac{ds}{dt}\right)^2=\frac{2gr^2}{s}-\frac{2gr^2}{5427r}\ \text{or}\ 2gr^2\left(\frac{1}{s}-\frac{1}{a}\right).$$

$$4.\quad dt=\sqrt{\frac{a}{2gr^2}}\cdot\frac{sds}{\sqrt{as-s^2}}\ \text{solving } dt \text{ in (3).}$$

$$5.\quad t=\sqrt{\frac{a}{2gr^2}}\int_a^R\frac{sds}{\sqrt{as-s^2}}\ \text{for } t=0 \text{ when } s=a.$$

$$6.\quad t=\sqrt{\frac{a}{2gr^2}}\left[\left(\sqrt{as-s^2}-\tfrac{1}{2}a\text{vers}^{-1}\frac{2s}{a}+C\right)\right]_a^R.$$

$$7.\quad t=\sqrt{\frac{a}{2gr^2}}\left(\sqrt{aR-R^2}-\tfrac{1}{2}a\text{vers}^{-1}\frac{2R}{a}+\frac{\pi a}{2}\right)\ \text{where } a=5427R.$$

8. $t=11.35+$ years, by substituting values and reducing.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let t=time, R=3963 miles=20924640 feet=radius of the earth, g
feet=gravity, $a=90(60R)=5400R$=distance the old woman was tossed.

$$\therefore t=\sqrt{\frac{a}{2gR^2}}\left(\sqrt{aR-R^2}-\tfrac{1}{2}a\text{vers}^{-1}\frac{2R}{a}+\frac{\pi a}{2}\right).$$

$$t=30\sqrt{\frac{3}{gR}}\{R\sqrt{5399}-2700R\text{vers}^{-1}\tfrac{1}{2700}+2700\pi R\}.$$

$t=355287708.316$ seconds = 11 years, 3 months, 7 days, 3 hours, 1 minute, 16 seconds.

Also solved by *J. C. CORBIN.*

47. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington,

What is the focus of the convex surface of a plano-convex lens, index μ, which will 'erge parallel monochromatic rays to a given focus, the rays entering the lens on the e side ?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let f=the given focal length. F=the focal length required,
u=distance of origin of ray from lense,
r, s, the radii of the first and second surfaces of the lense respectively,
t=the thickness, and regard all distances as measured from the posterior ıce.

Then we have for for a double convex lense,

$$\frac{1}{\frac{1}{f}+\frac{\mu-1}{s}}-\frac{1}{\frac{1}{u}-\frac{\mu-1}{r}}=\frac{t}{\mu}.$$

Parkinson's Optics, Art. 100, Cor. I, page 91).

Let $u=r=\infty$.

$$\therefore \frac{1}{\frac{1}{f}+\frac{\mu-1}{s}}=\frac{t}{\mu} \quad \text{.......(1).}$$

This is the plano-convex lense with light incident upon plane surface.

Write F for f, and let $s=u=\infty$.

$$\therefore \frac{1}{\frac{1}{F}}+\frac{1}{\frac{\mu-1}{r}}=\frac{t}{\mu} \quad \text{.......(2).}$$

This is the plano-convex lense with light incident upon the convex sur-
Since we are using the same lense, $r=s$.

$$\therefore\ r=s=\frac{(\mu-1)ft}{\mu f-t}, \text{ from (1).}$$

This value of r in (2) gives, $F=-\dfrac{t^2(\mu-1)}{(\mu f-t)^2}$.

$\therefore$ F is found independent of the radius of convexity.

48. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Two equal heavy rings connected by a string passing over a peg at the focus of a conic section will be in equilibrium at all points on the curve.

Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

An evident property of *any* curve which will be a curve of equilibrium for two weights thus attached is that the tension along the string shall be the same, wherever the weights are placed upon the curve. If this were not so, by altering the position of one of them, by changing the length of the string, the tension would be changed and the other would no longer be in equilibrium.

Call T the tension, W the weight, ϕ the angle which the curve makes with the horizontal, and θ the angle which the string makes.

Resolving along the curve,

$$W\cos\phi - T\cos(\theta-\phi)=0.$$

$$\frac{\cos(\theta-\phi)}{\cos\phi}=\frac{W}{T}=k.$$

$$\therefore\ \cos\theta+\sin\theta\tan\phi=k.$$

$$\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{x^2+y^2}}\cdot\frac{dy}{dx}=k.\qquad \frac{xdx+ydy}{\sqrt{x^2+y^2}}=kdx.$$

$$(x^2+y^2)^{\frac{1}{2}}=kx+c.\qquad x^2(1-k^2)+y^2-2kcx=c^2.$$

This is the equation of a conic with the origin at the focus.

$$k=e+c=a(1-e^2).$$

The above investigation refers to the case in which the tension is simply constant. The string may be attached to the fixed point.

If the string be now considered passing around the focus to the curve again and a weight W attached there also, the tension will be doubled.

Then $k=W/2T$.

If $W=2T$, or $T=\frac{1}{2}W$, the equation becomes $y^2-2cx=c^2$, a parabola.

If $W>2T$, or $T<\frac{1}{2}W$, the curve is an hyperbola.

If $W<2T$, or $T>\frac{1}{2}W$, the curve is an ellipse.

The above may be put in the following form:

In a parabola the tension is equal to half one of the equal weights; in an hyperbola it is less than half of the weight; and in an ellipse it is greater than the same.

AVERAGE AND PROBABILITY.

lucted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

43. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

In a circle whose radius is a, chords are drawn through a point distant b from the er. What is the average length of such chords, (1), if a chord is drawn from every t of the circumference, and (2), if they are drawn through the point at equal angular rvals?

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let $AC=CF=a$, $CD=b$, $\angle FDA=\theta$, $\angle FCA=\varphi$, $\tan\theta=m$.

Then the equation to DF is, $y=mx+mb$..........................(1).

The equation to the circle is, $x^2+y^2=a^2$.........................(2).

From (1) and (2) we easily get $EF=2\sqrt{a^2-\dfrac{m^2b^2}{1+m^2}}$,

$\therefore\ EF=2\sqrt{a^2-b^2\sin^2\theta}$. But $\sin\theta=\dfrac{a\sin\varphi}{\sqrt{a^2+b^2+2ab\cos\varphi}}$.

$$\therefore\ EF=\frac{2(a^2+ab\cos\varphi)}{\sqrt{a^2+b^2+2ab\cos\varphi}}=\frac{2(a^2+ab-2ab\sin^2\frac{1}{2}\varphi)}{\sqrt{(a+b)^2-4ab\sin^2\frac{1}{2}\varphi}}.$$

The limits of φ for $b>a$, are 0 and $\frac{1}{2}\pi+\sin^{-1}(a/b)=2\beta$.

The limits of φ for $b<a$, are 0 and $\frac{1}{2}\pi+\sin^{-1}(b/a)=2\gamma$.

The limits of θ for $b>a$, are 0 and $\sin^{-1}(a/b)=\theta'$.

The limits of θ for $b<a$, are 0 and $\frac{1}{2}\pi$.

Let Δ and Δ_1 be the average lengths required.

$\therefore\ \Delta=\int EFds/\int ds=\int EFd\varphi\,/\int d\varphi$,

$\int EFd\theta/\int d\theta$.

I. Let $\dfrac{4ab}{(a+b)^2}=e^2$, and $\frac{1}{2}\varphi=\gamma$.

$\therefore\ d\varphi=2d\gamma$.

Then $EF=\dfrac{2(a^2+ab-2ab\sin^2\gamma)}{(a+b)\sqrt{1-e^2\sin^2\gamma}}$.

$$\therefore\ EF=\frac{2a}{\sqrt{1-e^2\sin^2\gamma}}+\frac{2}{e\sqrt{ab}}\sqrt{1-e^2\sin^2\gamma}-\frac{2}{e\sqrt{ab}\sqrt{1-e^2\sin^2\gamma}}.$$

$$\therefore\int EF d\gamma=\frac{2}{e\sqrt{ab}}\{(ae\sqrt{ab}-1)F(e,\gamma)+E(e,\gamma)\}.$$

$$\therefore\ \Delta=\frac{2}{\beta e\sqrt{ab}}\{(ae\sqrt{ab}-1)F_0^{\beta}(e,\gamma)+E_0^{\beta}(e,\gamma)\},\ b>a.$$

$$\Delta=\frac{2}{\delta e\sqrt{ab}}\{(ae\sqrt{ab}-1)F_0^{\delta}(e,\gamma)+E_0^{\delta}(e,\gamma)\},\ b<a.$$

II. $\Delta_1=2\int_0^{\theta'}\sqrt{a^2-b^2\sin^2\theta}\,d\theta\Big/\int_0^{\theta'}d\theta=\dfrac{2}{\theta'}\int_0^{\theta'}\sqrt{a^2-b^2\sin^2\theta}\,d\theta.$

Let $\theta=\frac{1}{2}\pi+\lambda$. $\therefore\ \theta'-\frac{1}{2}\pi=\lambda$ to $-\frac{1}{2}\pi=\lambda$.

$$\therefore\ \Delta_1=\frac{2}{\theta'}\int_{-\frac{1}{2}\pi}^{\theta'-\frac{1}{2}\pi}\sqrt{a^2-b^2\cos^2\lambda}\,d\lambda=\frac{2}{\theta'}\int_{-\frac{1}{2}\pi}^{\theta'-\frac{1}{2}\pi}\sqrt{b^2\sin^2\lambda-(b^2-a^2)}\,d\lambda$$

$$=\frac{2\sqrt{(b^2-a^2)}}{\theta'}\int_{-\frac{1}{2}\pi}^{\theta'-\frac{1}{2}\pi}\sqrt{\frac{b^2}{b^2-a^2}\sin^2\lambda-1}\,d\lambda$$

$$=\frac{2\sqrt{(b^2-a^2)}}{\theta'}H_{-\frac{1}{2}\pi}^{\theta'-\frac{1}{2}\pi}\left(\frac{b}{\sqrt{b^2-a^2}},\ \lambda\right),\ b>a.$$

$$\Delta_1=2\int_0^{\frac{1}{2}\pi}\sqrt{a^2-b^2\sin^2\theta}\,d\theta\Big/\int_0^{\frac{1}{2}\pi}d\theta=\frac{4a}{\pi}E_0^{\frac{1}{2}\pi}\left(\frac{b}{a},\ \theta\right),\ b<a.$$

46. Proposed by J. C. WILLIAMS, Boston, Massachusetts.

At the end of the fifth inning the base ball score stands 7 to 9. What is the probability of winning for either team?

Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

From the stated score we are able to estimate the respective skill of the two teams, and their respective probabilities of winning the game.

The respective probabilities are $\frac{7}{16}$ and $\frac{9}{16}$. We have now to find the probabilities of either team winning *at least* 3 games out of 4, granting, of course, 9 ıings to be played. These probabilities are respectfully, $(\frac{7}{16})^4+4(\frac{7}{16})^3\cdot\frac{9}{16}$ [illegible], and $(\frac{9}{16})^4+4.(\frac{9}{16})^3\cdot\frac{7}{16}=$ [illegible].

46. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, ısylvania.

Four men starting from random points on the circumference of a circular field and veling at different rates, take random straight courses across it; find the chance that at st two of them will meet.

Professor Heaton says: "If the men are considered points the chance is ' [A possible though difficult problem could be made of this one by using in-ad of men segments of straight lines moving along random secants of a circle, ı velocity of the segments all being different. Editor.]

47. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

What is the average length of the chords that may be drawn from one extremity of major axis of an ellipse to every point of the curve?

Solution by the PROPOSER.

The length of a single chord is

$$[(a-x)^2+y^2]^{\frac{1}{2}}=(1/a)[a^2(a^2-x^2)+b^2(a^2-x^2)]^{\frac{1}{2}}.$$

Put S=distance around the ellipse. Then the required average is $A=$

$$\frac{2}{S}\int_0^{\frac{1}{2}S}[a^2(a-x)^2-b^2(a^2-x^2)]^{\frac{1}{2}}\,dS=$$

$$\frac{2}{a^2S}\int_{-a}^{+a}\frac{[a^2(a-x)^2+b^2(a^2-x^2)]^{\frac{1}{2}}\,[a^2(a^2-x^2)+b^2x^2]^{\frac{1}{2}}\,dx}{(a^2-x^2)^{\frac{1}{2}}}=$$

$$\frac{2}{a^2S}\int_{-a}^{+a}\frac{[a(a^2+b^2)-(a^2-b^2)x]^{\frac{1}{2}}\,[a^4-(a^2-b^2)x^2]^{\frac{1}{2}}\,dx}{(a+x)^{\frac{1}{2}}}$$

This is readily reducible to elliptic functions of the first and second order, t the expressions I have been able to obtain are involved radicals.

Also solved by *G. B. M. ZERR* and *J. F. SCHEFFER*.

NOTE ON PROBLEM 391

BY LEWIS NEIKIRK, BOULDER, COLORADO.

The man starts at O moving in a *perfectly random* manner. After t sec-ls suppose him at P and that during the next instant dt he travels through ds m at an angle θ with the line OP. Let $PM=dr=ds\cos\theta=v\cos\theta dt$, since $ds=$. He will escape from the desert if $\int dr>R$ (the radius) the limits of integration being those which correspond to 0 and T of t; that is, if $\int_0^T v\cos\theta dt > R$. But this integral depends upon two independent variables. Indeed, θ, being wholly discontinuous from point to point according to the conditions of the problem, can not be considered a variable at all. If however, we assume θ constant (i. e. if the "perfectly random" motion of the problem means motion in a logarithmic spiral) then the condition above reduces to $vT\cos\theta > R$; or $\theta > \cos^{-1}(R/vT)$, agreeing with Professor Anthony.

NOTE ON PROBLEM 39.

BY J. BURKETT WEBB, C. E., PROFESSOR OF MATHEMATICS AND MECHANICS, STEVENS INSTITUTE OF TECHNOLOGY, HOBOKEN, NEW JERSEY.

It seems to me that every such problem should have a complete and intelligible *physical* idea behind it, and further that a solution should be a development of the *physical ideas* of the problem, mathematics being simply the grammatical language of physics.

If Professor Anthony has a complete idea in the problem it is not intelligible to me and so it may be best to state the difficulties which appear to me.

It is to be inferred from the solution that the "perfectly random manner" means that the path consists of differential elements of equal length and all possible directions arranged in a chance succession.

If so the man will never reach the edge of the desert, or, stated otherwise, he will have but one chance in an infinite number of doing so.

In the solution the *rate of approach to the circumference* is spoken of; in random movements there would be no such rate except as the average of actual rates and this is not the use made of it.

The solution also supposes the man at each instant to go *within the angle* MPK, but this he does not need to do to get off in the time; so the deduced chance seems not to follow.

In fact the chance C=etc., is the answer to a different problem, as I see the matter, namely: Of all logarithmic spirals joining the center and circumference, having their origins at the center of the circle and differing from each other by equal increments of the angle between the radius vector and curve, what is the chance of choosing at random one whose included arc shall be less than Tv?

To make the problem apply to the case, for which it was I suppose, intended, of a wanderer in a desert I think one of two things will be needed. Either a certain finite length of step, taken at random must be fixed, or a law established to make large changes of direction less likely than small ones.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

77. **Proposed by F. S. ELDER, Professor of Mathematics, Oklahoma University, Norman, Oklahoma.**

For how many seconds must I count the clicking of the rails under a train that the umber of rails counted may be equal to the speed of the train in miles per hour, a rail eing 30 feet long?

78. **Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.**

Solve by pure arithmetic, no algebraic symbols: A Texan farmer owns 5169 cattle; here are 3 times as many horses as cows, plus 569, and 4 times as many cows as sheep, ninus 126; how many has he of each? [From *Brooks' Higher Arithmetic.*]

79. **Proposed by F. M. PRIEST, St. Louis, Missouri.**

How many $20 gold pieces can be put in a room 20 feet long, 18 feet wide, and 9 feet high?

GEOMETRY.

77. **Proposed by CHARLES C. CROSS, Laytonsville, Maryland.**

A line is drawn perpendicular to BC, of the triangle ABC, whose sides are $BC=a$, $A=b$, and $AB=c$, through A to D, a distance d, (d being equal to or greater than $a+b$); om D a line is drawn to E, a distance e, (e being equal to or greater than $a+b+c$) on BC xtended. Required the area of the ellipse which is isogonal conjugate to the straight line E with respect to the triangle ABC.

78. **Proposed by J. A. MOORE, Professor of Mathematics, Millsaps College, Jackson, Mississippi.**

Required the number of normals that can be drawn from any point (a, b) to the parbola $y^2=2px$.

77. **Proposed by JOHN MACNIE, Professor of Mathematics, University of North Dakota, University, rth Dakota.**

To construct a quadrilateral of given area, the diagonals, one of which is given, cutng each other in given ratios and at a given angle.

MECHANICS.

55. **Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, iversity P. O., Mississippi.**

Three equal heavy spheres, each of weight W, are placed on a rough ground just not uching each other. A fourth sphere of weight nW is placed on the top touching l three. Show that there is equilibrium if the coefficient of friction between two heres is greater than $\tan\frac{1}{2}\alpha$, and that between a sphere and the ground is greater than $n\frac{1}{2}\alpha n/(n+3)$, where α is the inclination to the vertical of the straight line joining the nters of the upper and one lower sphere.

56. **Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, iladelphia, Pennsylvania.**

"Hey-diddle-diddle, the cat and the fiddle,
The cow jumped over the moon."

Taking the weight of the cow to be 600 pounds, the initial resistance of the air to be) pounds and varying as the square of the velocity, find the initial and final velocities d the times of rising and falling.

57. Proposed b J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering, Agricultural and Mechanical College of Texas, College Station, Texas.

Over the intersection of two inclined planes slides a cord of uniform mass throughout its length. Find the equation to the path described by its center of gravity.

AVERAGE AND PROBABILITY.

54. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

A man is at the center of a circle whose diameter is equal to three of his steps. If each step is taken in a perfectly random direction, what is the probability, (1), that he will step outside the circle at the second step, and, (2), that he will step outside at the third step?

55. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

It has been clear for 15 consecutive days, what is the chance of the 16th day being cloudy?

56. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

Find the chance that the center of gravity of a triangle lies inside the triangle formed by three points taken at random within the triangle. [From *Williamson's Integral Calculus*.]

NOTES.

NOTE ON MR. BECHER'S ARTICLE IN OCTOBER NUMBER OF MONTHLY.

BY J. B. BALDWIN, DAVENPORT, IOWA.

In Franklin A. Becher's article for the October number, (Vol. III), I notice he says, "Multiplying an infinite number by another gives us infinity of a higher power, or dividing gives us infinity of a lower power."

How does he reconcile the latter part of this statement with Wallis's expression for the value of π,

$$\tfrac{1}{2}\pi=\frac{2.2.4.4.6.6.8.8\ \ldots\ldots}{1.3.3.5.5.7.7.9\ \ldots\ldots}\ ?$$

In this expression, we have the quotient of two infinite numbers equal to a finite number.

Mr. Lilley's criticism (MONTHLY, Vol. III., No. 3.) of the solution IV. (MONTHLY, Vol. II., page 190) is undoubtedly valid, but the statement that "Todhunter failed to produce a direct proof of it" is probably incorrect. The theorem is given by Todhunter (Euclid, page 316) and in a note at the bottom of page 317 he says, "For the history of this theorem see *Lady's and Gentlemen's Diary* for 1859, page 88." If any reader of the MONTHLY has the Diary for that year I should be very much pleased to see the history of this theorem published in the MONTHLY. Todhunter's proof of the theorem is indirect, but that does not argue that he was unable to discover a direct proof. I remember that many years ago when reading Todhunter's Euclid I attempted a direct proof of this theorem but failed. The proof on page 157, Vol. II. of the MONTHLY is a direct proof and, with the exception of a few mistakes in lettering, seems to be free from objection. A slight simplification may be made by proving the equality of the triangles *ADB*, *BFA*, instead *ADF*, *BDF*. WM. E. HEAL.

MULTI-DIRECTIONAL GEOMETRY.

BY JOHN N. LYLE, PH. D., BENTONVILLE, ARKANSAS.

The concept plane, rectilineal angle implies that there are straight lines and also, that they are located in different directions.

Hence, no system of plane geometry or of spherical geometry for that matter, is free from assumptions regarding "direction."

In some geometrical systems, however, larger use is made of "direction," both word and thing, than in others. Euclid, by his three geometrical axioms and his three postulates places restriction upon "directional geometry" which cannot be relaxed without endangering these axioms and postulates.

According to the Euclidean geometry there is but *one straight path* from the point *A* to the point *B*. That path marks the direction from *A* to *B*. A body moving in this direction on this path approaches *B* until *B* is reached. A body moving in the opposite direction along the same straight path recedes farther and farther from *B*.

This is the pure Euclidean doctrine, clear and strong, free from the suspicion even of a hypothetical "point at infinity" where ungeometrical deeds are reported to be done.

According to the Euclidean view, then, there is but one direction from *A* to *B*. According to Olans Henrici's view as given in the Article on Geometry in the Ninth Edition of the Encyclopœdia Britannica there are at least two directions diametrically opposite to each other from *A* to *B*; one direct and finite in length; the other roundabout via "the point at infinity."

This latter route can hardly be called "air line." Let us notice just one logical difficulty. Every path that reaches *B* drawn from *A* must be continuous. But a continuous line with two ends *A* and *B* must be finite. Hence, the hypothesis that a continuous line, infinite in length, can be drawn between two points *A* and *B* is a flagrant violation of the logical law of Non-Contradiction. By the way, this logical law is the bed rock on which the *reductio ad absurdum* process of reasoning is founded.

Another species under the genus Directional hypothesis is the Multi-Directional hypothesis. According to this hypothesis *B* may be so located with respect to *A* that myriads of different straight lines may be drawn between the two points. That is, *B* is myriads of different directions from *A*. This result seems to me to be absurd. Hence, for that reason, I would reject it. Many modern mathematicians, however, regard their hypotheses as beyond the reach of *reductio ad absurdum* method and the fundamental laws of thought.

EDITORIALS.

Professor Colaw was called away from home during the greater part of the past month, which fact will explain why his departments have been omitted in this issue.

We are happy to announce that a series of short elementary expository articles on Lie's Transformation Groups by Dr. E. O. Lovett, Baltimore, Maryland, will begin in the May number.

Professor Ollis Howard Kendall died last week at his home in Philadelphia. Professor Kendall was for a number of years Assistant Professor of Mathematics at the University of Pennsylvania, at the same time that his father occupied the Chair of Mathematics.

BOOKS AND PERIODICALS.

Algebra Reviews. By Edward Rutledge Robbins, Master in Mathematics and Physics, The Lawrenceville School. Paper Back, 44 pages. Chicago: Ginn & Co.

The object of this little book is to present the essentials of Elementary Algebra in a form sufficiently complete as to be helpful to teachers and students at the time of review. The exercises are various and well selected. Teachers desiring such a book, will find this one well suited to their needs. B. F. F.

Thoughts on Religion. By the late George John Romanes, M. A., LL. D., F. R. S., Canon of Westminster. Edited by Charles Gore, M. A., Canon of Westminster. Cloth, gilt top, 184 pages. Price, $1.25. Chicago: The Open Court Publishing Co.

The value and importance of this work on the thought and conscience of the world cannot be overestimated. Coming as it does from one of the foremost agnostics and scientific thinkers of his time, it comes as a revelation to all classes of readers. In this book can be studied the evolution of a master mind from adhering to the doctrine of agnosticism to that of a full acceptance of the religion of Jesus Christ. B. F. F.

Darwin, and After Darwin. An Exposition of the Darwinian Theory and a Discussion of the Post-Darwinian Questions. By George John Romanes, M. A., LL. D., F. R. S., Honorary Fellow of Gonville and Caius College, Cambridge. I. The Darwinian Theory. Second Edition. Cloth, gilt top, xiv and 460 pages. Price, $2.00. Chicago: The Open Court Publishing Co.

The first volume contains ten chapters. Chapter I, Introductory; Chapter II, Classification; Chapter III, Morphology; Chapter IV, Embryology; Chapter V, Palæontology; Chapter VI, Geographical Distribution; Chapter VII, The Theory of Natural Selection;

Chapter VIII, Evidences of the Theory of Natural Selection; Chapter IX, Criticisms of the Theory of Natural Selection; Chapter X, The Theory of Sexual Selection, and concluding remarks. A more earnest and convincing argument in favor of the Theory of Evolution has not appeared since Darwin's time. Dr. Romanes' grasp of thought and power of cogent reasoning appears in this volume with telling effect. No one with a fair knowledge of the methods of scientific investigations can fail, after having read this book, to be convinced of the truth of the theory. B. F. F.

University Algebra. By C. A. Van Velzer and Chas. S. Slichter, Professors in the University of Wisconsin. Pages 732. Madison, Wisconsin: Tracy, Gibbs and Company. 1893.

This book is now too well known to need any commendation from us. The authors are able and progressive teachers and in this text on algebra have introduced several new and valuable features. There are valuable chapters on mathematical induction, theory of limits, derivatives, complex numbers, the rational integral function, special equations, separation of roots, numerical equations, decomposition of rational fractions, graphic representation of equations, and determinants. The convergence and divergence of series is admirably treated. The accurate "historical notes" which are appended to the treatment of many of the topics will be appreciated. Every teacher of algebra has need of this work in his library whether he uses it as a class text-book or not. J. M. C.

Text-Book of Dynamics. University Tutorial Series. By William Briggs, M. A., F. C. S., F. R. A. S., and G. H. Bryan, M. A. Cloth, 105 pages. Price, 50 cents. Cambridge, England: W. B. Clive. New York Depot: Hinds & Noble, 4 Cooper Institute.

We called attention in a previous number to the text-book on *Hydro-Statics* by the same authors. The treatise on *Dynamics* deserves the same commendation. Due prominence is given to the principles of the subject, and in the solution of problems results are deduced as far as possible from these principles themselves. Worked examples are freely inserted, and hints relating to special difficulties are given where needed. The examples are numerous and practical, the examination papers well selected, and the summary of results after each chapter of special value in reviews. The book may be open to criticism on some minor points, but there are few text-books on this subject which are so well suited to the needs of beginners. J. M. C.

Theoretical Mechanics: Fluids. By J. Edward Taylor, M. A., B. Sc. 222 pages. Price, 80 cents. London and New York: Longmans, Green & Co.

Although intended to meet the Science and Art Department and London Matriculation requirements, this book may be used successfully in any school where a good text-book of its grade is required. One of the special features of the book is the large number of model examples which are fully worked out. The author believes they serve to fix the subject matter on the mind much more than simply reading over the text. This feature also makes it a valuable book to private students. The text is supplied with numerous graduated examples. J. M. C.

A Treatise on Elementary Hydrostatics. By John Greaves, M. A. Price, $1.10. 204 pages. Cambridge Press. New York: Macmillan & Co.

The author aims to treat the subject as fully as possible without using the Calculus, except in alternative proofs when by its aid results are more easily obtained or more concisely expressed. The mathematical element of the book is strong, and the book more advanced than the title and proposed method of treatment would indicate. It is well printed and furnished with sets of carefully selected exercises, while there are many excellent illustrative solutions. The topical index is helpful for ready reference. J. M. C.

Geometry of the Similar Figures and the Plane. By C. W. C. Barlow, M. A., B. Sc., and G. H. Bryan, M. A. Price, 60 cents. 123 pages. University Tutorial Series. Cambridge: W. B. Clive. New York Depot: Hinds & Noble.

This little book contains the Sixth and Eleventh Books of Euclid, together with a summary of Book V., and many important additional propositions and applications relating to the Geometry of Similar Figures and the Plane. Euclid's order has been closely followed, while the additional matter is mostly in the form of illustrative examples. The properties of centers of similitude and homologous points are collected in a supplement at the end of Book VI. In addition to the illustrative examples, numerous exercises for solution follow the propositions on which they depend. The feature of giving many alternative proofs enables the teacher to make his own choice of methods. It is a very satisfactory book in a useful series. J. M. C.

Modern Plane Geometry. By G. Richardson, M. A., and A. S. Ramsay, M. A. Price, $1.00. 202 pages. London and New York: Macmillan & Co.

This treatise includes chapters on properties of a triangle, quadrangle, and circle, harmonic and anharmonic ratio, geometrical maxima and minima, involution, reciprocation, inversion, and projection. It gives all that is best in the recent geometry on these subjects and is an excellent introduction to the more advanced books of Cremona and others. In arrangement the sequence of propositions recommended by the Association for the improvement of Geometrical Teaching has been followed. The triangle has been very fully and satisfactorily treated. The book will serve as an excellent sequel to Euclid, and as a means of proceedure from Euclidean Geometry to the higher descriptive Geometry of Conics and of imaginary points. J. M. C.

Our Notions of Number and Space. By Herbert Nichols, Ph. D., assisted by W. E. Parsons, A. B. 201 pages. Price, $1.00. Ginn & Company, Boston.

This book is an experimental contribution to the "Genetic Theory of Mind." It aims to trace out the origin and development of our present perceptions of number and space from the nature of our past experiences. The experiments were conducted with great care and patience, and the results are worthy of being placed in this permanent and accessable form. The general survey and summary at the end of the book are helpful and valuable. J. M. C.

Business Forms, Customs, and Accounts. By Seymour Eaton. Price of Exercise Manual, 50 cents; price of Book of Forms, $1.00. American Book Company, New York and Chicago.

This manual provides a course of instruction in business which may be used to advantage in schools of all grades where the principles of business are taught. The principles of double entry bookkeeping are taught, but the application of principles to the needs of each particular business are left to be learned in that business. The work is planned to encourage original effort. The exercises are drawn largely from actual transactions. The questions are practical and suggestive. The excellent Book of Forms which accompanies the Exercise Manual will serve to make the teaching of this study both easy and effective. J. M. C.

Spencerian System of Penmanship: Common School Course. No. 10, "Connected Business Forms;" No. 11, "Double Entry Bookkeeping." Price, 8 cents each. American Book Company, New York and Chicago.

These books afford the pupil exercise in penmanship, and also familiarize him in a practical manner with ordinary business forms. J. M. C.

Patriotic Citizenship. By Thomas J. Morgan, LL. D. Price, $1.00. 8 pages. American Book Company, New York, Cincinnate, and Chicago. 95.

The method of this book is a catechism of about 140 questions with as many concise l comprehensive answers by the author. The text of the answers is followed by brief ations from a wide range of authorities chiefly American. Here is found collected ıch of the finest literature on the selected topics, so arranged as to explain and enforce text. The book is designed primarily for the public schools following a course in U. S. tory, but it is also a good book for the citizen, reading circle, or family. We think the ɔk lacks some of the helps, in the way of outline, contents or index, showing the relan of selected topics to central theme, etc., which would have secured better adaptation m a teaching point of view. The study of this book will give good results in stimuing patriotism and promoting good citizenship. J. M. C.

Elementary Lessons in Algebra. By Stewart B. Sabin and Charles D. wry. Price, 50 cents. 128 pages. New York, Cincinnati, and Chicago: ıerican Book Company.

This little book was prepared to meet the demand for a text-book exactly suited to roduce the study of Algebra into Grammar Schools. The development is inductive, l in arrangement, method, problems and exercises, it is well adapted for its purpose.
J. M. C.

Elements of Plane Geometry. By John Macnie, A. M., author of "Theory Equations." Edited by Emerson E. White, A. M., LL. D., author of Vhite's Series of Mathematics." Price, 75 cents. 240 pages. 1895. New ırk, Cincinnati, and Chicago: American Book Company.

In this edition the Plane Geometry is bound separately. We reviewed the Plane l Solid Geometry as bound together in our issue of June, 1895, and a further examinan gives us no reason to withdraw the favorable comments made on this book in that tice. J. M. C.

Inductive Studies in English Grammar. By William R. Harper, Ph. D., esident of the University of Chicago, and Isaac M. Burgess, A. M., Professor the University of Chicago. Cloth, 12mo, 96 pages. Price, 40 cents. New ırk, Cincinnati, and Chicago: American Book Company.

This book presents in a brief compass a systematic course in English Grammar, th special reference to its relation and analogy to other languages. The essential facts the language are briefly and concisely stated, while the terminology and method of sentation are more closely adapted to that used in Latin Grammars. The pupil's owledge is tested by requiring him to pick out concrete examples of its application from ections of connected English, instead of giving rules with classified groups of examples. e book is scholarly and has many strong points, and is excellently adapted for a review ırse in English preparatory to the study of the Ancient or Modern Languages. We beve it will meet with a wider use for this purpose and as supplemental to other gramırs than as an independent class-book. J. M. C.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edd by John Brisben Walker. Price, $1.00 per year in advance. Single copies, cents. Irvington-on-the-Hudson, New York.

What is probably the most important discussion of the educational question ever d, has been opened in the April *Cosmopolitan.* President Gilman of the Johns Hopkins

University will follow the introductory article, and the leading educators of the day will contribute articles upon this most important inquiry: "Does Modern Education Educate, in the Broadest and Most Liberal Sense of the Term?" Those interested in the instruction of youth. either as teacher or parent, can not afford to miss this remarkable symposium, intended to review the mistakes of the nineteenth century, and signalize the entrance of the twentieth by advancing the cause of education. President Dwight of Yale, President Schurman of Cornell, Bishop Potter and President Morton are among those who have already agreed to contribute to what promises to be the most significant series of educational papers ever printed. The aim is to consider existing methods in the light of the requirements of the life of to-day, and this work has never been undertaken on a scale in any degree approaching that outlined for *The Cosmopolitan*. Write to us for subscriptions.

B. F. F.

The Arena. An Illustrated Monthly Magazine. Edited by John Clark Redpath and Helen H. Gardner. Price, $3.00 per year, in advance. Single Number, 25 Cents. Boston: The Arena Co.

The April number of *The Arena* is fully up to the average. In the opening article Governor Pingree, Mayor of Detroit, continues the discussion of Municipal Reform begun in the March number by Mayor Quincy, of Boston. Mayor Pingree, in his breezy paper, affirms that "contracts are the centre and almost the entire circumference of municipal government," and that "almost all the bribes of serious influence in municipalities are given for contracts." His remedy is the letting of contracts by referendum, or direct popular vote.

Under the title of "Lincoln and the Matson Negroes," Jesse W. Weik details the history of a curious slave case, the records of which he has recently unearthed, in which Lincoln was concerned, and which was tried in the circuit court in Illinois, in 1847, during the old fugitive-slave day. None of the numerous biographies of Lincoln makes mention of his part in the affair. B. F. F.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single Number, 25 cents. The Review of Reviews Co., 13 Astor Place, New York City.

In the "Progress of the World" department of the April *Review of Reviews*, the editor comments on the change of administration at Washington, on the tariff bill, and other measures before the extra session of Congress, and on President McKinley's diplomatic appointments; the Greco-Cretan situation is carefully reviewed, and other recent developments in foreign politics are treated with the thoroughness and impartiality to which the *Review's* readers have grown accustomed. B. F. F.

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DISCUSSION OF MERIT CONTESTS IN COLLEGE EXAMINATIONS BY THE METHOD OF LEAST SQUARES.

By CHARLES H. KUMMELL, U. S. Coast and Geodetic Survey, Washington, D. C.

[A paper read before the Philosophical Society of Washington.]

It is well known that it is customary in schools and colleges to estimate merit on the basis of 100 being perfect. If then a number of students 1, 2, 3, ...n receive from judges A, B, C, ...L (number$=m$) the estimates of merit a_1, b_1, c_1, ... l_1 ; a_2, b_2, c_2, ...l_2 ; ...... respectively, it is required to find from these discrepant data the most probable merit of each student as well as the personal error of each judge.

Take the case in which three judges A, B, C, have given estimates of merit to seven students, and let the estimates for the first student be:

$a_1=87$; $b_1=70$; $c_1=70$.

Let M_1=true or most probable merit.

Δa_1=error of judge A,

Δb_1=error of judge B,

Δc_1=error of judge C,

then we have undoubtedly,

$$M_1=\tfrac{1}{3}(a_1+b_1+c_1)\pm0.6745\sqrt{\frac{\Delta a_1^2+\Delta b_1^2+\Delta c_1^2}{3\times2}}$$

$$=75.7\pm3.82 \text{ with weight } p_1=1.56.$$

If there was but this one student, then this would be the final ans the question and the personal errors of the judges must be taken,

$\Delta a_1 = -11.3,$

$\Delta b_1 = +5.7,$

$\Delta c_1 = +5.7.$

If these same judges give estimates of merit to more than one stude shall have more or less discrepant values of the errors of the judges from a mean personal error may be determined. Now we have the following in ual results for the seven students in the example,

$M_1 = 87 - 11.3 = 70 + 5.7 = 70 + 5.7 = 75.7 \pm 3.82$; $p_1 = 1.56,$

$M_2 = 92 - 4.3 = 76 + 11.7 = 95 - 7.3 = 87.7 \pm 3.98$; $p_2 = 1.44,$

$M_3 = 80 - 11.7 = 60 + 8.3 = 65 + 3.3 = 68.3 \pm 4.05$; $p_3 = 1.38,$

$M_4 = 93 - 8.3 = 68 + 16.7 = 93 - 8.3 = 84.7 \pm 5.62$; $p_4 = 0.71,$

$M_5 = 85 - 11.7 = 67 + 6.3 = 68 + 5.3 = 73.3 \pm 3.94$; $p_5 = 1.47,$

$M_6 = 95 - 7.3 = 78 + 9.7 = 90 - 2.3 = 87.7 \pm 3.40$; $p_6 = 1.98,$

$M_7 = 96 - 9.0 = 80 + 7.0 = 85 + 2.0 = 87.0 \pm 3.19$; $p_7 = 2.24.$

In examining these results we notice that judge *A* always over-estin judge *B* under-estimates, and that judge *C* is the least consistent of the judges, as can be roughly seen from their ranges, being 7.4 for *A*, 11.0 for *l* 14.0 for *C*. To determine the personal errors of the judges, the arithn mean of their errors might be taken, but it is more rigorous to take their wei mean, using the above weights, which are reciprocally proportional to the sc of the probable errors. We have thus,

$$\Delta a = \frac{[p\Delta a]}{[p]} \pm 0.6745 \sqrt{\frac{[p\Delta a^2] - [p]\Delta a^2}{[p](n-1)}}$$

=personal error of judge *A*, and similarly for the other judges. We then the numerical results :

$\Delta a = -9.0 \pm 0.70$; $P_a = 1.53.$

$\Delta b = +8.7 \pm 0.79$; $P_b = 1.20.$

$\Delta c = +0.4 \pm 1.31$; $P_c = 0.44.$

It is obvious that if we correct the original estimates by these quar the corrected merits by judge *A* will be nearest the truth, those of *B* will b best, and those of *C* will hardly be improved ; hence the merits must be r mined by the formula,

$$=a_r+\Delta a+\Delta^2 a_r=b_r+\Delta b+\Delta^2 b_r=c_r+\Delta c+\Delta^2 c_r,$$

$$=\frac{P_a(a_r+\Delta a)+P_b(b_r+\Delta b)+P_c(c_r+\Delta c)}{P_a+P_b+P_c}$$

$$\pm 0.6745\sqrt{\frac{P_a(\Delta a^2+\Delta^2 a_r^2)+P_b(\Delta b^2+\Delta^2 b_r^2)+P_c(\Delta c^2+\Delta^2 c_r^2)}{(P_a+P_b+P_c)(3-1)}}$$

ich gives the following numerical results :

$M_1=78.0-0.8=78.7-1.5=70.4+6.8=77.2\pm 4.14,$

$M_2=83.0+2.4=84.7+0.7=95.4-10.0=85.4\pm 4.33,$

$M_3=71.0-1.7=68.7+0.6=65.4+3.9=69.8\pm 4.03,$

$M_4=84.0-1.5=76.7+5.8=93.4-10.9=82.5\pm 4.72,$

$M_5=76.0-1.2=75.7-0.9=68.4+6.4=74.8\pm 4.12,$

$M_6=86.0+0.9=86.7+0.2=90.4-3.5=86.9\pm 3.98,$

$M_7=87.0+0.4=88.7-1.3=85.4+2.0=87.4\pm 3.97.$

These results show, as they should, that the corrected estimates of judge ıme nearest to the correct value, *B's* next best, and *C's* hardly improved ; r ranges being 4.1, 7.3 and 17.7 respectively. We also notice that now con- ınt 7 has the highest merit while in the first approximation 2 and 6 came out with a tie. The reason for this is that 2 and 6 received very high marks ı judge *C*, which have very small weight in the second approximation. re is an apparent paradox in this, that the best values of the merits have ertheless larger probable errors than those of the first approximation. ' each of the arithmetic means of the first approximation involves only three rs out of the 27. By correcting the estimates by the personal errors of the es, a secondary effect of the remaining 24 errors is added in each case.

The second approximation, although sufficiently close to the true values however, yet be improved. For we now have more correct actual errors of udges as follows :

$\Delta a_1=-9.8$; $\Delta b_1=+7.2$; $\Delta c_1=+7.2,$

$\Delta a_2=-6.6$; $\Delta b_2=+9.4$; $\Delta c_2=-9.6,$

$\Delta a_3=-10.7$; $\Delta b_3=+9.3$; $\Delta c_3=+4.3,$

$\Delta a_4=-10.5$; $\Delta b_4=+14.5$; $\Delta c_4=-10.5,$

$\Delta a_5=-10.2$; $\Delta b_5=+7.8$; $\Delta c_5=+6.8,$

$\Delta a_6=-8.1$; $\Delta b_6=+8.9$; $\Delta c_6=-3.1,$

$\Delta a_7=-8.6$; $\Delta b_7=+7.4$; $\Delta c_7=+2.4.$

The weighted means of these, weights being given according to their probable errors, will be new values of the personal errors Δa, Δb, Δc of the judges and applying these we obtain new values for the merits. It is easily seen however that this can only affect the 0.01 and though theoretically speaking an infinite number of approximations is required to obtain the most probable values of the merits we may safely regard the second approximation as sufficient.

ON THE CIRCULAR POINTS AT INFINITY.

By R. D. ROE, JR., Associate Professor of Mathematics in Oberlin College, Oberlin, Ohio.

I. The Coordinate System. In the following discussion, in addition to the usual Cartesion coördinates, homogeneous point and line coördinates will be used. They are related to Cartesian coördinates as follows :*

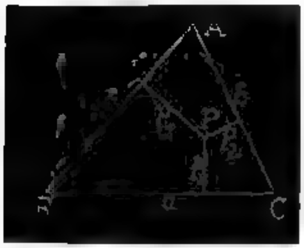

Fig. 1.

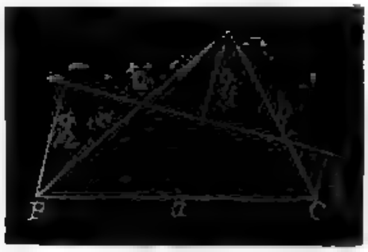

Fig. 1′.

In figure

1, p_1, p_2, p_3 are the perpendicular distances of a point P, from the sides of the coördinate triangle.

1′, q_1, q_2, q_3 are the perpendicular distances of a line Q, from the vertices of the coördinate triangle.

The three

point coördinates of P are expressed as follows :

$$\begin{aligned} \rho x_1 &= p_1 \varkappa_1 \\ \rho x_2 &= p_2 \varkappa_2 \\ \rho x_3 &= p_3 \varkappa_3 \end{aligned} \qquad (1)$$

line coördinates of Q are expressed as follows :

$$\begin{aligned} \sigma u_1 &= q_1 \lambda_1 \\ \sigma u_2 &= q_2 \lambda_2 \\ \sigma u_3 &= q_3 \lambda_3 \end{aligned} \qquad (1)'$$

The $\varkappa$'s and λ's are six constants which might be chosen at pleasure, but for convenience are chosen in a particular way.

*For a fuller treatment see Clebsch, Vorlesungen ueber Geometrie. S. 27-29, S. 62-73.

In Cartesian

int coördinates the equations of the les of the triangle may be

$$a_1x+b_1y+c_1=0 \text{ for } BC$$
$$a_2x+b_2y+c_2=0 \text{ for } CA \qquad (2)$$
$$a_3x+b_3y+c_3=0 \text{ for } AB$$

line coördinates the equations of the opposite vertices will be

$$A_1u+B_1v+C_1=0 \text{ for } A$$
$$A_2u+B_2v+C_2=0 \text{ for } B \qquad (2)'$$
$$A_3u+B_3v+C_3=0 \text{ for } C$$

We have then

$$p_1 = \frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}$$
$$p_2 = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \qquad (3)$$
$$p_3 = \frac{a_3x+b_3y+c_3}{\sqrt{a_3^2+b_3^2}}$$

$$q_1 = \frac{A_1u+B_1v+C_1}{C_1\sqrt{u^2+v^2}}$$
$$q_2 = \frac{A_2u+B_2v+C_2}{C_2\sqrt{u^2+v^2}} \qquad (3)'$$
$$q_3 = \frac{A_3u+B_3v+C_3}{C_3\sqrt{u^2+v^2}}$$

We choose

$$\varkappa_1=\sqrt{a_1^2+b_1^2},\ \varkappa_2=\sqrt{a_2^2+b_2^2},$$
$$\varkappa_3=\sqrt{a_3^2+b_3^2}$$

$$\lambda_1=C_1,\ \lambda_2=C_2,\ \lambda_3=C_3$$

and write

$$\rho x_1=a_1x+b_1y+c_1$$
$$\rho x_2=a_2x+b_2y+c_2 \qquad (A)$$
$$\rho x_3=a_3x+b_3y+c_3$$

$$\sigma u_1=A_1u+B_1v+C_1$$
$$\sigma u_2=A_2u+B_2v+C_2 \qquad (A)'$$
$$\sigma u_3=A_3u+B_3v+C_3$$

Solving these equations, and writing r for (abc),

$$x = \frac{\rho(A_1x_1+A_2x_2+A_3x_3)}{r}$$
$$y = \frac{\rho(B_1x_1+B_2x_2+B_3x_3)}{r}$$
$$1 = \frac{\rho(C_1x_1+C_2x_2+C_3x_3)}{r}$$

$$u = \frac{\sigma r(a_1u_1+a_2u_2+a_3u_3)}{r^2}$$
$$v = \frac{\sigma r(b_1u_1+b_2u_2+b_3u_3)}{r^2}$$
$$1 = \frac{\sigma r(c_1u_1+c_2u_2+c_3u_3)}{r^2}$$

or

$$x = \frac{A_1x_1+A_2x_2+A_3x_3}{C_1x_1+C_2x_2+C_3x_3}$$
$$y = \frac{B_1x_1+B_2x_2+B_3x_3}{C_1x_1+C_2x_2+C_3x_3} \qquad (5)$$

$$u = \frac{a_1u_1+a_2u_2+a_3u_3}{c_1u_1+c_2u_2+c_3u_3}$$
$$v = \frac{b_1u_1+b_2u_2+b_3u_3}{c_1u_1+c_2u_2+c_3u_3} \qquad (5)'$$

Upon our choice of the $\varkappa$'s and λ's depends the following important alt :

(6)

$$\rho\sigma(u_1x_1+u_2x_2+u_3x_3)=\tau(ux+vy+1).$$

Hence $u_1x_1+u_2x_2+u_3x_3$ vanisnes whenever $ux+vy+1$ vanishes. W may note that

(7)

$C_1x_1+C_2x_2+C_3x_3=0$ is the equation of the line at infinity. It gives the condition that $x=y=\infty$.

(7)′

$c_1u_1+c_2u_2+c_3u_3=0$, is the equati of the origin of coördinates. It giv the condition that $u=v=\infty$.

II. The Circular Points.*

A. Proof that all circles whatsoever pass through two points at infinity.

The equations of any two circles may be written,

$$\begin{aligned} x^2+y^2+2gx+2fy+c=0. \\ x^2+y^2+2g'x+2f'y+c'=0. \end{aligned} \tag{8}$$

In homogeneous point coördinates,

$$x=\frac{A_1x_1+A_2x_2+A_3x_3}{C_1x_1+C_2x_2+C_3x_3}=\frac{A}{C}$$

by (5) and our equations become,

$$y=\frac{B_1x_1+B_2x_2+B_3x_3}{C_1x_1+C_2x_2+C_3x_3}=\frac{B}{C}.$$

$$\begin{aligned} A^2+B^2+2gAC+2fBC+cC^2=0. \\ A^2+B^2+2g'AC+2f'BC+c'C^2=0. \end{aligned} \tag{9}$$

The lines passing through their points of intersection are, by subtraction,

$$C[2(g-g')A+2(f-f')B+(c-c')C]=0. \tag{10}$$

Of these the line $C=0$, is the one that interests us. It is the equation o the line at infinity. The infinite points are found by solving $C=0$, wit the equation of either circle, and thus we find them from $C=0$, and $A^2+B^2=0$ or in Cartesian coördinates from the equation of the line at infinity an $x^2+y^2=0$; this would give the same solution always for any two circles; there fore every circle passes through two points at infinity.

B. Cartesian equation of the points in line coördinates.

The equations of the points in Cartesian line coördinates may be readil obtained.

*See Clebsch, Vorlesungen ueber Geometrie, S. 145-149. Salmon's Conic Sections, pages 2[illegible], [illegible] Fiedler's Salmon, Analytische Geometrie der Kegelschnitte, S. 208.

As $x=\frac{A}{C}$, $y=\frac{B}{C}$, the equation of a point $ux+vy+1=0$, becomes

$$Au+Bv+C=0.$$

$x+yi=0$, gives for any point on the line $A+Bi=0$.
We must then have for one of the points $Au+Bv+C=0$, all true.

$$A+Bi=0$$
$$C=0$$

Hence $\begin{vmatrix} u & v & 1 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{vmatrix}=0$, or $ui-v=0$,

similarly for the other point $-ui-v=0$.

For the pair we have $u^2+v^2=0$. (11)

C. Coördinates of the circular points.

1. *Homogeneous rectangular coördinates.* We saw that we could find the coördinates of the circular points by solving the equations of the line at infinity, $x^2+y^2=0$. The equation of the line at infinity is $0x+0y+c=0$. $x^2+y^2=(x+yi)(x-yi)=0$, a pair of imaginary straight lines through the origin. We will find the intersections of a line $ax+by+c=0$ with $x+yi=0$, and $x-yi=0$.

Solving $ax+by=-c$, we get $x=\frac{-ci}{ai-b}$.

$$x+yi=0 \qquad y=\frac{c}{ai-b}.$$

Or $\frac{x}{-i}=\frac{y}{1}=\frac{c}{ai-b}$. If now we introduce a third coördinate c to make our rectangular coöordinates homogeneous, and consider the ratios of x, y, and c as coördinates, we see that the coördinates of one imaginary circular point are given by $x:y:c=-i:1:0$,
and of the other by $x:y:c=i:1:0$. (12)

2. *Homogeneous point coöordinates.* The coördinates of the circular points however assume a more convenient form when expressed in the general point coödinates. We shall obtain them in proving the following highly interesting proposition:

A circle, with fixed center in the finite region, whose radius becomes indefinitely great, degenerates into the two circular points at infinity.

We will obtain the equation of the circle in homogeneous line coördinates. We express that a line u is always at a distance r from the point x', the center of the circle.

Let $u_1x_1+u_2x_2+u_3x_3=0$ be the point equation of the line [see (6)].

This by (4) is

$$u_1(a_1x+b_1y+c_1)+u_2(a_2x+b_2y+c_2)+u_3(a_3x+b_3y+c_3)=0$$

or $(u_1a_1+u_2a_2+u_3a_3)x+(u_1b_1+u_2b_2+u_3b_3)y+(u_1c_1+u_2c_2+u_3c_3)=0.$ (13)

Putting this in cosine form, and taking the square of the distance from $x'y'$ to the line equal to r^2, we get,

$$\frac{[(u_1a_1+u_2a_2+u_3a_3)x'+(u_1b_1+u_2b_2+u_3b_3)y'+(u_1c_1+u_2c_2+u_3c_3)]}{(u_1a_1+u_2a_2+u_3a_3)^2+(u_1b_1+u_2b_2+u_3b_3)^2}=r^2.$$

or (14)*

$$\frac{(u_1x_1'+u_2x_2'+u_3x_3')^2}{\varkappa_1^2u_1^2+\varkappa_2^2u_2^2+\varkappa_3^2u_3^2-2\varkappa_1\varkappa_2u_1u_2\cos C-2\varkappa_2\varkappa_3u_2u_3\cos A-2\varkappa_3\varkappa_1u_3u_1\cos B}=r^2$$

The reductions in the denominator depend on the following:

$$a_1^2+b_1^2=\varkappa_1^2, \text{ etc.} \quad a_1a_2+b_1b_2=\varkappa_1\varkappa_2\left(\frac{a_1b_2}{\varkappa_1\varkappa_2}+\frac{b_1b_2}{\varkappa_1\varkappa_2}\right)$$

$$=\varkappa_1\varkappa_2(\cos\alpha_1\cos\alpha_2-\sin\alpha_1\sin\alpha_2)=\varkappa_1\varkappa_2\cos C, \text{ etc.},$$

where α_1, α_2, α_3 are the angles that p_1, p_2, p_3 make with the axis of x, and the origin is taken within the coördinate triangle. (14) is the line equation of the circle. We notice now that the expression in the denominator may be factored, for considering the variables as, $\varkappa_1u_1$, $\varkappa_2u_2$, $\varkappa_3u_3$, its discriminant is

$$\begin{vmatrix} 1 & -\cos C & -\cos B \\ -\cos C & 1 & -\cos A \\ -\cos B & -\cos A & 1 \end{vmatrix}$$

and that this is zero, may be shown as follows:

From trigonometry we have the three equations:

$$\begin{aligned} a-b\cos C-c\cos B&=0\\ -a\cos C+b-c\cos A&=0\\ -a\cos B-b\cos A+C&=0 \end{aligned}$$

whence it follows that the determinant of the coefficients of a, b, and c, vanishes. Put $\varkappa_1u_1=l$, etc. Then

$$l^2+m^2+n^2-2lm\cos C-2mn\cos A-2nl\cos B=(l\alpha+m\beta+n\gamma)(l\alpha'+m\beta'+n\gamma')$$

$$=l^2\alpha\alpha'+m^2\beta\beta'+n^2\gamma\gamma'+lm(\alpha\beta'+\alpha'\beta)+mn(\beta\gamma'+\beta'\gamma)+nl(\alpha\gamma'+\alpha'\gamma),$$

and we must have:

*Compare Salmon's Conic Sections, page 128, Ex. 6.

$\alpha\alpha'=1.$ Take $\alpha=\cos B-i\sin B,\ \ \gamma=-1,$
$\beta\beta'=1.\qquad \alpha'=\cos B+i\sin B,\ \ \gamma'=-1,$
$\gamma\gamma'=1.\qquad \beta=\cos A+i\sin A,$
$\beta'=\cos A-i\sin A,$

the first three conditions are satisfied. Also,

$+\alpha'\beta=[\cos(A+B)-i\sin(A+B)+\cos(A+B)+i\sin(A+B)]$
$=2\cos(A+B)=-2\cos C.$

$+\beta'\gamma=-\cos A-i\sin A-\cos A+i\sin A=-2\cos A.$

$+\alpha'\gamma=-\cos B+i\sin B-\cos B-i\sin B=-2\cos B.$

Our expression therefore has the two factors, viz :

$$[(\cos B-i\sin B)x_1u_1+(\cos A+i\sin A)x_2u_2-x_3u_3]=L.$$

$$[(\cos B+i\sin B)x_1u_1+(\cos A-i\sin A)x_2u_2-x_3u_3]=M.$$

Also write $u_1x_1+u_2x_2+u_3x_3=u_x$, then our line equation of the circle may written

$$LM=\left(\frac{u_{x'}}{r}\right)^2. \qquad (15)$$

If r becomes indefinitely great $u_{x'}$ does not become indefinitely great, for x''s are finite, the coördinates of the fixed center, and the u's by (4)$'$ are al- rs finite, since u and v are always finite, and for a line which is moved off to nity approach zero together. It follows that $\lim_{r\doteq\infty}\left(\frac{u_{x'}}{r}\right)=0.$

Hence the equation of a circle whose radius is infinite, and whose center ı the finite region is in line coördinates,

$$LM=0. \qquad (16)$$

But this is also the equation of a point-pair, and since we have proved that y circle whatsoever contains the two imaginary circular points at infinity, it ıws that the two points into which this circle has degenerated are themselves two imaginary circular points at infinity. As we might just as well have ıred our expression LM in two other ways, in which the two angles B and C, ' and A, play the same part as A and B, we may write the coördinates of the points in the three ways, as follows:

$$\begin{array}{llll}
(1). & -x_1, & x_2e^{-iC}, & x_3e^{iB}. \\
 & -x_1, & x_2e^{iC}, & x_3e^{-iB}. \\
(2). & x_1e^{iC}, & -x_2, & x_3e^{-iA}. \\
 & x_1e^{-iC}, & -x_2, & x_3e^{iA}. \\
(3). & x_1e^{-iB}, & x_2e^{iA}, & -x_3. \\
 & x_1e^{iB}, & x_2e^{-iA}, & -x_3.
\end{array} \qquad (17).$$

Before going on to the next division of our discussion, we will recall* that if

$$a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2+2a_{12}x_1x_2+2a_{23}x_2x_3+2a_{31}x_3x_1=0$$

where $a_{ik}=a_{ki}$ is the point equation of a conic, with non-vanishing discriminant,

then

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & u_1 \\ a_{21} & a_{22} & a_{23} & u_2 \\ a_{31} & a_{32} & a_{33} & u_3 \\ u_1 & u_2 & u_3 & 0 \end{vmatrix}=0. \qquad (18)$$

is the line equation of the same conic.

and that always

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & x_1 & x_1' \\ a_{21} & a_{22} & a_{23} & x_2 & x_2' \\ a_{31} & a_{32} & a_{33} & x_3 & x_3' \\ x_1 & x_2 & x_3 & 0 & 0 \\ x_1' & x_2' & x_3' & 0 & 0 \end{vmatrix}=0 \qquad (19)$$

is the equation of the pair of tangents from the point x' to the same conic.

$$A_{11}u_1^2+A_{22}u_2^2+A_{33}u_3^2+2A_{12}u_1u_2+2A_{23}u_2u_3+2A_{31}u_3u_1=0$$

where $A_{ik}=A_{ki}$ is the line equation of a conic, with non-vanishing discriminant,

then

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} & x_1 \\ A_{21} & A_{22} & A_{23} & x_2 \\ A_{31} & A_{32} & A_{33} & x_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix}=0. \qquad (18)'$$

is the point equation of the same conic.

and that always

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} & u_1 & u_1' \\ A_{21} & A_{22} & A_{23} & u_2 & u_2' \\ A_{31} & A_{32} & A_{33} & u_3 & u_3' \\ u_1 & u_2 & u_3 & 0 & 0 \\ u_1' & u_2' & u_3' & 0 & 0 \end{vmatrix}=0 \qquad (19)'$$

is the equation of the pair of points where the line u' cuts the same conic.

III. Fundamental Geometrical Relations Defined in Terms of the Circular Points at Infinity.

A. The equation of a circle in terms of the circular points.

Our line equation of the circle (14) may be written:

$$(r^2\varkappa_1^2-x_1^2)u_1^2+(r^2\varkappa_2^2-x_2'^2)u_2^2+(r^2\varkappa_3^2-x_3'^2)u_3^2$$
$$-2(r^2\varkappa_1\varkappa_2\cos B+x_1'x_2')u_1u_2-2(r^2\varkappa_2\varkappa_3\cos A+x_2'x_3')u_2u_3$$
$$-2(r^2\varkappa_3\varkappa_1\cos B+x_3'x_1')u_3u_1=0.$$

Therefore by (18)', its point equation is,

$$\begin{vmatrix} r^2\varkappa_1^2-x_1'^2 & -(r^2\varkappa_1\varkappa_2\cos C+x_1'x_2') & -(r^2\varkappa_2\varkappa_3\cos B+x_1'x_3') & x_1 \\ -(r^2\varkappa_1\varkappa_2\cos C+x_1'x_2') & r^2\varkappa_2^2-x_1'^2 & -(r^2\varkappa_2\varkappa_3\cos A+x_2'x_3') & x_2 \\ -(r^2\varkappa_1\varkappa_3\cos B+x_1'x_3') & -(r^2\varkappa_2\varkappa_3\cos A+x_2'x_3') & r^2\varkappa_3^2-x_3' & x_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix}=0.$$

The coefficients of x_1^2 and x_1x_2 changed in sign will be:

*Clebsch, Vorlesungen ueber Geometrie, S. 118.

f x_1^2, $(r^2\varkappa_2^2-x_2'^2)(r^2\varkappa_3^2-x_3'^2)-(r^2\varkappa_2\varkappa_3\cos A+x_2'x_3')^2$

$$=r^4\varkappa_2^2\varkappa_3^2-r^2(\varkappa_3^2x_2'^2+\varkappa_2^2x_3'^2)-r^4\varkappa_2^2\varkappa_3^2\cos^2A-2r^2\varkappa_2\varkappa_3x_2'x_3'\cos A,$$

f x_1x_2, $2(r^2\varkappa_3^2-x_3'^2)(r^2\varkappa_1\varkappa_2\cos C+x_1'x_2')$

$$+2(r^2\varkappa_2\varkappa_3\cos A+x_2'x_3')(r^2\varkappa_1\varkappa_3\cos B+x_1'x_3')$$

$$=2(r^4\varkappa_1\varkappa_2x_3^2\cos C-r^2\varkappa_1\varkappa_2\cos Cx_3'^2+r^2\varkappa_3^2x_1'x_2'+r^4\varkappa_1\varkappa_2\varkappa_3\cos A\cos B$$
$$+r^2\varkappa_2\varkappa_3x_1'x_3'\cos A+r^2\varkappa_1\varkappa_3x_2'x_3'\cos B).$$

Similarly for the other terms. Put terms containing r^2 on right hand side f equation, divide by r^2, arrange, and reduce, and we get finally,

$$r^2(\varkappa_2\varkappa_3\sin Ax_1+\varkappa_3\varkappa_1\sin Bx_2+\varkappa_1\varkappa_2\sin Cx_3)^2$$

$$=\varkappa_3^2(x_2'x_1-x_2x_1')^2+\varkappa_1^2(x_3'x_2-x_3x_2')^2+\varkappa_2^2(x_1'x_3-x_1x_3')^2$$

$$-2\varkappa_2\varkappa_3(x_3x_1'-x_3'x_1)(x_1x_2'-x_1'x_2)\cos A-2\varkappa_3\varkappa_1(x_1x_2'-x_1'x_2)(x_2x_3'-x_2'x_3)\cos B$$

$$-2\varkappa_1\varkappa_2(x_2x_3'-x_2'x_3)(x_3x_1'-x_3x_1)\cos C.^* \qquad (20)$$

From what we did with LM of (14) it is clear that the right member of 20) can be factored. Put $(x_3'x_2-x_3x_2')=l$, etc. We then have as factors,

$l\varkappa_1e^{-iB}+m\varkappa_2e^{iA}-n\varkappa_3$ or if $\rho\xi_1=\varkappa_1e^{-iB}$, $\rho\xi_1'=\varkappa_1e^{iB}$.
$l\varkappa_1e^{iB}+m\varkappa_2e^{-iA}-n\varkappa_3$ $\qquad\rho\xi_2=\varkappa_2e^{iA}$, $\rho\xi_2'=\varkappa_2e^{-iA}$.
$\qquad\rho\xi_3=-\varkappa_3$, $\rho\xi_3'=-\varkappa_3$.

The factors become,

$$\rho(l\xi_1+m\xi_2+n\xi_3)$$
$$\rho(l\xi_1'+m\xi_2'+n\xi_3')$$

d by supplying the values of l, m, and n, these become

$$\rho\begin{vmatrix}\xi_1 & x_1 & x_1'\\ \xi_2 & x_2 & x_2'\\ \xi_3 & x_3 & x_3'\end{vmatrix},\qquad \rho\begin{vmatrix}\xi_1' & x_1 & x_1'\\ \xi_2' & x_2 & x_2'\\ \xi_3' & x_3 & x_3'\end{vmatrix}$$

Also the left member is $r^2(C_1x_1+C_2x_2+C_3x_3)^2$ since

$$\varkappa_2\varkappa_3\sin A=\varkappa_2\varkappa_3\sin(\alpha_3-\alpha_2)=\varkappa_2\varkappa_3\left(\frac{a_2b_3-a_3b_2}{\varkappa_2\varkappa_3}\right)=C_1,\text{ etc.},$$

*Compare Salmon's Conic Sections, page 128, Ex. 6.

and (20) takes the form

$$r^2(C_1x_1+C_2x_2+C_3x_3)^2=\rho^2(\xi xx')(\xi' xx'). \qquad (21)$$

As a check on our determination of the coördinates of the circular points at infinity, let us see if the coördinates, $\rho\xi$, $\rho\xi'$ satisfy this equation of the circle. They certainly reduce the right member to zero, for $(\xi\xi x')=0$, and $(\xi'\xi' x')=0$. They also reduce the left member to zero, for it is reduced to zero, by the coördinates of any infinitely distant points [see (7)]. We thus confirm the results of (17), and (21) is the equation of a circle, whose center is at x', in terms of the coördinates of the circular points at infinity.*

B. General distance formula in terms of the circular coördinates.

The preceding result (21) may be used as a formula for the distance between two points x and x'. It must first however be made homogeneous in all the coördinates. It is clear that,

$$C_1x_1+C_2x_2+C_3x_3=\kappa\rho^2\begin{vmatrix} x_1 & \xi_1 & \xi_1' \\ x_2 & \xi_2 & \xi_2' \\ x_3 & \xi_3 & \xi_3' \end{vmatrix},$$

for the vanishing of both expressions signifies a line through two infinitely distant points. To determine κ, let $x_2=x_3=0$.

Then $C_1x_1=\kappa\rho^2\begin{vmatrix} x_1 & \xi_1 & \xi_1' \\ 0 & \xi_2 & \xi_2' \\ 0 & \xi_3 & \xi_3' \end{vmatrix}$

$$=\kappa\rho^2x_1\frac{(-\kappa_2\kappa_3(\cos A+i\sin A)+\kappa_2\kappa_3(\cos A-i\sin A)}{\rho^2}$$

$$=-2\kappa x_1\kappa_2\kappa_3 i\sin A$$

$$=-2\kappa x_1 iC_1.$$

$\therefore\ \kappa=-\dfrac{1}{2i}=\tfrac{1}{2}i$, and we have

$$C_1x_1+C_2x_2+C_3x_3=\tfrac{1}{2}(i\rho^2)(x\xi\xi')=$$

a constant by the first solution of (4).

By this fact we can make r^2 homogeneous. For

$$r^2=c\frac{(\xi xx')(\xi' xx')}{(x\xi\xi')^2(x'\xi\xi')^2}$$

where c is some constant. To determine it let the distance between a and a' be

*A question might arise as to the constant ρ^2. That can be disposed of as is done in the next division.

unity. Substituting the value of c so obtained we have

$$r^2=\frac{(\xi xx')(\xi' xx')(a\xi\xi')^2(a'\xi\xi')^2}{(x\xi\xi')^2(x'\xi\xi')^2(\xi aa')(\xi' aa')}, \quad (22)$$

which is seen to be homogeneous and of degree zero in all the coördinates. It is also clear that it is an absolute invariant expression in ternary forms, for on account of the multiplication law of determinants any determinant of the form $(x'y'z')$ in terms of the new variables becomes equal under linear transformation, to $M(xyz)$ where M is the modulus of the transformation, and the transformation equations are:

$$\begin{aligned}\rho x_1' &= a_{11}x_1+a_{12}x_2+a_{13}x_3\\ \rho x_2' &= a_{21}x_1+a_{22}x_2+a_{23}x_3\\ \rho x_3' &= a_{31}x_1+a_{32}x_2+a_{33}x_3 \; *\end{aligned}$$

Our distance is thus defined projectively with respect to the circular points.

C. The equation of the pair of lines from a point x to the two circular points.

These lines will certainly be given by $(xx'\xi)(xx'\xi')=0$. (23).

By (16) the equation of the circular points in line coördinates is

$$\varkappa_1^2u_1^2+\varkappa_2u_2^2+\varkappa_3u_3^2-2\varkappa_1\varkappa_2u_1u_2\cos C-2\varkappa_2\varkappa_3u_2u_3\cos A-2\varkappa_3\varkappa_1u_3u_1\cos B=0.$$

Now the equation of the pair of tangents from x', to the points will be by (19),

$$\begin{vmatrix} \varkappa_1^2 & -\varkappa_1\varkappa_2\cos C & -\varkappa_1\varkappa_3\cos B & x_1 & x_1' \\ -\varkappa_1\varkappa_2\cos C & \varkappa_2^2 & -\varkappa_2\varkappa_3\cos A & x_2 & x_2' \\ -\varkappa_1\varkappa_3\cos B & -\varkappa_2\varkappa_3\cos A & \varkappa_3^2 & x_3 & x_3' \\ x_1 & x_2 & x_3 & 0 & 0 \\ x_1' & x_2' & x_3' & 0 & 0 \end{vmatrix}=0. \quad (24)$$

It follows that this determinant is equal to the left member of (23) multiplied by a constant, since their vanishing represents the same geometrical form. Let $\varkappa$ denote this constant. Put $x_1=a_4$, $x_2'=b_5$, above. $\varkappa_3^2=c_3$, $x_1=d_1$, $x_2'=e_2$, below. On the left hand the term containing $x_1^2x_2'^2$ will be represented by $a_4b_5c_3d_1e_2$. The number of inversions of order $j=3+3+2=8$. On the right hand $\varkappa x_1^2 \begin{vmatrix} x_2' & \\ & \xi_3 \end{vmatrix} \begin{vmatrix} x_2' & \\ & \xi_3' \end{vmatrix} = \frac{\varkappa x_1^2 x_2' \varkappa_3^2}{\rho^2}$.

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*The result of this division is found in Klein's First Lecture, Winter Semester, 1889-90, on the "Nicht-Euclidsche Geometry," S. 40.

$$\therefore\ x_1^2 x_2'^2 \varkappa_3^2 = \frac{\varkappa}{\rho^2} x_1^2 x_2'^2 \varkappa_3^2. \quad \therefore\ \varkappa = \rho^2,$$

and we obtain the interesting result in determinants,*

$$\begin{vmatrix} \varkappa_1^2 & -\varkappa_1\varkappa_2\cos C & -\varkappa_1\varkappa_3\cos B & x_1 & x_1' \\ -\varkappa_1\varkappa_2\cos C & \varkappa_2^2 & -\varkappa_2\varkappa_3\cos A & x_2 & x_2' \\ -\varkappa_1\varkappa_3\cos B & -\varkappa_2\varkappa_3\cos A & \varkappa_3^2 & x_3 & x_3' \\ x_1 & x_2 & x_3 & 0 & 0 \\ x_1' & x_2' & x_3' & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & x_1' & \varkappa_1 e^{-iB} \\ x_2 & x_2' & \varkappa_2 e^{iA} \\ x_3 & x_3' & -\varkappa_3 \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_1' & \varkappa_1 e^{iB} \\ x_2 & x_2' & \varkappa_2 e^{-iB} \\ x_3 & x_3' & -\varkappa_3 \end{vmatrix} \qquad (25)$$

D. The angle between the lines.

Let us take the four lines,

$$x+iy=0. \quad 1. \qquad x-iy=0. \quad 2.$$
$$x+\lambda y=0. \quad 3. \qquad x+\lambda' y=0. \quad 4.$$

The double ratio of these is, taking them in the order named using the ratio,

$$\alpha = \frac{(\mu_1-\mu_3)(\mu_4-\mu_2)}{(\mu_3-\mu_2)(\mu_1-\mu_4)}, \dagger$$

where $\mu_1=i$, $\mu_2=-i$, $\mu_3=\lambda$, $\mu_4=\lambda'$, we have

$$\frac{(i-\lambda)(\lambda'+i)}{(\lambda+i)(i-\lambda')} = r+5i,$$

$$\text{or } i\lambda'-1-\lambda\lambda'-i\lambda = ri\lambda-r-r\lambda\lambda'-ri\lambda'-s\lambda-si-s\lambda\lambda' i+s\lambda.$$

Or, equating the real parts, and the imaginary parts, we have,

$$\lambda'-\lambda = r(\lambda-\lambda')-s(1+\lambda\lambda'),\ (\lambda-\lambda')(r+1) = s(1+\lambda\lambda').$$

$$1+\lambda\lambda' = r(1+\lambda\lambda')+s(\lambda-\lambda'),\ s(\lambda-\lambda') = (1-r)(1+\lambda\lambda').$$

$$\therefore\ \frac{\lambda-\lambda'}{1+\lambda\lambda'} = \frac{s}{1+r} = \frac{1-r}{s}.$$

And we see that r and s are restricted to the relation : $r^2+s^2=1$. denote the angle between the lines 3 and 4,

*Compare Salmon's Conic Sections, page 133, Ex. 2.
†Clebsch, Vorlesungen ueber Geometrie, S. 88.

$\tan\phi = \frac{s}{1+r} = \pm\sqrt{\frac{1-r}{1+r}}$. $\therefore r = \cos 2\phi,\ s = \pm\sin 2\phi$.

If we denote our double ratio of the four lines by (DR), and choose the lower sign for s, we have

$$(DR) = \cos 2\phi - i\sin 2\phi, \text{ or}$$
$$(DR) = e^{-2\phi i},$$
$$\phi = \tfrac{1}{2}i\log(DR). \quad (26)$$

We could have obtained this result in another way. If the equations of 3 and 4 were written more generally

$$ux + vy + 1 = 0.$$
$$u'x + v'y + 1 = 0.$$

$\tan\phi = \frac{uv' - u'v}{uu' + vv'}$. If $z = x + yi$, we have,

$$\log(x+yi) = \tfrac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x).$$
$$\log(x-yi) = \tfrac{1}{2}\log(x^2+y^2) - i\tan^{-1}(y/x).$$
$$\log\frac{x+yi}{x-yi} = 2i\tan^{-1}(y/x) = 2i\omega, \text{ where } \tan\omega = y/x.$$

$$\omega = \tfrac{1}{2}i\log\frac{x-yi}{x+yi}.$$

Using this as a formula for expressing ϕ, we get

$$\phi = \tfrac{1}{2}i\log\left(\frac{uu'+vv'+i(u'v-uv')}{uu'+vv'-i(u'v-uv')}\right)$$

$$= \tfrac{1}{2}i\log\left(\frac{uu'+vv'+\sqrt{(uu'+vv')^2-(u^2+v^2)(u'^2+v'^2)}}{uu'+vv'-\sqrt{(uu'+vv')^2-(u^2+v^2)(u'^2+v'^2)}}\right).^* \quad (27)$$

But the expression under the logarithm is the quotient of the roots of the equation in λ,

$$u^2 + v^2 + 2\lambda(uu'+vv') + \lambda^2(u'^2+v'^2) = 0,^\dagger$$

n equation obtained by substituting in the line equation of the circular points, 11), the values $u+\lambda u'$, $v+\lambda v'$, so that the ratio of the two roots of λ is again

*First given by Laguerre: Nouvelles Annales de Math. 1853.
†See Clebsch, Vorlesungen ueber Geometrie, S. 148.

the double ratio of the four lines. Klein in the before mentioned lecture on Non-Euclidean Geometry obtains the same result in still another way. The angle between two lines is thus also defined projectively with reference to the two fixed circular points at infinity, for the double ratio of four lines is an absolute invariant under linear transformation. Some special results in angle determination may interest us.*

1. The angle that a line to either of the two circular points makes with any other line of the finite region is to be regarded as infinite. For the tangent of that angle is given by

$$\frac{\tan\psi - i}{1 + i\tan\psi} = -i,$$

i, and ψ, being the tangents of the two lines. Now we have,

$$\tan x = \frac{1}{i}\tanh xi = \frac{1}{i}\cdot\frac{e^{2xi}-1}{e^{2xi}+1}.\quad \lim_{x=\infty}\tan x = \frac{1}{i},\ (\tan x)_{x=\infty} = -i.$$

The above angle between the two lines must therefore be regarded as an infinite one. Similarly for the line to the other circular point. By our new definition of angle, the matter is simpler still, for then in this case $\lambda = i$, or $-i$, and $(DR) = r + si = 0$, or ∞, whence $\phi = \infty$.

2. Two lines are perpendicular to each other when the double ratio (26) is equal to -1, that is when the four lines form a harmonic quadrupel. For using $-\pi i$ as $\log(-1)$ we get from (26), $\phi = \frac{1}{2}\pi$. Also above, put $r = -1$, $s = 0$, and obtain the same result.

3. Two lines are parallel when $r = 1$, $s = 0$, that is when the double ratio of the four lines is unity.

4. Two lines make an angle of 45°, or 135° when $r = 0$, $s = \pm l$; that is when the double ratio is equal to $\pm i$.

5. All angles inscribed in a circle and intercepting the same arc are equal for the double ratio of four rays from some variable point in a circle to four fixed points in constant. Here the four fixed points are the two finitepoints at the ends of the arc, and the two fixed circular points at infinity. But if the double ratio is constant r and s are constant, therefore,

$$\frac{\lambda - \lambda'}{1 + \lambda\lambda'} = \frac{s}{1+r} = \frac{1-r}{s}$$

is constant, and the inscribed angle is constant.

IV. Relation of the Circular Points to Non-Euclidean Geometry.

What we have established in the preceding seems to suggest the way for investigations and generalizations of the greatest importance. And such was the course of history on the analytic side of the passage from Euclidean to Non-Euc-

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†Clebsch, Vorlesungen ueber Geometrie, S. 147-149.

lidean geometry. It only remained to make the generalization that, $\Sigma xx = \Sigma a_{ik}x_i x_k = 0$, being the equation of the fundamental form in point or in line coördinates as might be needed, the expression

$$\varkappa \log\left(\frac{\Sigma xx' + \sqrt{(\Sigma xx')^2 - \Sigma xx'.\Sigma x'x}}{\Sigma xx' - \sqrt{\Sigma xx' - \Sigma xx'.\Sigma x'x}}\right)$$

should be in general the distance between the points, or the angle between two lines. If $\varkappa = \frac{1}{2}i$, and $\Sigma xx = u^2 + v^2$ we have the ordinary Euclidean angle between two lines. If $\Sigma xx'$ is not equal to $u^2 + v^2$, we evidently have something quite different from that angle, $\varkappa$ times the logarithm of the double ratio of the two lines and the pair of tangents to the conic from their point of intersection.

The derivation of the Euclidean distance formula is not so simple, a case of limits being involved. According as this fundamental conic is an actual one, a point pair, or an imaginary one, we get hyperbolic, parabolic, or elliptic metrical determination. Cayley seems to have given the first valuable suggestions tending towards analytic methods. Klein has built up an admirable analytic treatment, using what he calls the "Cayley'schen Maassbestimmung" as a basis. In his illustrations of elliptic, and hyperbolic geometry of the plane, he uses as fundamental conics $x^2 + y^2 = -r^2$, and $x^2 + y^2 = r^2$ respectively. It is interesting to note that the square of the element of length in each is,

$$ds_1^2 = \frac{dx^2 + dy^2 + \frac{(ydx - xdy)^2}{r^2}}{\left(1 + \frac{x^2 + y^2}{r^2}\right)^2}, \quad ds_2^2 = \frac{dx^2 + dy^2 - \frac{(ydx - xdy)^2}{r^2}}{\left(1 - \frac{x^2 + y^2}{r^2}\right)^2}.$$

Now if r becomes indefinitely great, we have as the limit of both ds_1^2 and ds_2^2, $ds_3^2 = dx^2 + dy^2$, the square of the element of length in the ordinary Euclidean plane. This affords incidentally confirmation of our proposition under II. C, 2. that when the radius of a circle whose center is in the finite region becomes indefinitely great, the circle degenerates into the two circular points at infinity.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

74. Proposed by JOHN T. FAIRCHILD, Principal of Crawfis College, Crawfis College, Ohio.

When U. S. Bonds are quoted in London at $108\frac{3}{4}$ and in Philadelphia at $112\frac{1}{2}$, exchange $\$4.48\frac{1}{2}$, gold quoted at 107, how much more was a \$1000 U. S. bond worth in London than in Philadelphia?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

If I understand the problem correctly, the exchange price is not necessary for the solution.

$\$1000\times1.12\frac{1}{2}=\$1122.50$, price in Philadelphia.

$\$1000\times1.08\frac{3}{4}=\$1087.50$, price in London.

But one dollar of London gold is worth \$1.07 of Philadelphia currency.

$\therefore\ \$1087.50\times1.07=\$1163.62\frac{1}{2}$, price of London bond in U. S. currency.

$\therefore\ \$1163.62\frac{1}{2}-\$1122.50=\$41.12\frac{1}{2}$, the amount the London bond cost an American more than the Philadelphia bond.

To find the difference in cost to an Englishman in London, we proceed as follows:

$\$1000\times1.12\frac{1}{2}=\$1122.50$.

$\$1122.53+1.07=\$1049.06\frac{58}{107}$, price of the Philadelphia bond in English gold.

$\$1000\times1.08\frac{3}{4}=\$1087.50$.

$\$1087.50-\$1049.06\frac{58}{107}=\$38.43\frac{49}{107}$.

$\$38.43\frac{49}{107}\div\$4.89\frac{1}{2}=7£\ 17s\ .433d$.

[We believe Dr. Zerr's view of this problem to be the correct one. EDITOR.]

77. Proposed by F. S. ELDER, Professor of Mathematics, Oklahoma University, Norman, Oklahoma.

For how many seconds must I count the clicking of the rails under a train that the number of rails counted may be equal to the speed of the train in miles per hour, a rail being 30 feet long.

I. Solution by FREDERIC R. HONEY, Ph. B., New Haven, Connecticut, and CHAS. C. CROSS, Laytonsville, Maryland.

This problem is similar to the one proposed in the July-August number, Vol. III. The result is independent of the number of rails counted and of the number of miles per hour the train is running.

In the problem referred to, the answer is $3a/88$ minutes during which the poles are counted, where a equals the number of yards the polls are apart.

In the present case, $a=10$ yards. Hence, substituting, $3a/88$ minutes$=$ $30/88$ minutes$=20\frac{5}{11}$ seconds.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, and the PROPOSER.

Let t=number of seconds, n=number of miles per hour.

∴ $5280n/3600=22n/15$ feet per second=speed of train. Also in t seconds ain goes $30n$ feet.

∴ $30n/t$=number of feet in one second.

∴ $30n/t=22n/15$. ∴ $t=20\frac{5}{11}$ seconds.

78. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.

Solve by pure arithmetic, no algebraic symbols: A Texan farmer owns 5169 cattle; iere are 3 times as many horses as cows, plus 569, and 4 times as many cows as sheep, iinus 126; how many has he of each? [From *Brooks' Higher Arithmetic*.]

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, and J. C. CORBIN, Principal of Schools, ine Bluff, Arkansas.

$5169+126-569=4726$=number of cattle when there are 4 times as many ows as sheep and 3 times as many horses as cows.

Every time he takes 1 sheep, he takes 4 cows and 12 horses, or 17 in all.

∴ he has as many lots of 1 sheep, 4 cows, 12 horses, as 17 is contained in 4726. ∴ $4726\div17=278$.

∴ 278×1=number of sheep= 278

$278\times4-126$=number of cows= 986

$278\times12+569$=number of horses=3905

Total=5169

This problem was solved with a different view of its enunciation by Frederic R. Honey, and O. S. Westcott, A. M., Sc. D., Principal North Division High School, Chicago, Illinois.

[NOTE. P. S. Berg and H. C. Wilkes should each have received credit in the last number for solving problems 75 and 76. EDITOR.]

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

70. Proposed by J. A. CALDERHEAD, A. B., Professor of Mathematics in Curry University, Pittsburg, ennsylvania.

Given $\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=\sqrt[3]{c}$ to find x.

I. Solution by J. MARCAS BOORMAN, Consultative Mechanician, Counselor at Law, Inventor, Etc., ewlett, Long Island, New York; EDWARD R. ROBBINS, Master in Mathematics and Physics in Lawrenceville chool, Lawrenceville, New Jersey; E. L. SHERWOOD, A. M., Principal of City Schools, West Point, Mississippi; O. W. ANTHONY, M. Sc., Columbian University, Washington, D. C.; A. H. HOLMES, Brunswick, Maine; J. SCHEFFER, A. M., Hagerstown, Maryland.

Cubing, transposing, etc.,

$$(a^2-x^2)^{\frac{1}{3}}[(a+x)^{\frac{1}{3}}+(a-x)^{\frac{1}{3}}]=(c-2a)/3, \text{ or } (a^2-x^2)^{\frac{1}{3}}[c^{\frac{1}{3}}]=(c-2a)/3.$$

$$\therefore\ x=\pm\sqrt{\left(a^2-\frac{(c-2a)^3}{27c}\right)}.\quad \text{If } c=2a,\ x=a.$$

II. **Solution by J. H. DRUMMOND, LL. D., Portland, Maine; A. H. BELL, Hillsboro, Illinois; and CHAS. C. CROSS, Laytonsville, Maryland.**

Transposing $\sqrt[3]{(a-x)}$, cubing and reducing, we have,

$$(a-x)^{\frac{2}{3}}-c^{\frac{1}{3}}(a-x)^{\frac{1}{3}}=(2a-c)/3c^{\frac{1}{3}}.$$

Completing the square, we find,

$$(a-x)^{\frac{1}{3}}=\tfrac{1}{2}\left[\ c^{\frac{1}{3}}\pm\sqrt{\left(\frac{8a-c}{3c^{\frac{1}{3}}}\right)}\right].$$

Hence $x=a-\frac{1}{8}\left[\ c^{\frac{1}{3}}\pm\sqrt{\left(\frac{8a-c}{3c^{\frac{1}{3}}}\right)}\right]^3$; or transposing $\sqrt[3]{(a+x)}$, and proceeding as before,

$$x=\tfrac{1}{8}\left[\ c^{\frac{1}{3}}\pm\sqrt{\left(\frac{8a-c}{3c^{\frac{1}{3}}}\right)}\right]^3-a.$$

Messrs. Bell and Cross let $y=\sqrt[3]{(a+x)}$, substitute, and then solve as above.

III. **Solution by H. C. WHITAKER, M. Sc., Ph. D., Professor of Mathematics in the Manual Training School, Philadelphia, Pennsylvania.**

For convenience in writing, denote $\sqrt[3]{c}$ by b, $\sqrt[3]{(a+x)}$ by y and $\sqrt[3]{(a-x)}$ by z. Of the following equations, (1) is given and the others are assumed. (Take $\alpha^3=1$).

$y+z-b=0$ .... (1).

$\alpha y+z-b=A$ ....(2).

$\alpha^2y+z-b=B$ ....(3).

$y+\alpha z-b=C$ .... (4).

$\alpha y+\alpha z-b=D$ ....(5).

$\alpha^2y+\alpha z-b=E$ ....(6).

$y+\alpha^2z-b=F$ ....(7).

$\alpha y+\alpha^2z-b=G$ ....(8).

$\alpha^2y+\alpha^2z-b=H$ ....(9).

We have, by multiplying these three at a time,

$$[y^3+(z-b)^3][y^3+(\alpha z-b)^3][y^3+(\alpha^2z-b)^3]=0.$$

Or, completing the multiplications,

$$(y^3+z^3-b^3)^3+27b^3y^3z^3=0.$$

Restoring the original values of y, z, and b, we get,

$$27c(x^2-a^2)=(2a-c)^3.$$

Hence $x=\sqrt{\left(\dfrac{(2a-c)^3+27a^2c}{27c}\right)}$.

This may be the root of the given equation or the root of any of the assumed equations, depending on the various values of a and c.

[This example is found in Bonnycastle's Algebra (1845), page 97.]
Also solved by *P. S. BERG*, *H. C. WILKES*, and *G. B. M. ZERR*.

71. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

When $x=0$, find the limit of the expression

$$u=\left(\frac{m+x}{m-x}\right)^{\frac{1}{x}}+\left(\frac{m-x}{m+x}\right)^{\frac{1}{x}}.$$

I. Solution by O. W. ANTHONY, M. Sc., Columbian University, Washington, D. C., and G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let $u=u_1+u_2$. $\therefore u_1=\left(\dfrac{m+x}{m-x}\right)^{\frac{1}{x}}$,

$$\log u_1=(1/x)\{\log(m+x)-\log(m-x)\}$$

$$=(1/x)\{[\log m+(x/m)-(x^2/2m^2)+(x^3/3m^3)-\ldots\ldots]-[\log m-(x/m)-(x^2/2m^2)-(x^3/3m^3)-\ldots\ldots]\}$$

$$=\frac{2}{x}\left(\frac{x}{m}+\frac{x^3}{3m^3}+\frac{x^5}{5m^5}+\ldots\ldots\right)=2\left(\frac{1}{m}+\frac{x^2}{3m^3}+\frac{x^4}{5m^5}+\ldots\ldots\right)$$

$=2/m$, when $x=0$. $\log u_2=-\log u_1=-2/m$, when $x=0$.

$\therefore u_1=e^{2/m}$, $u_2=e^{-2/m}$. $\therefore u=e^{2/m}+e^{-2/m}$ when $x=0$.

II. Solution by H. C. WHITAKER, M. Sc., Ph. D., Professor of Mathematics in Philadelphia Manual Training School, Philadelphia, Pennsylvania.

Since $(1+x)^{1/x}=e$ when $x=0$, we have

$$\left(\frac{m+x}{m-x}\right)^{\frac{1}{x}}=\left[\left(1+\frac{2x}{m-x}\right)^{\frac{m-x}{2x}}\right]^{\frac{2}{m-x}}=e^{2/m}\text{ when }x=0.$$

In the same way $\left(\frac{m-x}{m+x}\right)^{\frac{1}{x}}=e^{-2/m}$ when $x=0$. Hence $u=e^{2/m}+e^{-}$

Also solved by *J. SCHEFFER*.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent t

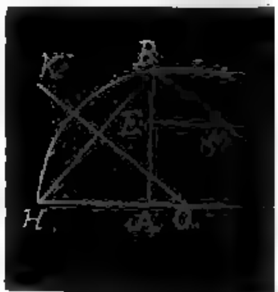
SOLUTIONS OF PROBLEMS.

69. Proposed by WILLIAM SYMMONDS, M. A., Professor of Mathematics and Astronomy in P lege, Santa Rosa, California; P. O., Sebastopol, California.

To divide a square card into right-lined sections in a manner, that a rectan given breadth can be formed from the sections; likewise, form a square from a re lar card.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

(1). Let $ABCD$ be the square. Produce DA to H making AH eq given width of the rectangle, join HB, and draw KO perpendicular to HI mid-point, then O is the center of the circle through HB. Produce AD to meet circle at G; AG is the length of the required rectangle. Take $AE=AH$ and complete the rectangle $AEFG$.



Now the right triangle AHB=right triangle BCN=right triangle MFG.

$\therefore CN=AE$ and $DN=BE$;

$\therefore \triangle BEM=\triangle DNG$.

$\therefore ABCD=ADNME+BCN+BEM$ $=ADNME+MFG+NDG=AEFG$.

(2). Let $AEFG$ be the given rectangle. Produce GA to H $AH=AE$. Upon HG describe the semi-circle. Then AB is a side of the ed square. Complete the square $ABCD$ and draw BG. The rest of the the same as above.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

(1). Let $ABCD$ represent the square card. From A lay off on the width of the rectangle successively as many times as possible, as AE, EF.

Then from the opposite corner C, lay off *one width only* of the rectangle, as CG. Now cut through on line GB. Then cut FH and EI parallel to DA.

Then will *EFGHI* coincide with *DIMKL*, *FBH* with *LKO*, and *BCG* with *O*; and we have the rectangle *AENO*, of the given width *AE*, equivalent to square *ABCD*.

(2). Let *AENO* represent a rectangular card. Find the side of a square valent to the given rectangle. (Any geometry will show construction.)

Now from *A* lay off on *OA* the side of the square successively as many s as possible, as *AD*, *DL*. From *N*, lay off $NM=AD$.

Now, cut through on line *OM*. Then cut *LK* and *DI* perpendicular to *AO*.

Then will the sections of rectangle form a square as shown in first part of tion. The proof is evident.

III. Solution by O. H. WILSON, New York, New York.

Let *ABCD* be the given square, and *EF* the breadth of the required recte. Find *GH*, a third proportional to *EF* and *AB*.

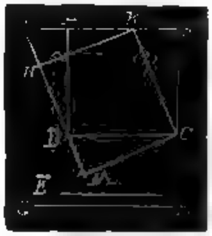


Fig. 1.

With *C* as a center and radius equal to *GH*, describe rc cutting *AB* at *K*. (If $GH<BC$, use *EF* as radius). off the triangle *CBK* and attach it in the position *DAL*. *KN* perpendicular to *LD*. Cut of $\triangle LKN$ and attach the position *DCM*. *NKCM* is the required rectangle, it is equivalent to the square, and its area equals $\times NK=GH\times NK$. But by construction $GH\times EF=\overline{AB}^2$. $\therefore EF=NK$ (*Ax*).

Fig. 2.

If *K* falls beyond *A*, attach *ARCB* in position *TSDA*, cut off *KAR* and attach it in position *LTS*.

Then proceed as before.

Reverse operation (Fig. 1.) by finding mean proportional between *MN* and *MC*. With it as a radius and *C* as a center, describe an arc cutting *MN* in *D*. off $\triangle MDC$ and attach it as *NLK*. Draw *DA* perpendicular to *KL*, cut off *AD* and attach it as *KBC*.

IV. Solution by the PROPOSER.

On *AC*, a side of the given square *ABCD*, draw a -circle *AEC*. Measure from *A* the chord *AE* equal to given breadth of the rectangle. Produce *CE*, marking in *S*. On *CD* take *CM*, *MN*, etc., equal to *AS*.

Through *m*, *n*, etc., draw *ml*, *nr*, etc., parallel to *CS*. se will trace the lines of division required.

From the figure it is plain the rectangle *AEGH* can rmed by joining fragments, *AEC*, *AES*, *SM*, *m*, *Drn*.

(2). On one of the longer sides *AH* of the given recle *AEGH* describe the semi-circle *ABH*.

From *A* lay off chord *AB* equal in length to side of ired square.

Join BH. Take BO, OR, etc., equal to AB. Through O, R, etc., draw OM, RL, etc., parallel to BA, marking the lines of division MN, LR, etc.

Hence the square $ABCD$ forms the parts of the rectangle, AES, SM, MR, RLH, HGF.

COROLLARY. When AS is greater than AB, or conversely, when AB is less than AS, the construction is quite simple.

V. Solution by J. W. SCROGGS, Principal of Rogers Academy, Rogers, Arkansas.

Let $ABCD$ be the given square, $AB=a+b$, $AM=MK=OD=GE=a$, $OK=OC=DE=b$, and $EF=b/r$.

Then area of $ADOM=a^2+ab$, and area of $BNKM=ab$.

$\therefore$ Their sum$=a^2+2ab$.

Let $b/a=r$. Then $a=br$, and area of $EFHG=b/r\times br=b^2$. $\therefore$ $AFHM=ABCD$.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

40. Proposed by B. F. BURLESON, Oneida Castle, New York.

Find (1) in the leaf of the strophoid whose axis is a the axis of an inscribed leaf of the lemniscata, the node of the former coinciding with the crunode of the latter. Find (2) in a leaf of the lemniscata whose axis b the axis of a of an inscribed leaf of the strophoid, the node of the former also coinciding with the crunode of the latter.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

CASE I.

The equation of strophoid with origin at node is,

$$y^2=\frac{x(x-a)^2}{2a-x}\text{..........................(1).}$$

The equation of lemniscate with origin at crunode is,

$$(x^2+y^2)^2=b^2(x^2-y^2)\text{... ....(2).}$$

In order that the latter may be inscribed in the former we must have tangency. $\therefore$ x, y, and dy/dx must be equal for each curve.

Substituting y from (1) in (2) we get after reduction,

$$x^3-4ax^2+\left(\frac{9a^2b^2-a^4}{2b^2}\right)x-a^3=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

$$\frac{dy}{dx}=\frac{b^2x-2x^3-2xy^2}{y(b^2+2x^2+2y^2)}=\frac{4ax^2-4a^2x+a^3-x^3}{y(2a-x)^2} \quad \ldots\ldots\ldots\ldots (4).$$

Substituting y from (1) in (4) and reducing we get,

$$x^4-6ax^3+12a^2x^2-\left(\frac{17a^3b^2-2a^5}{2b^2}\right)x+a^4=0\ldots\ldots\ldots\ldots (5).$$

(5) may be written as follows :

$$x^4-5ax^3+8a^2x^2-\left(\frac{8a^3b^2-a^5}{2b^2}\right)x-a\{x^3-4ax^2+\left(\frac{9a^2b^2-a^4}{2b^2}\right)x-a^3\}=0.$$

(3) in the last equation gives,

$$x^3-5ax^2+8a^2x-\frac{8a^3b^2-a^5}{2b^2}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

(3)−(6) gives, $x^2-\frac{7ab^2+a^3}{2b^2}x-\frac{a^4-6a^2b^2}{2b^2}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$

$\frac{8b^2-a^2}{2b^2}$ times (3)−(6) gives,

$$(6b^2-a^2)x^2-(22ab^2-4a^3)x+\frac{40a^2b^4-17a^4b^2+a^6}{2b^2}=0\ldots\ldots (8).$$

$(6b^2-a^2)$ times (7)−(8) gives,

$$x=\frac{4ab^4-5a^3b^2}{2b^4-7a^2b^2+a^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

$\frac{a^2b^4-17a^4b^2+a^6}{2b^2}$ times $(7)+\frac{a^4-6a^2b^2}{2b^2}$ times (8) gives,

$$x=\frac{16ab^6+13a^3b^4-18a^5b^2+a^7}{8b^6-10a^2b^4} \quad \ldots\ldots\ldots\ldots\ldots (10).$$

From (9) and (10) we get,

$$6b^8+161a^2b^6-141a^4b^4+25a^6b^2-a^8=0\ldots\ldots\ldots\ldots (11).$$

Let $a^2/b^2=u$, then (11) becomes,

$$u^4-25u^3+141u^2-161u-6=0\dots\dots\dots\dots(12).$$

$\therefore\ u=1.586892.\quad \therefore\ a^2=1.586892b^2.$

$\therefore\ a=1.2597b.\quad \therefore\ b=.7938a.$

CASE II.

The equation of the strophoid with origin at crunode is,

$$y^2=\frac{x^2(a+x)}{a-x}\dots\dots\dots\dots(13).$$

The equation of the lemniscate with origin at node is,

$$\{(x+b)^2+y^2\}^2=b^2\{(x+b)^2-y^2\}\dots\dots\dots\dots(14).$$

In order that the former may be inscribed in the latter we must have x, y, dy/dx equal for both curves.

(13) in (14) gives,

$$(2a^2+b^2-4ab)x^3+(4a^2b-5ab^2+b^3)x^2+(4a^2b^2-2ab^3)x+a^2b^3=0\dots\dots(15).$$

$$\frac{dy}{dx}=\frac{a^2x+ax^2-x^3}{y(a-x)^2}=-\frac{2x^3+6bx^2+5b^2x+b^3+2(x+b)y^2}{y(2x^2+4bx+3b^2+2y^2)}\dots\dots(16).$$

(13) in (16) gives,

$$(8ab-4a^2-2b^2)x^4+(8a^3-20a^2b+9ab^2-b^3)x^3$$
$$+(12a^3b-15a^2b^2+3ab^3)x^2+(8a^3b^2-3a^2b^3)x+a^3b^3=0\dots(17).$$

(17) may be written

$$(8ab-4a^2-2b^2)x^4+(6a^3-16a^2b+8ab^2-b^3)x^3$$
$$+(8a^3b-10a^2b^2+2ab^3)x^2+(4a^3b^2-a^2b^3)x$$
$$+a[(2a^2+b^2-4ab)x^3+(4a^2b-5ab^2+b^3)x^2+(4a^2b^2-2ab^3)x+a^2b^3]=0\dots(18).$$

(15) in (18) gives,

$$2(2a^2+b^2-4ab)x^3+(16a^2b-8ab^2+b^3-6a^3)x^2$$
$$+(10a^2b^2-8a^3b-2ab^3)x+a^2b^3-4a^3b^2=0\dots(19).$$

(19)—2 times (15) gives,

$$^2b+2ab^2-b^3-6a^3)x^2+(2a^2b^2-8a^3b+2ab^3)x-a^2b^3-4a^3b^2=0 \;\ldots\ldots(20).$$

b times (19)—$(b-4a)$ times (15) gives,

$$^3+b^3-14a^2b)x^2+(ab^3+10a^3b-8a^2b^2)x+8a^3b^2-2a^2b^3=0 \;\ldots\ldots\ldots\ldots(21).$$

$^3+b^3-14a^2b)$ times (20)—$(8a^2b+2ab^2-b^3-6a^3)$ times (21) gives,

$$x=\frac{14a^3b^3+16a^5b+8a^2b^4-28a^4b^2-3ab^5}{4a^6-38a^3b^3+8ab^4+18a^2b^4-3b^6} \;\ldots\ldots\ldots\ldots(22).$$

$(8a-2b)$ times (20)+$(4a+b)$ times (21) gives,

$$x=\frac{10a^3b^2-24a^4b+8a^2b^3-3ab^4}{14a^2b^2+16a^4+8ab^3-28a^3b-3b^4} \;\ldots\ldots\ldots\ldots\ldots(23).$$

From (22) and (23) we get,

$$9b^5+16ab^4-148a^2b^3-144a^3b^2+468a^4b-176a^5=0 \ldots\ldots\ldots(24).$$

Let $b/a=v$, then (24) becomes,

$$9v^5+16v^4-148v^3-144v^2+468v-176=0.$$

$\therefore\ v=1.12257.\quad \therefore\ b=1.12257a.\quad \therefore\ a=.89085.$

The published solution assumes that both curves coincide at their crunodes. is is not what the problem calls for. The above solution realizes in every re- ct the demands of the problem.

MECHANICS.

aducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

44. Proposed by O. W. ANTHONY, M. Sc., Columbian University, Washington, D. C.

There is a triangle whose sides repulse a center of force within the triangle with an nsity that varies inversely as the distance of the center of force from each point of sides of the triangle. What is the position of equilibrium of the center?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

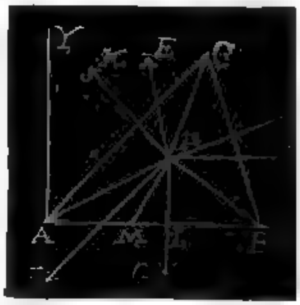
Let $BC=a$, $AC=b$, $AB=c$, $AP=h$, $BP=k$, $CP=l$, $AL=m$, $PL=$
$\angle APB=\beta$, $\angle BPC=\gamma$, $\angle APC=\delta$.

Then $\beta=\cos^{-1}\left(\dfrac{k^2+h^2-c^2}{2hk}\right)$, $\gamma=\cos^{-1}\left(\dfrac{k^2+l^2-a^2}{2kl}\right)$, $\delta=\cos^{-1}\left(\dfrac{h^2+l^2-b^2}{2hl}\right)$.

$$h^2=m^2+n^2,\ k^2=(c-m)^2+n^2,\ l^2=(b\cos A-m)^2+(b\sin A-n)^2.$$

If P be the required point, A origin, AX, AY, axes, $AM=x$, ρ=densi and κ area of cross section of the lines perpendicular to their length, ν=mass P, μ=some constant, then the attraction on M is,

$$\frac{\mu\nu\rho\kappa dx}{PM}=\frac{\mu\nu\rho\kappa dx}{\{n^2+(m-x)^2\}^{\frac{1}{2}}}.$$



Call $\mu\nu\rho\kappa$, Q. Then the resolved part of the force parallel to axis of X is,

$$\frac{Q(m-x)dx}{n^2+(m-x)^2},$$

and the resolved part parallel to the axis of Y is,

$$\frac{Qndx}{n^2+(m-x)^2}.$$

$$\therefore\ PD=-Q\int_0^c\frac{(m-x)dx}{n^2+(m-x)^2}=\tfrac{1}{2}Q\log\left\{\frac{n^2+(c-m)^2}{n^2+m^2}\right\}.$$

$$\therefore\ PD=Q\log(k/h),\ PE=Q\log(l/k),\ PF=Q\log(h/l).$$

$$PG=-Q\int_0^c\frac{ndx}{n^2+(m-x)^2}=-Q\{\tan^{-1}(m/n)+\tan^{-1}\frac{c-m}{n}\}.$$

$$\therefore\ PG=-Q\beta,\ PH=-Q\gamma,\ PK=-Q\delta.$$

If the resultant of two forces is equal to the third, the three forces are i equilibrium.

Regarding PD, PE, PF as three forces and PG, PH, PK as three forces we get,

$$\left.\begin{array}{l}\{\log(k/h)\}^2+\{\log(l/k)\}^2+2\log(k/h)\log(l/k)\cos B=\{\log(h/l)\}^2\\ \beta^2+\gamma^2-2\beta\gamma\sin B=\delta^2\end{array}\right\}\ \ldots\ldots\ldots\ldots(1)$$

$$\left.\begin{array}{l}\log(l/k)\}^2+\{\log(h/l)\}^2+2\log(l/k)\log(h/l)\cos C=\{\log(k/h)\}^2 \\ \gamma^2+\delta^2-2\gamma\delta\sin C=\beta^2\end{array}\right\}\ldots\ldots\ldots\ldots(2).$$

$$\left.\begin{array}{l}\log(h/l)\}^2+\{\log(k/h)\}^2+2\log(h/l)\log(k/h)\cos A=\{\log(l/k)\}^2 \\ \beta^2+\delta^2-2\beta\delta\sin A=\gamma^2\end{array}\right\}\ldots\ldots\ldots\ldots(3).$$

Substituting the values of h, k, l, β, γ, δ in terms of m, n in either (1), ?), or (3), we get, in either case, two equations in m and n from which, if possi-le, the values of m and n may be found and the point P determined.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

60. Proposed by CHARLES C. CROSS, Laytonsville, Maryland.

From a cask containing 10 gallons of wine, a servant drew off 1 gallon each day, for ...days, each time supplying the deficiency by adding a gallon of water. Afterwards, ...ng detection, he again drew off a gallon a day for five days, adding each time a gallon ...ine. How many gallons of water still remained in the cask? [From *Quackenbos' Ar-...etic.*]

61. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, ...field, Missouri.

How far will a body fall in the first second on the sun, the density of the sun being ...nes that of the earth and its diameter 886400 miles?

62. Proposed by CHARLES C. CROSS, Laytonsville, Maryland.

Two men, A and B, started from the same point at the same time; A traveled south-...st for 10 hours and at the rate of 10 miles per hour, and B due south for the same time, ...ing 6 miles per hour; they then turned and traveled directly towards each other at the ...me rates respectively, till they met. How far did each man travel?

DIOPHANTINE ANALYSIS.

55. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washing-...n, D. C.

Construct a general Magic Square whose sum is $3m$.

56. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

If $\phi(R)$ is the number of integers which are less than R and prime to it, and if y is ...rime to R, show that $y^{\phi(R)}-1\equiv0(\text{mod.}R)$.

57. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

Each of *five* of the digits may be the terminal figure of a perfect integral square. ...ach of *eighteen* combinations of two digits may be the *two* terminal figures of an integral ...uare. Each of *one hundred and nineteen* combinations of three digits may be the *three* ...rminal figures of an integral square. *Under these conditions*, what is the greatest number ...' arrangements of the nine digits, all taken together, whose three terminal figures shall ...e those of a square number?

MISCELLANEOUS.

53. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

(a). What is the highest north latitude in which the sun will shine in at the north window of a building at least once in a year?

(b). How many days will it shine in at the north window of a building in latitude 41° N.?

54. Proposed by S. HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, New York.

On latitude 40° N.$=\gamma$, when the moon's declination is 5° 23′ N.$=\delta$, and the sun's 5° 52′ S.$=-\delta$, how long after sunset, will the two horns or cusps of the moon's crescent (recently new) set at the same moment, the crescent with its back *down* having touched the horizon first? Semi-diameters, refraction, and parallax not considered.

55. Proposed by J. M. COLAW, A. M., Monterey, Virginia.

Multiply 6 by 4. Is the problem legitimate when both symbols represent pure number?

[NOTE. "A measured or numbered quantity may be divided into a number of parts, or taken a number of times; but no number can be multiplied or divided into parts."—*McLellan and Dewey's Psychology of Number*. "The astounding thesis is maintained that number is not a magnitude, does not possess quantity at all, and that 'no number can be multiplied or divided into parts'."—*Lefevre's Number and Its Algebra*.]

BOOKS.

Theory of Discrete Manifoldness. By F. W. Franklin. Pamphlet, 12 pages. Published by the Author.

Recent Books on Quaternions (from "Science," Vol. V, pages 699—701). By Dr. Alexander Macfarlane.

This is a concise criticism on the following works: *Theorie der Quaternionen*, Von Dr. P. Molenbroke; *Anwendung der Quaternionen auf die Geometrie*, by same author; *The Outlines of Quaternions*, by Lieut. Col. H. W. L. Hime; *A Primer of Quaternions*, by A. S. Hathaway; and *Utility of Quaternions in Physics*, by A. McAulay. B. F. F.

Application of Hyberbolic Analysis to the Discharge of a Condenser. By Dr. Alexander Macfarlane. Pamphlet, 16 pages.

A paper presented at the annual meeting of the American Institute of Electrical Engineers, May 18th, 1897. B. F. F.

Introduction to American Literature, including Illustrative Selections with Notes. By F. V. N. Painter, A. M., D. D., Professor of Modern Languages in Roanoke College, Author of a History of Education, Introduction to English Literature, etc. 8vo. Cloth, 498 pages. Boston : Leach, Shewell and Sanborn.

This is the best prepared work on American Literature with which we are acquainted. Only the very best productions from the best authors are selected. The selections for special notice, which are chosen to illustrate the distinguishing characteristics of each author are supplied with explanatory notes. The short sketch of our leading writers is written in an intensely interesting manner, all the kernels having been preserved and all the chaff thrown away. Each sketch is preceded by an excellent portrait. B. F. F.

JAMES JOSEPH SYLVESTER

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BIOGRAPHY.

JAMES JOSEPH SYLVESTER, LL. D., F. R. S.

BY GEORGE BRUCE HALSTED.

ON Monday, March 15, 1897, in London, where, September 3, 1814, he was born, died the most extraordinary personage for half a century in the mathematical world.

James Joseph Sylvester was second wrangler at Cambridge in 1837. When we recall that Sylvester, Wm. Thomson, Maxwell, Clifford, J. J. Thomson were all second wranglers, we involuntarily wonder if any senior wrangler except Cayley can be ranked with them.

Yet it was characteristic of Sylvester that not to have been first was always bitter to him.

The man who beat him, Wm. N. Griffin, also a Johnion, afterwards a modest clergyman, was tremendously impressed by Sylvester, and honored him in a treatise on optics where he used Sylvester's first published paper, "Analytical development of Fresnel's optical theory of crystals," *Philosophical Magazine*, 1837.

Sylvester could not be equally generous, and explicitly rated above Griffin the fourth wrangler George Green, justly celebrated, who died in 1841.

Sylvester's second paper, "On the motion and rest of fluids," *Philosophical Magazine*, 1838 and 1839, also seemed to point to physics.

In 1838 he succeeded the Rev. Wm. Ritchie as professor of natural philosophy in University College, London.

His unwillingness to submit to the religious tests then enforced at Cambridge and to sign the 39 articles not only debarred him from his degree and from competing for the Smith's prizes, but, what was far worse, deprived him of the Fellowship morally his due. He keenly felt the injustice.

In his celebrated address at the Johns Hopkins University his denunciation of the narrowness, bigotry and intense selfishness exhibited in these compulsory creed tests, made a wonderful burst of oratory. These opinions were fully shared by De Morgan, his colleague at University College. Copies I possess of the five examination papers set by Sylvester at the June examination, session of 1839-40, show him striving as a physicist, but it was all a false start. Even his first paper shows he was always the Sylvester we knew. To the "Index of Contents" he appends the characteristic note: "Since writing this index I have made many additions more interesting than any of the propositions here cited, which will appear toward the conclusion." Ever he is borne along helpless but ecstatic in the ungovernable flood of his thought.

A physical experiment never suggests itself to the great mental experimenter. Cayley once asked for his box of drawing instruments. Sylvester answered, "I never had one." Something of this irksomeness of the outside world, the world of matter, may have made him accept, in 1841, the professorship offered him in the University of Virginia.

On his way to America he visited Rowan Hamilton at Dublin in that observatory where the maker of quaternions was as out of place as Sylvester himself would have been. The Virginians so utterly failed to understand Sylvester, his character, his aspirations, his powers, that the Rev. Dr. Dabney, of Virginia, has seriously assured me that Sylvester was actually deficient in intellect, a sort of semi-idiotic calculating boy. For the sake of the contrast, and to show the sort of civilization in which this genius had risked himself, two letters from Sylvester's tutors at Cambridge may here be of interest.

The great Colenso, Bishop of Natal, previously Fellow and Tutor of St. John's College, writes: "Having been informed that my friend and former pupil, Mr. J. J. Sylvester, is a candidate for the office of professor of mathematics, I beg to state my high opinion of his character both as a mathematician and a gentleman.

"On the former point, indeed, his degree of Second Wrangler at the University of Cambridge would be, in itself, a sufficient testimonial. But I beg to add that his powers are of a far higher order than even that degree would certify."

Philip Kelland, himself a Senior Wrangler, and then professor of mathematics in the University of Edinburgh, writes: "I have been requested to express my opinion of the qualifications of Mr. J. J. Sylvester, as a mathematician.

"Mr. Sylvester was one of my private pupils in the University of Cambridge, where he took the degree of Second Wrangler. My opinion of Mr. Sylvester then was that in originality of thought and acuteness of perception he had never been surpassed, and I predicted for him an eminent position among the mathematicians of Europe. My anticipations have been verified. Mr. Sylves-

; published papers manifest a depth and originality which entitles them to high position they occupy in the field of scientific discovery. They prove ı to be a man able to grapple with the most difficult mathematical questions are satisfactory evidence of the extent of his attainments and the vigor of his ıtal powers."

The five papers produced in this year, 1841, before Sylvester's departure Virginia, show that now his key note is really struck. They adumbrate ıe of his greatest discoveries.

They are: "On the relation of Sturm's auxiliary functions to the roots of ılgebraic equation," British Assoc. Rep. (pt. 2), 1841; "Examples of the di- ıic method of elimination as applied to ternary systems of equations," Camb. Jour. II., 1841; On the amount and distribution of multiplicity in an alge- ic equation," Phil. Mag. XVII., 1841; "On a new and more general theory ıultiple roots," Phil. Mag. XVIII., 1841; "On a linear method of elimina- ; between double, treble and other systems of algebraic equations," Phil. Mag. III., 1841; "On the dialytic method of elimination," Phil. Mag. XXI., Irish d. Proc. II.

This was left behind in Ireland, on the way to Virginia. Then suddenly ırs a complete stoppage in this wonderful productivity. Not one paper, not word, is dated from the University of Virginia. Not until 1844 does wounded bird begin again feebly to chirp, and indeed it is a whole decade ıre the song pours forth again with mellow vigor that wins a waiting world.

Disheartening was the whole experience; but the final cause of his sudden ndonment of the University of Virginia I gave in an address entitled, "Orig- Research and Creative Authorship the Essence of University Teaching," ıted in *Science*, N. S., Vol. I., pp. 203-7, February 22, 1895.

On the return to England with heavy heart and dampened ardor, he takes for his support the work of an actuary and then begins the study of law. In 7 we find him at 26 Lincoln's Inn Fields, "eating his terms." On Novem- 22, 1850, he is called to the bar and practices conveyancing.

But already in his paper dated August 12, 1850, we meet the significant ıes Boole, Cayley, and harvest is at hand.

The very words which must now be used to say what had already happen- ınd what was now to happen were not then in existence. They were after- 'd made by Sylvester and constitute in themselves a tremendous contribution. he himself says: "Names are, of course, all important to the progress hought, and the invention of a really good name, of which the want, not pre- ısly perceived, is recognized, when supplied, as having ought to be felt, is en- ;d to rank on a level in importance, with the discovery of a new scientific ɔry."

Elsewhere he says of himself: "Perhaps I may without immodesty lay m to the appellation of the Mathematical Adam, as I believe that I have m more names (passed into general circulation) to the creatures of the math- ıtical reason than all the other mathematicians of the age combined."

In one year, 1851, Sylvester created a whole new continent, a new in the universe of mathematics. Demonstration of its creation is given l Glossary of New Terms which he gives in the *Philosophical Transaction* 143, pp. 543-548.

Says Dr. W. Franz Meyer in his exceedingly valuable Bericht über di schritte der projectiven Invariantentheorie, the best history of the subject (

"Als äusseres Zeichen für den Umfang der vorgeschrittenen Entwick mag die ausgedehnte, grösstenteils von *Sylvester* selbst herrührende Termin dienen, die sich am Ende seiner grossen Abhandlung über Sturm'sche Fur en (1853) zusammengestellt findet."

Using then this new language, let us briefly say what had happened decade when Sylvester's genius was suffering from its Virginia wound. birth-day of the giant *Theory of Invariants* is April 28, 1841, the date at by George Boole to a paper in the Cambridge *Mathematical Journal* wh not only proved the invariantive property of discriminants generally also gave a simple principle to form simultaneous invariants of a system functions. The paper appeared in November, 1841, and shortly after, in I ary, 1842, Boole showed that the polars of a form lead to a broad class of iants. Here he extended the results of the first article to more than two F Boole's papers led Cayley, nearly three years later (1845), to propose to h the problem to determine *a priori* what functions of the coefficients of an tion possess this property of invariance, and he discovered its possession by functions besides discriminants, for example the quadrinvariants of binary tics, and in particular the invariant S of a quartic.

Boole next discovered the other invariant T of a quartic and the e sion of the discriminant in terms of S and T. Cayley next (1846) publis symbolic method of finding invariants. Early in 1851 Boole reproduced additions, his paper on Linear Transformations; then at last began Sylv He always mourned what he called "the years he lost fighting the world" after all, it was he who made the Theory of Invariants.

Says Meyer: "sehen wir in dem Cyklus *Sylvester*'scher Publicat (1851–1854) bereits die Grundzüge einer allgemeinen Theorie erstehen, w die Elemente von den verschiedenartigsten Zweigen der späteren Dis umfasst." "Sylvester beginnt damit, die Ergebnisse seiner Vorgänger einem einzigen Gesichtspunkte zuvereinigen."

With deepest foresight Sylvester introduced, together with the or variables, those dual to them, and created the theory of contravariants and mediate forms. He introduced, with many other processes for producing i iantive forms, the principle of mutual differentiation.

Hilbert attributes the sudden growth of the theory to these process producing and handling invariantive creatures. "Die Theorie dieser G erhob sich, von speciellen Aufgaben ausgehend, rasch zu grosser Allgeme —dank vor Allem dem Umstande, dass es gelang, eine Reihe von beson der Invariantentheorie eigenthümlichen Prozessen zu entdecken, deren wendung die Aufstellung und Behandlung invarianter Bildungen beträchtlich erleichterte."

"Was die Theorie der algebraischen Invarianten anbetrifft so sind die ersten Begründer derselben, *Cayley* und *Sylvester*, zugleich auch als die Vertreter der naiven Periode anzusehen : an der Aufstellung der einfachsten Invarianten-bildungen und an den eleganten Anwendungen auf die Auflösung der Gleichungen der ersten 4 Grade hatten sie die unmittelbare Freude der ersten Entdeckung." It was Sylvester alone who created the theory of canonic forms and proceeded to apply it with astonishing power. What marvelous mass of brand new being he now brought forth!

Moreover he trumpeted abroad the eruption. He called for communications to himself in English, French, Italian, Latin or German, so only the "Latin character" were used.

From 1851 to 1854 he produces forty-six different memoirs. Then comes a dead silence of a whole year, broken in 1856 by a feeble chirp called "A Trifle on Projectiles."

What has happened? Some more "fighting the world." Sylvester declared himself a candidate for the vacant professorship of geometry in Gresham College, delivered a probationary lecture on the 4th of December, 1854, and was ignominiously "turned down." Let us save a couple of sentences from this lecture:

"He who would know what geometry is must venture boldly into its depths and learn to think and feel as a geometer. I believe that it is impossible to do this, to study geometry as it admits of being studied, and I am conscious it can be taught, without finding the reasoning invigorated, the invention quickened, the sentiment of the orderly and beautiful awakeded and enhanced, and reverence for truth, the foundation of all integrity of character, converted into a fixed principle of the mental and moral constitution, according to the old and expressive adage '*abeunt studia in mores.*' "

But this silent year concealed still another stunning blow of precisely the same sort, as bears witness the following letter from Lord Brougham to The Lord Panmure:

"Brougham,
Private. 28 Aug. 1855.

My dear P.

My learned excellent friend and brother mathematician Mr. Sylvester is again a candidate for the professorship at Woolwich on the death of Mr. O'Brian who carried it against him last year.

I entreat once more your favorable consideration of this eminent man who has already to thank you for your great kindness.

Yours sincerely,
H. Brougham.

On this third trial, backed by such an array of credentials as no man ever presented before, he barely scraped through, was appointed professor of mathematics at the Royal Military Academy, and served at Woolwich exactly 14 years, 9 months, and 15 days.

A single sentence of his will best express his greatest achievement there and his manner of exit thence:

"If Her most Gracious Majesty should ever be moved to recognise the palmary exploit of the writer of this note in the field of English science as having been the one successfully to resolve a question and conquer an algebraic difficulty which had exercised in vain for two centuries past, since the time of Newton, the highest mathematical intellects in Europe (Euler, Lagrange, Maclaurin, Waring among the number), by conferring upon him some honorary distinction in commemoration of the deed, he will crave the privilege of being allowed to enter the royal presence, not covered, like De Courcy, but barefooted, with rope around his waist, and a *goose*-quill behind his ear, in token of repentant humility, and as an emblem of convicted simplicity in having once supposed that on such kind of success he could found any additional title to receive fair and just consideration at the hands of Her Majesty's Government when quitting his appointment as public professor at Woolwich under the coercive operation of a non-Parliamentary retrospective and utterly unprecedented War Office enactment." Athenæum Club, January 31, 1871. Of course this means a row of barren years, 1870, 1871, 1872, 1873.

The fortunate accident of a visit paid Sylvester in the autumn of 1873 by Pafnuti Lvovich Chebyshev, of the Universty of St. Petersburg, reawakened our genius to produce in a single burst of enthusiasm a new branch of science.

On Friday evening, January 23, 1874, Sylvester delivered at the Royal Institution a lecture entitled "On Recent Discoveries in Mechanical Conversion of Motion," whose ideas, carried on by two of his hearers, H. Hart and A. B. Kempe, have made themselves a permanent place even in the elements of geometry and kinematics. A synopsis of this lecture was published, but so curtailed and twisted into the third person that the life and flavor are quite gone from it. I possess the unique manuscript of this epoch-making lecture as actually delivered. A few sentences will show how characteristic and inimitable was the original form:

"The air of Russia seems no less favorable to mathematical acumen than to a genius for fable and song. Lobacheffsky, the first to mitigate the severity of the Euclidean code and to beat down the bars of a supposed adamantine necessity, was born (a Russian of Russians), in the government of Nijni Novgorod; Tchebicheff [Chebyshev], the prince and conqueror of prime numbers, able to cope with their refractory character and to confine the stream of their erratic flow, their progression, within algebraic limits, in the adjacent circumscription of Moscow; and our own Cayley was cradled amidst the snows of St. Petersburg." [Sylvester himself contracted Chebyshev's limits for the distribution of primes.] "I think I may fairly affirm that a simple direct solution of the problem of the duplication of the cube by mechanical means was never accomplished down to this day. I will not say but that, by a merciful interpretation of his oracle, Apollo may have put up with the solution which the ancient geometers obtained by means of drawing two parabolic curves; but of this I feel assured that had I been alive, and could have shown my solution, which I am about to exhibit to Apollo would have leaped for joy and danced (like David before the ark), my triple cell in hand, in place of his lyre, before his own duplicated altar."

That in the very next year Sylvester was taking a more active part than hitherto been known in the organization of the incipient Johns Hopkins Uni- ty is seen from the following letter to him in London from the great Joseph ry:

SMITHSONIAN INSTITUTION,
August 25, 1875.

EAR SIR:

Your letter of the 18th inst. has just been received and in reply I have to say that I written to President Gilman of the Hopkins University giving my views as to what ght to be and have stated that if properly managed it may do more for the advance of ture and science in this country than any other institution ever established; it is en- independent of public favor and may lead instead of following popular opinion.

I have advised that liberal salaries be paid to the occupants of the principal chairs hat to fill them the best men in the world who can be obtained should be secured.

I have mentioned your name prominently as one of the very first mathematicians of ay; what the result will be, however, I can not say.

The Trustees are all citizens of Baltimore and among them I have some personal ls; the President, Mr. Gilman, and one of them, came to Washington a few weeks ago t from me any suggestions that I might have to offer.

It is to be regretted that in this country the Trustees, who control the management quests of this character, think it important to produce a palpable manifestation of the ution to be established by spending a large amount of the bequest in architectural ays. Against this custom I have protested and have asserted that if the proper men the necessary implements of instruction are provided, the teaching may be done in abins.

It would give me great pleasure to have you again as my guest, and I will do what I o secure your election.

Very truly your friend,
JOSEPH HENRY.

We know the result.

Sylvester was offered the place; demanded a higher salary; won; came.

I was his first pupil, his first class, and he always insisted that it was I brought him back to the Theory of Invariative Forms. In a letter to me of ember 24, 1882, he writes: "Nor can I ever be oblivious of the advantage h I derived from your well-grounded persistence in inducing me to lecture he Modern Algebra, which had the effect of bringing my mind back to this ect, from which it had for some time previously been withdrawn, and in h I have been laboring, with a success which has considerably exceeded my ipations, ever since."

He made this same statement at greater length in his celebrated address e Johns Hopkins on February 22, 1877: "At this moment I happen to be ged in a research of fascinating interest to myself, and which, if the day only onds to the promise of its dawn, will meet, I believe, a sympathetic response the professors of our divine algebraical art wherever scattered through the d.

"There are things called Algebraical Forms; Professor Cayley calls them

Quantics. These are not, properly speaking, Geometrical Forms, although capable, to some extent, of being embodied in them, but rather schemes of processes, or of operations for forming, for calling into existence, as it were, algebraic quantities.

"To every such Quantic is associated an infinite variety of other forms that may be regarded as engendered from and floating, like an atmosphere around it; but infinite in number as are these derived existences, these emanations from the parent form, it is found that they admit of being obtained by composition, by mixture, so to say, of a certain limited number of fundamental forms, standard rays, as they might be termed, in the Algebraic Spectrum of the Quantic to which they belong; and, as it is a leading pursuit of the physicists of the present day to ascertain the fixed lines in the spectrum of every chemical substance, so it is the aim and object of a great school of mathematicians to make out the fundamental derived forms, the Covariants and Invariants, as they are called, of these Quantics.

"This is the kind of investigation in which I have, for the last month or two, been immersed, and which I entertain great hopes of bringing to a successful issue.

"Why do I mention it here? It is to illustrate my opinion as to the invaluable aid of teaching to the teacher, in throwing him back upon his own thoughts and leading him to evolve new results from ideas that would have otherwise remained passive or dormant in his mind.

"But for the persistence of a student of this university in urging upon me his desire to study with me the modern algebra I should never have been led into this investigation; and the new facts and principles which I have discovered in regard to it (important facts, I believe) would, so far as I am concerned, have remained still hidden in the womb of time. In vain I represented to this inquisitive student that he would do better to take up some other subject lying less off the beaten track of study, such as the higher parts of the Calculus or Elliptic Functions, or the theory of Substitutions, or I wot not what besides. He stuck with perfect respectfulness, but with invincible pertinacity, to his point. He would have the New Algebra (Heaven knows where he had heard about it, for it is almost unknown on this continent), that or nothing. I was obliged to yield, and what was the consequence? In trying to throw light upon an obscure explanation in our text-book my brain took fire; I plunged with requickened zeal into a subject which I had for years abandoned, and found food for thoughts which have engaged my attention for a considerable time past, and will probably occupy all my powers of contemplation advantageously for several months to come."

Another specific instance of the same thing he mentions in his paper, "Proof of the Hitherto Undemonstrated Fundamental Theorem of Invariants," dated November 13, 1877:

"I am about to demonstrate a theorem which has been waiting proof for the last quarter of a century and upwards. It is the more necessary that this

ould be done, because the theorem has been supposed to lead to false conclu-
ons, and its correctness has consequently been impugned. Thus in Professor
là de Bruno's valuable *Théorie des formes binaires*, Turin, 1876, at the foot of
ge 150 occurs the following passage: "Cela suppose essentiellement que les
uations de condition *soient toutes indépendantes entr'elles, ce qui n'est pas tou-
ors le cas*, ainsi qu'il résulte des recherches du Professor Gordan sur les nom-
es des covariants des formes quintique et sextique."

The reader is cautioned against supposing that the consequence alleged
ove does result from Gordan's researches, which are indubitably correct. This
pposed consequence must have arisen from a misapprehension, on the part of
. de Bruno, of the nature of Professor Cayley's rectification of the error of
soning contained in his second memoir on Quantics, which had led to results
scordant with Gordan's. Thus error breeds error, unless and until the per-
cious brood is stamped out for good and all under the iron heel of rigid demon-
ation. In the early part of this year Mr. Halsted, a fellow of Johns Hopkins
niversity, called my attention to this passage in M. de Bruno's book; and all I
uld say in reply was that 'the extrinsic evidence in support of the independ-
ce of the equations which had been impugned rendered it in my mind as cer-
in as any fact in nature could be, but that to reduce it to an exact demonstra-
on transcended, I thought, the powers of the human understanding.'"

In 1883 Sylvester was made Savilian professor of geometry at Oxford, the
st Cambridge man so honored since the appointment of Wallis in 1649.

To greet the new environment, he created a new subject for his researches
-Reciprocants, which has inspired, among others, J. Hammond, of Oxford; Mc-
ahon, of Woolwich; A. R. Forsyth, of Cambridge; Leudesdorf, Elliott and
alphen.

Sylvester never solved exercise problems such as are proposed in the *Ed-
ucational Times*, though he made them all his life long down to his latest years.
or example, *unsolved* problems by him will be found even in Vol. LXII. and
ol. LXIII. of the *Educational Times* reprints (1895). If at the time of meeting
s own problem he met also a neat solution he would communicate them to-
ther, but he never solved any. In the meagre notices that have been given of
ylvester the strangest errors abound. Thus C. S. Pierce, in the *Post*, March
th, speaks of his accepting, "with much diffidence," a word whose meaning he
ver knew; and gives 1862 as the date of his retirement from Woolwich, which
eight years wrong, as this forced retirement was July 31, 1870, after his 55th
rthday. Cajori, in his inadequate account (History of Mathematics, p. 326),
ts the studying of law before the professorship at University College and the
ofessorship at the University of Virginia, both of which it followed. Effect
ust follow cause. And strange, that of the few things he ascribes to Sylvester,
should have hit upon something not his, "the discovery of the partial differ-
tial equations satisfied by the invariants and covariants of binary quantics."
t Sylvester has explicitly said in Section VI. of his "Calculus of Forms:" "I
uded to the partial differential equations by which every invariant may be de-

fined. M. Aronhold, as I collect from private information, was the first to think of the application of this method to the subject; but it was Mr. Cayley who communicated to me the equations which define the invariants of functions of two variables."

Surely he needs nothing but his very own, this marvellous man who gave so lavishly to every one devoted to mathematics, or, indeed, to the highest advance of human thought in any form.

University of Texas.

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from March Number.]

III. Proofs Resulting from Comparison of Areas.

Note. Under this head, only a few varieties in connection with each type of figure will be given. The possible number of varieties of "dissection proofs" is absolutely unlimited.

XXXIII. Fig. 25.

Rectangle AM is equivalent to $2\triangle FAC$ is equivalent to $2\triangle EAB$ is equivalent to square EC. Similarly, rectangle BM is equivalent to square KC.

$\therefore$ Adding, square AH is equivalent to square EC+square KC.

Euclid's Proof. Prop. 47, Book I.

Fig. 25.

XXXIV. Fig. 25.

Rectangle AM is equivalent to parallelogram An=parallelogram AO is equivalent to square AD. Similarly, rectangle BM is equivalent to square BL.

$\therefore$ Adding, square AH is equivalent to square AD+square BL.

Edwards's Geometry, page 160.

XXXV. Fig. 25.

$AB\cdot Ad$ is equivalent to $ABOE$ is equivalent to $ACDE$.

$AB\cdot Be$ is equivalent to $ABKP$ is equivalent to $BKLC$.

$\therefore$ Adding, $AB(Ad+Be)$ is equivalent to $AB(Ad+dB)$ is equivalent to $AB\cdot AF=ABHF$ is equivalent to $ACDE+BKLC$.

XXXVI. Fig. 25.

AFMN is equivalent to *AFaC* is equivalent to $AC \cdot Af = ACDE$. Similarly, *BHMN* is equivalent to *BKLC*.

∴ Adding, *ABHF* is equivalent to *ACDE*+*BKLC*.

Vieth, 1805.

XXXVII. Fig. 25.

$FMa = AdE$; $AfF = EDO$; $ANaf = AdOc$.

∴ *AFMN* is equivalent to *ACDE*. Similarly, *BHMN* is equivalent to *BKLC*.

∴ *ABHF* is equivalent to *ACDE*+*BKLC*.

E. von Littrow, 1839.

XXXVIII. Fig. 25.

$AVaF = RUCA$; $FaH = AER$; $HhB = PLK$ is equivalent to $RDU + CLKT$; $BhV = KBT$.

∴ *ABHF* is equivalent to *ACDE*+*BKLC*.

XXIX. Fig. 25.

$AVaF = RUCA$; $FaH = AER$; $RDU = HmW$; $BhmW = BKLS$; $BhV = BCS$.

∴ *ABHF* is equivalent to *ACDE*+*BKLC*.

XL. Fig. 26.

AFMR is equivalent to *ACNO* is equivalent to *ACDE*. So, *BHMR* is equivalent to *BKLC*.

∴ *ABHF* is equivalent to *ACDE*+*BKLC*.

Sechhio, 1753.

XLI. Fig. 26.

AFMR is equivalent to $AFaC = ESLD$ is equivalent to *ACDE*. So, *BHMR* is equivalent to *BKLC*.

∴ *ABHF* is equivalent to *ACDE*+*BKLC*.

Edwards's Geometry, page 158.

XLII. Fig 26.

$CAFU = ACNE$, and $CaU = CND$.

∴ *FaCA* (is equivalent to *AFMR*) is equivalent to *ACDE*. So, *BHMR* is equivalent to *BKLC*.

∴ *ABHF* is equivalent to *ACDE*+*BKLC*.

XLIII. Fig. 26.

$Fa = AC = Va$.

∴ $Fa \cdot Va$ (is equivalent to $AFaC$) $= ACDE$.

∴ *ARMF* is equivalent to *ACDE*. So, *BHMR* is equivalent to *BKLC*.

∴ *ABHF* is equivalent to *ACDE*+*BKLC*.

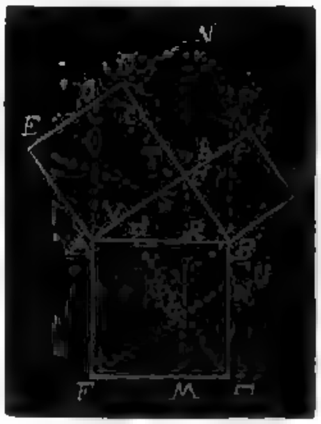


Fig. 26.

XLIV. Fig. 26.

AFHd=*OABc* is equivalent to *AEOcC*.

BcH=*KBT* is equivalent to *KBCb*+*ODc*. *Bcd*=*KLb*.

∴ *ABHF* is equivalent to *ACDE*+*BKLC*.

XLV. Fig. 26.

HaW=*KLb*. *AFHaWB* is equivalent to *ACBWF* is equivalent to *ONPfA* is equivalent to *ONCA*+*NPfC* is equivalent to *ACDE*+*BKbC*.

∴ *ABHF* is equivalent to *ACDE*+*BKLC*.

[To be Continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from April Number.]

PROPOSITION XXVII. *If a straight AX (Fig. 32.) drawn at any however small angle from the point A of AB, must at length meet (anyhow at an infinite distance) any perpendicular BX, which is supposed erected at any distance from this point A upon the secant AB: I say there will then be no more place for the hypothesis of acute angle.*

PROOF. From any point *K* chosen at will in *AB* near the point *A*, the perpendicular *KL* is erected to *AB*, which certainly (from Cor. II. of the preceding proposition) meets *AX* at a finite or terminated distance in some point *L*. But now it holds that there may be assumed in *KB* portions *KK* each equal to a certain assignable length *R*, and these more than any assignable finite number; since indeed the point *B* can be situated, in accordance with the present supposition, at however great a distance from this point *A*.

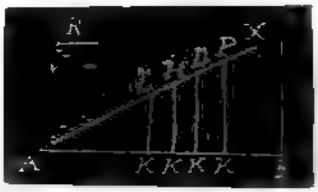


Fig. 32.

And accordingly from the other points *K* are erected to *AB* perpendiculars *KH*, *KD*, *KP*, which all (from the aforesaid corollary) meet the straight *AX* in certain points *H*, *D*, *P*; and so about the remaining points *K* uniformly designated toward the point *B*.

It holds secondly (from Eu. I. 16) that the angles at the points *L*, *H*, *D*, *P* will all be obtuse toward the parts of the points *X*; and just so (from Eu. I. 13) the angles at the aforesaid points will all be acute toward the point *A*.

Therefore (from Cor. II. after 8 of this) the side KH will be greater than the side KL; the side KD greater than the side KH; and so always proceeding towards the points X.

It holds thirdly that the four angles together of the quadrilateral $KLHK$ will be greater than the four angles together of the quadrilateral $KHDK$: for this in like case has already been demonstrated in XXIV of this.

It holds fourthly that the same is valid likewise of the quadrilateral $KHDK$ in relation to the quadrilateral $KDPK$; and so on always, proceeding to quadrilaterals more remote from this point A.

Since therefore are present (as in XXV of this) as many quadrilaterals described in the aforesaid mode, as there are, except the first LK, perpendiculars let fall from points of AX to the straight AB, it will hold uniformly (if we assume nine perpendiculars of this sort let fall, besides the first) the sum of all the angles which are comprehended by these nine quadrilaterals will exceed 35 right angles; and therefore the four angles together of the first quadrilateral $KLHK$, which indeed in this regard has been shown the greatest of all, will fall short of four right angles by less than the ninth part of one right angle. Wherefore, these quadrilaterals being multiplied beyond any assignable finite number, proceeding always toward the parts of the points X, it holds in the same way (as in the same already recited theorem) that the four angles together of this stable quadrilateral $KHLK$ will fall short of four right angles less than any assignable little portion of one right angle.

Therefore these four angles together will be either equal to four right angles, or greater.

But then (from XVI of this) is established the hypothesis either of right angle or of obtuse angle; and therefore (from V and VI of this) is destroyed the hypothesis of acute angle.

So then it holds, that there will be no place for the hypothesis of acute angle, if the straight AX drawn under however small angle from the point A of AB must at length meet (anyhow at an infinite distance) any perpendicular BX, which is supposed erected at any distance from this point A upon this secant AB.

Quod erat etc.

SOME DIVISIBILITY TESTS.

By WM. E. HEAL, Member of the London Mathematical Society, Marion, Indiana.

In the *Educational Times* for March, 1897, Professor Sylvester proposed the following problem: "If the digits r in number of any integer N read from left to right be multiplied repeatedly by the first r terms of the recurring series

1, 4, 3, −1, −4, −3 ; $\dot{1}$, $\dot{4}$, $\dot{3}$, $-\dot{1}$, $-\dot{4}$, $-\dot{3}$, show that, if the sum of these products be divisible by 13, so N will be, and not otherwise." The reason for the rule is apparent when we notice that 1, 4, 3, −1, −4, −3 are the remainders in reverse order of 10^1, 10^2, 10^3, 10^4, 10^5, 10^6 mod. 13 ; or what is the same thing in the development of $\frac{1}{13}$ as a circulating decimal.

Since we may prefix any number of ciphers to any number, it is clear that we may start with any number of the series only being careful to preserve the cyclical order. For example, we might equally as well write the series 3, −1, −4, −3, 1, 4.

Example. 11140640173 is divisible by 13 because 1(1)+4(1)+3(1)−1(4) −4(0)−3(6)+1(4)+4(0)+3(1)−1(7)−4(3)=−26=−2(13).

728 is divisible by (13) because 3(7)−1(2)−4(8)=−13.

The reason for the rule suggests its extension to any number whatever.

Thus $\frac{1}{3}$ developed in a circulating decimal gives the constant remainder 1 and we have the well known rule that a number is divisible by 3 if the sum of its digits is so. $\frac{1}{7}$ developed in a circulating decimal gives the series 2, 3, 1, −2, −3, −1. Thus 6028620892 is divisible by 7 because 2(6)+3(0)+1(2)−2(8) −3(6)−1(2)+2(0)+3(8)+1(9)−2(2)=7.

For 11 the remainders are 1, −1, and we have the known rule for divisibility by 11. For 13 the rule is as stated by Sylvester. For 17 we find the series 1, −5, 8, −6, −4, 3, 2, 7, −1, 5, −8, 6, 4, −3, −2, −7. Thus 442 is divisible by 17 because 3(4)+2(4)+7(2)=34=2(17).

For 19 we have the series 1, 2, 4, 8, −3, −6, 7, −5, 9, −1, −2, −4, −8, 3, 6, −7, 5, −9. It is clear that in this way we can find similar tests of divisibility for any number whatever, but it does not seem worth while to push the matter further except in special cases.

A simple rule for divisibility by 37 may be found in this way. The remainders are 1, −11, 10. Thus 343619 is divisible by 37 because 1(3)−11(4) +10(3)+1(6)−11(1)+10(9)=74=2(37).

May 7, 1897.

INTRODUCTION TO DIFFERENTIATION.

By JOHN MACNIE, A. M., Professor of Mathematics, University of North Dakota.

1. In the identity $\frac{r^n-1}{r-1}=r^{n-1}+r^{n-2}+\dots\dots r+1$, ...(1),

Since r may have any value, let $r=\frac{x^{1/m}}{z^{1/m}}$; then, by substituting this value for r in (1), multiplying both members by $z^{\frac{n-1}{m}}$, and simplifying, we obtain

$$\frac{x^{n/m}-z^{n/m}}{x^{1/m}-z^{1/m}}=x^{\frac{n-1}{m}}+x^{\frac{n-2}{m}}z^{\frac{1}{m}}+\;\ldots\ldots x^{\frac{1}{m}}z^{\frac{n-2}{m}}+z^{\frac{n-1}{m}}\;\ldots\ldots\ldots(2).$$

Dividing both members of (2) by the factor that rendered $x^{1/m}-z^{1/m}$ rational, we obtain, since, by (2), $x-z=(x^{1/m}-z^{1/m})(x^{\frac{n-1}{m}}+\ldots\ldots z^{\frac{n-1}{m}})$,

$$\frac{x^{n/m}-z^{n/m}}{x-z}=\frac{x^{\frac{n-1}{m}}+z^{\frac{n-2}{m}}z^{\frac{1}{m}}+\ldots\ldots x^{\frac{1}{m}}z^{\frac{n-2}{m}}+z^{\frac{n-1}{m}}}{x^{\frac{m-1}{m}}+x^{\frac{m-2}{m}}z^{\frac{1}{m}}+\ldots\ldots x^{\frac{1}{m}}z^{\frac{m-2}{m}}+z^{\frac{n-1}{m}}}\;\ldots\ldots\ldots(3),$$

which, as m may have any value, ± 1 included, is a general expression for the *ratio of the difference of two like powers to the difference of their bases.*

In (3), if we suppose $z=x$, since, then, there are in the numerator of the second member n terms, each $=x^{\frac{n-1}{m}}$, and in the denominator m terms, each $=x^{\frac{m-1}{m}}$ we obtain, for $z=x$,

$$\left[\frac{x^{\frac{n}{m}}-z^{\frac{n}{m}}}{x-z}\right]_{x=z}=\frac{0}{0}=\frac{nx^{\frac{n-1}{m}}}{mx^{\frac{m-1}{m}}}=\frac{n}{m}x^{(n/m)-1}$$

the first member assuming the indeterminate form on account of the presence in numerator and denominator of the factor $x^{1/m}-z^{1/m}$, which becomes zero by hypothesis. Hence, as m may have any value, the formula

$$\left[\frac{x^{n}-z^{n}}{x-z}\right]_{x=z}=nx^{n-1}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

holds true for every value of n. For the sake of simplicity of statement we shall suppose in what immediately follows $m=1$, and n=a positive integer.

Then (3) becomes

$$\frac{x^{n}-z^{n}}{x-z}=x^{n-1}+x^{n-2}z+\ldots\ldots xz^{n-2}+z^{n-1}\;\ldots\ldots\ldots\ldots(3').$$

2. Now, instead of regarding x and z in (3′) as unknown constants, we may regard them as denoting different values of the same variable z, as it varies from $z=0$, through $z=x$, toward $z=+\infty$. From this point of view we see that, assigning any two values to x and z, each member of (3′) expresses *the ratio of the increment of the power to the increment of the base, between these values;* or, briefly expressed, gives the *rate of increase of z^n.* For example, let $z=0$, $x=a$; then both members of (3′) become a^{n-1}, the *average* rate of increase of z^n while z increases from 0 to a; i. e. while z has increased by a units, z^n has increased a^{n-1} times as fast. We say "average rate" because, as will be seen by giving different values to a, the rate of increase of z^n is continually accel-

erating, just as the velocity or rate of motion of a falling body is contin[u]
accelerating.

3. If now we suppose $z=x-h$, h being infinitely small, the second m[em-]
ber of (3′) will be *less* than nx^{n-1} by a difference infinitely small ; and if we [sup-]
pose $z=x+h$, the second member of (3′) will be *greater* than nx^{n-1} by a differ[ence]
infinitely small ; we infer, accordingly (the values of that second member b[eing]
continuous) that nx^{n-1} represents the *rate of increase of z^n when z is pas[sing]*
through x. For, if nx^{n-1} does not represent the rate of increase of z^n when [z is]
passing through x, for what value of z does it represent the rate ?

The difficulty that is here experienced arises from the fact that we h[ave]
here to deal, not with a constant ratio, as in algebra, but with a ratio that [var-]
ies continuously as its terms vary, ratios of frequent occurence in phy[sics]
and kindred sciences. Thus, when we say that a falling body at a certain p[oint]
in its descent has a velocity of 50 feet a second, we do not mean that the b[ody]
moves at that rate during any assignable period of time, but *would* descend t[hat]
distance in a second, *if the motion continued uniform.* In the same way, n[...]
does not mean the rate of increase of z^n during an interval of increase of z but
rate at which z^n would increase if the rate became constant from x.

From the limitation of our faculties, we are unable to realize the *absol[ute]*
as, for example, to draw or even conceive a straight line absolutely with[out]
breadth. Yet, while admitting this inability, we ignore in our reasonings u[pon]
straight lines all that is inconsistent with their definition. Similarly, whil[e in]
our conception of a variable, a changing velocity for example, we can not h[elp]
thinking of the *element* of change as constant for some interval, however min[ute,]
we here, again, ignore whatever is inconsistent with the definition of a vari[able]
as changing continuously. There is no objection, then, to our assisting our g[rasp]
of the idea by regarding a power of a variable as changing by infinitely sm[all]
constant* *elements*, as long as we ignore inconsistent consequences.

4. Def. *Function*, as usual. Example, x^n a function of x.

5. Def. A variable being supposed to change by infinitely small [ele-]
ments, such an element is called the *differential* of the variable. The differen[tial]
of a variable is denoted by the symbol d prefixed to the symbol of the varia[ble.]
Thus dx, $d(x^n)$, are read respectively, *the differential of x, the differential of x^n*

It has already been seen that $d(x^n)=nx^{n-1}dx$, that is when the variab[le is]
passing through the value x, the power is changing nx^{n-1} times as fast as the [var-]
iable. Hence nx^{n-1} is called *the differential coefficient* of x^n, etc.

6. (Here would follow the demonstration of the rules for algebraic s[ums]
of variables, found much as usual. The rule for the differential of products [may]
be found as follows, without the intervention of series.)

7. *To find the differential of the product of two variables*, say xy.

$\because\ 2xy=(x+y)^2-x^2-y^2$.

$\therefore\ 2d(xy)=2(x+y)\times d(x+y)-d(x^2)-d(y^2)$.

*That is, constant during an infinitely small interval.

$$2d(xy)=2(x+y)(dx+dy)-2xdx-2ydy.$$

i. e. $d(xy)=xdx+xdy+ydx+ydy-xdx-ydy.$

i. e. $d(xy)=xdy+ydx.$

From this may be derived rules for $d(xyz)$, etc., and $d(x/y)$.

8. Here would follow demonstration by differentials of Binomial Formula all values of n, with exercises.

9. Here would follow the algebraic deduction of some such formula as:

$$\log(1+z)=M(z-\tfrac{1}{2}z^2+\tfrac{1}{3}z^3-\ldots\ldots \text{ ad inf.})$$

$$\text{whence } d(\log 1+z)=M(1-z+z^2-\ldots\ldots\text{ad inf.})dx$$

$$d(\log 1+z)=M.\frac{1}{1+z}.dx$$

putting x for $1+z$ we have

$$d\log x=M(dx/x).$$

Whence may be derived $d(a^x)=a^x\log a$, etc.

10. Here would follow the algebraic deduction of

$$\sin x=x-(x^3/3\,!)+(x^5/5\,!)-(x^7/7\,!)+\ \ldots\ldots \qquad (1)$$

$$\text{and } \cos x=1-(x^2/2\,!)+(x^4/4\,!)-(x^6/6\,!)+\ldots\ldots \qquad (2).$$

From (1), $d(\sin x)=\{1-(x^2/2\,!)+(x^4/4\,!)-(x^6/6\,!)+\ldots\ldots\}dx=\cos x dx$,

from (2), $d(\cos x)=-\sin x dx$, etc.

11. Then might follow applications to questions of maxima and minima, Then deduction of Taylor's Theorem, with applications.

ARITHMETIC.

ducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

79. Proposed by F. M. PRIEST, St. Louis, Mo.

How many $20 gold pieces can be put in a room 20 feet long, 18 feet wide, 9 feet high?

Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Spring-Springfield, Missouri.

A \$20 gold piece is about $\frac{8}{100}$ of an inch thick, and about $1\frac{7}{20}$ inches in diameter. By putting the pieces in cylindrical layers lengthwise of the room, we can place $(18\times12)\div1\frac{7}{20}$ or 160 cylinders in the first layer, each cylinder containing $(20\times12)\div\frac{8}{100}$ or 3000 \$20 gold pieces. By rectangular arrangement of the cylinders we can put in $(9\times12)\div1\frac{7}{20}$ or 80 layers. Hence, by this arrangement, we can put $80\times160\times3000=38,400,000$ pieces in the room.

By laying the cylinders of the second layer of cylinders between two cylinders of the first layer, the distance between the plane of centers of the first layer and the plane of centers of the second layer is $\sqrt{(\frac{27}{20})^2-(\frac{27}{40})^2}=\frac{27}{40}\sqrt{3}$. Hence, there can be placed in the room, by this arrangement, $(9\times12)\div\frac{27}{40}\sqrt{3}$ or 92 layers $+.376$ of a layer.

In these 92 layers 46 layers would contain 160 cylinders and 46 would cantain 159. But since there is still room at the top the last layer can be placed in so as to contain 160 cylinders.

Hence, there will be 47 layers of 160 cylinders and 45 layers of 159.

Since each cylinder contains 3000 \$20 gold pieces, there can be placed in the room by this method $(47\times160+45\times159)\times3000=44,025,000$ pieces.

It is possible that by considering other dimensions in the same way as the width in this solution a still larger number may be placed in the room.

Charles C. Cross obtained as his answer 38,400,000.

80. Proposed by CHARLES C. CROSS, Laytonsville, Maryland.

From a cask containing 10 gallons of wine, a servant drew off 1 gallon each day, for five days, each time supplying the deficiency by adding a gallon of water. Afterwards, fearing detection, he again drew off a gallon a day for five days, adding each time a gallon of wine. How many gallons of water still remained in the cask? [From *Quackenbos' Arithmetic.*]

Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Let 10 gallons$=a$, 1 gallon$=b$, the quantity of water or wine added after each draught, $\frac{1}{10}=b/a=1/n$, the part drawn off each time.

Then $a-a/n=a\left(\frac{n-1}{n}\right)$=quantity of wine left after first draught;

$a\left(\frac{n-1}{n}\right)-1/n$ of $a\left(\frac{n-1}{n}\right)=a\left(\frac{n-1}{n}\right)^2$=quantity of wine left after second draught;

$a\left(\frac{n-1}{n}\right)^2-1/n$ of $a\left(\frac{n-1}{n}\right)^2=a\left(\frac{n-1}{n}\right)^3$=quantity of wine left after third draught; and $a\left(\frac{n-1}{n}\right)^m$ =quantity left after the mth draught$=A$.

Then $a-A$=water in the cask.

$A+b$=quantity of wine in cask before the $(m+1)$th draught since b gallons of wine are added.

$A+b-[(A/n)+(b/n)]+b=A\left(\frac{n-1}{n}\right)+b\left(\frac{n-1}{n}\right)$=quantity of wine before the $(m+2)$th draught.

$A\left(\frac{n-1}{n}\right)+b\left(\frac{2n-1}{n}\right)-A\left(\frac{n-1}{n^2}\right)-b\left(\frac{2n-1}{n^2}\right)+b=A\left(\frac{n-1}{n}\right)^2-b\left(\frac{3n^2-3n+1}{n^2}\right)$

=quantity of wine before the $(m+3)$th draught.

$$\therefore A\left(\frac{n-1}{n}\right)^p+b\left(pn^{p-1}-\frac{p(p-1)}{1.2}n^{p-2}+\ldots\ldots\right)+b$$

$$=A\left(\frac{n-1}{n}\right)^p+b\left(\frac{n^p-(n-1)^p}{n^p}\right)$$

=quantity of wine left after $(m+p)$th draught$=a\left(\frac{n-1}{n}\right)^{m+p}+b\left(\frac{n^p-(n-1)^p}{n^p}\right)$

In the present case, $a=10$, $b=1$, $1/m=\frac{1}{10}$, $m=5$, and $p=5$. Hence, substituting, we have $10\left[\frac{10-1}{10}\right]^{10}+1.\left[\frac{10^5-(10-1)^5}{10^5}\right]=7.581884401$ gallons, the quantity of wine left after putting in the last gallon of wine, and, therefore, 2.4181155599 gallons=quantity of water in the cask.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

71. Proposed by ROBERT J. ALEY, A. M., Ph. D., Professor of Mathematics, Indiana University, Bloomington, Indiana.

Prove by pure geometry: A perpendicular at the middle point, M_a, of the side BC of the triangle ABC meets the circumcircle in A'. On this perpendicular A'' and A''' are taken so that $M_aA''=M_aA'$ and $A''A'''=AH$. (H is the orthocenter of triangle ABC.) Prove that A''' is on the circumicircle.

Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bor New Jersey, and the PROPOSER.

Let $M_aA_1=M_aA_2$, $A_2A_3=AH$, to prove A_3 on the circumference circle. Since A_1A_3 is a line through M, the center of the circle, the proposition is in effect to prove A_3 one extremity of the diameter through M_a.

By the conditions $AH=A_2A_3$, and is parallel to it, therefore AHA_3A_2 is a parallelogram.

Also triangles BHA and M_aMM_b are similar, hence since $2M_aM_b=AB$, we have $AH=2MM_a$.

Therefore, $A_1A_3=A_3A_2+A_2M_a+M_aA_1$

$=AH+2M_aA_1$

$=2M_aM+2M_aA_1$

$=2(MA_1)=2r$, hence A_3 is extremity of diameter.

Q. E. D.

Also solved by *CHAS. C. CROSS*, and *J. W. SCROGGS*.
Mr. Cross furnished two different solutions.

72. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, ton, D. C.

If a line with its extremities upon two curves move in any manner whateve line may vary in length), and P a point upon the line which divides it in the ratio scribe a curve, the area of this curve will be given by the formula—

$$A=\frac{(m^2+nm)A_1+(n^2+mn)A_2-mnA_3}{(m+n)^2}.$$

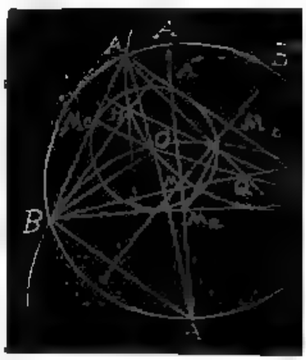
No solution of this problem has been received.

73. Proposed by ROBERT J. ALEY, A. M., Ph. D., Professor of Mathematics, Indiana University ington, Indiana.

Prove by pure geometry: (1) A', B', and C' are the middle points of the ar CA, and AB respectively. With these points as centers, circles are described through B and C, C and A, and A and B respectively. Prove that these circles int in O, the center of the incircle of the triangle ABC; (2) that O, the center of the ir is Nagel's point of the triangle formed by joining the middle points of the sides.

Solution by CHARLES C. CROSS, Laytonsville, Maryland, and the PROPOSER.

(1) AO cuts the circumcircle at A', for AO bisects angle A and also its subtending arc. $\measuredangle OBA'=\frac{1}{2}(A+B)$.



$\measuredangle BOA'=\frac{1}{2}(A+B)$ for it is exterior angle to triangle BOA.

$\therefore$ triangle $A'BO$ is isosceles.

$A'B=A'O$. By similar reasoning it is proved that $B'A=B'O$ and $C'A=C'O$.

$\therefore$ The circles intersect in O.

(2) It is a well known property of Nagel's point that AQ and OM_a, BQ and OM_b, CQ and OM_c are respectively parallel.

The triangle $M_aM_bM_c$ is similar to the triangle ABC.

$$\angle OM_aM_c = \angle QAC.$$
$$\angle OM_bM_c = \angle QBC.$$
$$\angle OM_cM_a = \angle QCA.$$

$\therefore$ O with respect to the triangle $M_aM_bM_c$, is located precisely as Q is with respect to the triangle ABC.

Hence O is Nagel's point of triangle $M_aM_bM_c$.

Also solved by *F. M. McGAW* and *G. B. M. ZERR*.

74. Proposed by ROBERT J. ALEY, A. M., Ph. D., Professor of Mathematics, Indiana University, Bloomington, Indiana.

Let O be the center of the inscribed circle. AO produced meets the circumcircle in A'. Find the ratio of AO to OA'.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics, Ohio University, Athens, Ohio.

The coördinates of A are $\left(\frac{2\triangle}{a}, 0, 0\right)$; of O, (r, r, r); and of A', those of the intersection of $\beta-\gamma=0$......(1), with $a\beta\gamma+b\alpha\gamma+c\alpha\beta=0$......(2), having the constant relation $a\alpha+b\beta+c\gamma=2\triangle$......(3). These give for the coördinates

$$A'\left(-\frac{(b+c)^2}{a^2}-\frac{2\triangle}{a},\quad \frac{(b+c)^2}{a^2}+\frac{2\triangle(b+c)}{a^2},\quad \frac{(b+c)^2}{a^2}+\frac{2\triangle(b+c)}{a^2}\right).$$

The distance d between $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ is given by

$$d^2=-\frac{abc}{4\triangle^2}\{a(\beta_1-\beta_2)(\gamma_1-\gamma_2)+b(\gamma_1-\gamma_2)(\alpha_1-\alpha_2)+c(\alpha_1-\alpha_2)(\beta_1-\beta_2)\}\quad ...(4).$$

Putting $\alpha_1=(2\triangle/a)$, $\beta_1=0$, $\gamma_1=0$; $\alpha_2=\beta_2=\gamma_2=r$,

$\overline{AO}^2=bcr(b+c-a)/2\triangle$ .....(5).

Putting α_1, β_1, γ_1 equal respectively to the coördinates of A', and $\alpha_2=\beta_2=\gamma_2=r$ as before, in (4), we get an expression for $\overline{OA'}^2$.

We can then express the ratio of OA to OA'.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

The point A' is evidently the middle point of arc BC. Since $\angle A'OC=\frac{1}{2}(A+C)$ and $\angle A'CO=\frac{1}{2}(A+B)$, $OA'=A'C=A'B$.

From Ptolemy's theorem, $ACA'B$ being a cyclic quadrilateral,

$$AB\times A'C+AC\times A'B=AA'\times BC, \text{ or}$$
$$c\times OA'+b\times OA'=(AO+OA')a.$$
$$\therefore OA : OA'=b+c-a : a=3-a : 2a.$$

Also solved by *G. B. M. ZERR* and *CHAS. C. CROSS*.

75. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

A plane passes through $(0,0,c)$ and touches the circle $x^2+y^2=a^2$, $z=0$; determine the locus of the ultimate intersections of the plane.

I. Solution by the PROPOSER.

Let the plane be $Ax+By+Cz+p=0$ ....(1).

Passing through $(0, 0, c)$, (1) gives $p=-cC$ ....(2),

and (1) becomes $Ax+By+Cz-cC=0$...... ....(3).

The x, y, z of (3) are those of $x^2+y^2=a^2$.........(4), $z=0$..... ...(5),

and also of $Ax+By-cC=0$...(6).

Making (4) homogeneous by aid of (6),

$$\left[\frac{1}{a^2}-\frac{A^2}{c^2C^2}\right]\frac{x^2}{y^2}-\frac{2AB}{c^2C^2}x/y+\left[\frac{1}{a^2}-\frac{B^2}{c^2C^2}\right]=0 \text{......(7)}.$$

For (3) to touch (7), the values of x/y from (7) must be equal, or

$$\left[\frac{1}{a^2}-\frac{A^2}{c^2C^2}\right]\left[\frac{1}{a^2}-\frac{B^2}{c^2C^2}\right]=\frac{A^2B^2}{c^4C^4} \text{..............(8)},$$

$$\text{or, } A^2/C^2+B^2/C^2-c^2/a^2=0 \text{...................(9)}.$$

From (3), $A/C=(z-c)/x-(y/x)(B/C)$............................(10).

Substituting (10) in (9), etc.,

$$\frac{x^2+y^2}{x^2}B^2/C^2-\frac{2y(z-c)}{x^2}B/C+\frac{(z-c)^2}{x^2}-c^2/a^2=0 \text{ ....(11)},$$

a quadratic in the undetermined constant B/C, giving the envelope

$$\frac{x^2+y^2}{a^2}=\frac{(z-c)^2}{c^2} \text{ ....(12)}.$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let the plane touch the circle at the point (x', y').

$\therefore \dfrac{xx'}{a^2}+\dfrac{yy'}{b^2}+z/c=1$, is the equation to the plane, but

$x'^2+y'^2=a^2$..............(1). $\therefore dy'/dx'=-x/y=-x'/y'$..............(2).

(2) in the equation to the plane gives,

$$x'=\frac{a^2x(c-z)}{c(x^2+y^2)},\quad y'=\frac{a^2y(c-z)}{c(x^2+y^2)}.$$

These values of x', y' in (1), $x^2+y^2-(a^2/c^2)(c-z)^2=0$, a cone of revolution as the locus.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

49. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

A rectangular stick of timber of known dimensions is placed upon a platform of given height in a vertical position with the center above the edge of platform, and slightly displaced from the vertical. Where and in what manner will it strike the ground?

I. Solution by the PROPOSER.

Any body rotating about the center of an end has the energy,

$$\tfrac{1}{6}(\omega^2)m(a^2+b^2).$$

If the body has fallen through the angle θ, the energy is

$$\tfrac{1}{2}b(1-\cos\theta)m.$$

$$\therefore\ 2\omega^2(a^2+b^2)=3b(1-\cos\theta)+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The body will leave the platform when the statical pressure=centrifugal force. The pressure=$m\cos\theta$. Centrifugal force=$\tfrac{1}{2}m\omega^2b$.

$$\therefore\ \omega^2b=2\cos\theta\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1) and (2), $\omega=\sqrt{\dfrac{6b}{4a^2+7b^2}}$ .... (3), and $\cos\theta=\dfrac{3b^2}{4a^2+7b^2}$ ......(4).

Take the edge of platform as origin. Let the axis of x be horizontal and the axis of y vertical. Resolve the angular velocity of the center of gravity into its vertical and horizontal components at the instant of the stick leaving the platform.

$$\left.\begin{aligned}V_x&=\tfrac{1}{2}b\omega\cos\theta\\ V_y&=\tfrac{1}{2}b\omega\sin\theta\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

For the accelerations we have $\frac{d^2x}{dt^2}=0$, $\frac{d^2y}{dt^2}=-g$.

Then $\frac{dx}{dt}=c$, and $\frac{dy}{dt}=-gt+c_2$.

Let us begin to reckon time from the instant that the body leaves the platform.

Then $\frac{dx}{dt}=V_x$, and $\frac{dy}{dt}=-gt+V_y$.

$x=V_xt+c_3$. $y=-\frac{1}{2}gt^2+V_yt+c_4$.

When $t=0$, $x=\frac{1}{2}b\sin\theta$, and $y=\frac{1}{2}b\cos\theta$.

$$\left.\begin{array}{l}\text{Then } x=V_xt+\frac{1}{2}b\sin\theta \\ \text{and } y=-\frac{1}{2}gt^2+V_yt+\frac{1}{2}b\cos\theta\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

These give the motion of the center of gravity.

Call T the time taken for one end to reach the ground. Then after leaving the platform it will have rotated through the angle $T\omega$.

It therefore makes an angle $\theta+T\omega$ with the vertical.

The center of gravity has fallen,

$$Y_1=-\frac{1}{2}gT^2+V_yT+\frac{1}{2}b\cos\theta. \quad \text{Also } X_1=V_xT+\frac{1}{2}b\sin\theta.$$

The center of gravity will be the distance $\frac{1}{2}b\cos(\theta+T\omega)$ from the ground.

Now $Y_1+\frac{1}{2}b\cos(\theta+T\omega)=H$, the height of tower.

Or, $\frac{1}{2}b\cos(\theta+T\omega)-\frac{1}{2}gT^2+V_yT+\frac{1}{2}b\cos\theta=H$ ....(7).

From this equation T may be determined. The horizontal distance from the foot of the tower will then be given by the equation,

$$X=X_1+\tfrac{1}{2}b\cos(\theta+T\omega)=V_xT+\tfrac{1}{2}b\sin\theta+\tfrac{1}{2}b\sin(\theta+T\omega).$$

II. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O. University, Mississippi.

The stick turns about its lower extremity until it reaches a horizontal position with an angular velocity given by $\omega^2=(3g/2a)$, $2a$ being its length.

Subsequently there are two motions which may be considered independently. One is that of rotation about the center of gravity with a constant angular velocity, ω; the other, that of translation, the center of gravity falling vertically with an initial velocity, $a\omega$.

Estimating the motion from the horizontal position, the stick is vertical when it has turned through an odd number of right angles; that is, at the end of $n\pi/2\omega$ seconds, n being any odd number. If S_v denote the distance from the level of the platform to the lowest point of the stick at the instants of verticality, the motion of translation gives,

$$S_v - a = a\omega \frac{n\pi}{2\omega} + \frac{1}{2}g\left(\frac{n\pi}{2\omega}\right)^2,$$

or, substituting the value of ω,

$$S_v = \{1 + \tfrac{1}{2}(\pi n)[1 + \tfrac{1}{4}(\pi n)]\}a, \ n \text{ being odd.}$$

Similarly the positions of horizontality are given by

$$S_h = \tfrac{1}{2}(\pi n)[1 + \tfrac{1}{4}(\pi n)]a, \ n \text{ being even.}$$

If any value of S_v or S_h equals D, the distance from the platform to the ground, the stick will strike the ground, in the one case vertically, in the other, horizontally.

The discussion might be continued in general terms. Instead of this, however, let $a=1$ foot, and $D=10$ feet.

Giving to n the values of 1, 2, and 3 in the proper equations, the first and second values of S_v are found to about 3.4 and 13.1, and the first value of S_h about 6.4. Consequently the stick will strike the ground in passing from a horizontal toward a vertical position.

Since in falling 6.4 feet a half revolution has been made, the time for this motion is π/ω seconds, and the velocity of the center of gravity when the stick is horizontal for the last time is $\omega + g(\pi/\omega)$, remembering that $a=1$. If the stick turns through an angle θ before striking the ground, the center of gravity falls through $(3.6 - \sin\theta)$ feet in θ/ω seconds, giving the equation,

$$3.6 - \sin\theta = [\omega = g(\pi/\omega)](\theta/\omega) + \tfrac{1}{2}g[(\theta/\omega)]^2,$$

which reduces to $3.6 - \sin\theta = 3.1\theta + \frac{1}{2}\theta^2$, approximately ; from which $\theta = 48^\circ\ 20'$, about.

The horizontal distance from the edge of the platform to the point at which the stick touches the ground is $1 + \cos\theta$, or 1 foot, 8 inches, approximately.

θ is, of course, the inclination of the stick to the horizontal at the instant of contact with the ground.

In the last part of this work the thickness of the stick has been neglected.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

49. Proposed by EDMUND FISH, Hillsboro, Illinois.

A rectangular field, whose length and breadth in rods are in whole numbers, is enclosed with a fence and subdivided by fences on both diagonals, the total length of the fences being 2204 rods; required the sides and area.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.; O. S. WESTCOTT, North Division High School, Chicago, and the PROPOSER.

Let $2xy$, x^2-y^2, and x^2+y^2 be the length, breadth and diagonal of the field, respectively; then $2x^2+2xy=1102$.

$\therefore x^2+xy=551$; whence $y=\dfrac{551}{x}-x, =\dfrac{19\times29}{x}-x$.

As x and y are known to be integral, $551/x$ must be integral, which can only be when $x=19$. Hence $y=10$.

$\therefore 2xy=380$; and $x^2-y^2=261$, breadth.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let x=length, y=breadth; then $2x+2y+2\surd(x^2+y^2)=2204$.

$\therefore 607202+xy=1102(x+y)$. $\therefore 1102-y=1102(551-y)/x$.

Let $1102-y=z$, then $z=1102(x-551)/x$.

$\therefore x=1102-(2.19^2.29^2/z)$. $\therefore z=29^2$. $\therefore x=380$, $y=261$.

Area=99180 square rods, =619 acres, 14 square rods.

III. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; JOSIAH H. DRUMMOND, LL. D., Portland, Maine; A. H. HOLMES, Brunswick, Maine; and P. S. BERG, Larimore, North Dakota.

As the field is a rectangle, the diagonals are equal, and the fences form the sides of two equal right triangles of which the legs and hypotenuse are respectively the sides and diagonal of the field.

Let a and b be the sides and c the diagonal of the field. Then $2a+2b+2c=2204$, and $a^2+b^2=c^2$. From the identity $(2mn)^2+(m^2-n^2)^2=(m^2+n^2)^2$, the formula for finding integral sides of right triangles, take $a=2mn$, $b=m^2-n^2$, and $c=m^2+n^2$. Then $2a+2b+2c=4m(m+n)=2204$. Whence $m(m+n)=551$. We now separate 551 into two factors, making the larger factor equal $m+n$, and the smaller equal m.

$551=19\times29$. Then $m+n=29$ and $m=19$; whence $n=10$. Substituting these values of m and n in the values for a, b, and c, we obtain $a=2mn=380=$ the length of the field; $b=m^2-n^2=261=$the breadth of the field; and $c=m^2+n^2=461=$the diagonal of the field. The area$=380\times261=99180$ square rods$=619\frac{7}{8}$ acres.

Also solved by *A. H. BELL.*

50. Proposed by SYLVESTER ROBINS, North Branch Depot, New Jersey.

The edges of a rectangular parallelopiped are within 1 of the proportion 3 : 9, and they are $2x\pm1$, $8x$ and $9x$, $(2x\mp1)^2+(3x)^2+(9x)^2$=the diagonal ared$=94x^2\mp4x+1=\square$. To find four integral values for x.

I. Solution by A. H. HOLMES, Box 963, Brunswick, Maine.

We may put it in the form : $90x^2+(2x\pm1)^2=\square$, or

$$m^2x^2-(m^2-90)x^2+(2x\pm1)^2=\square.$$

$\therefore\ 2m(2x\pm1)=(m^2-90)x$; $4mx\pm2m=m^2x-90x$.

$\therefore\ x=\pm(2m/(m^2-4m-90)$.

Let $m=nx$. Then $n^2x^2-4nx-90=\pm2n$; $n^2x^2-4nx+4=94\pm2n$.

Take plus sign and let $n=3$. $\therefore\ 3x=2+10=12$. $\therefore\ x=4$.

Now let $n=a/b^2$. $a^2x^2/b^4-4ax/b^2+4=94\pm2a/b^2=(94b^2\pm2a)/b^2$.

Now take $b=3$. $\therefore\ a=5/2$ and $a/b^2=5/18$.

$\therefore\ 5x/18=2+29/3$. $5x=36+174=210$. $\therefore\ x=42$.

Now let $b=10$. $\therefore\ a=9/2$ and $a/b^2=9/200$.

$\therefore\ 9x/200=2+97/10$. $9x=400+1940=2340$. $\therefore\ x=260$.

Now let $b=23$. $\therefore\ a=-3/2$ and $a/b^2=-3/1058$.

$\therefore\ -3x/1058=2-223/23$, or $3x=8142$. $\therefore\ x=2714$.

For $x=4$ we have : $94x^2+4x+1=\square$.

For $x=42$ we have : $94x^2-4x+1=\square$.

For $x=260$ we have : $94x^2+4x+1=\square$.

For $x=2714$ we have : $94x^2-4x+1=\square$.

II. Solution by A. H. BELL, Hillsboro, Illinois.

The equation readily reduces to : $t^2-94y^2=-90$..................(1), l $x=(t\mp2)/94$ ....(2). (1)÷9 gives $t'^2-94y'^2=-10$, and $3t'$, $y=3y'$..................(3).

One cycle.

No. of Frac's:		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16:
ients com- Denom'rs	1 :	13	6	5	9	10	3	15	2	15	3	10	9	5	6	13	1:
ients	9 :	1,	2,	3,	1,	1,	5,	1,	(8),	1,	5,	1,	1	3	2	1	18:
ergents	$\frac{1}{0}$, $\frac{9}{1}$:	$\frac{10}{1}$,	$\frac{29}{3}$,	$\frac{97}{10}$,	$\frac{126}{13}$,	$\frac{223}{23}$,	$\frac{1241}{128}$,	$\frac{1464}{151}$,	$\frac{12953}{1336}$,	$\frac{14417}{1487}$,	$\frac{85038}{8771}$,	$\frac{99455}{10258}$,	$\frac{184493}{19029}$,	$\frac{652934}{67345}$,	$\frac{1490361}{153719}$,	$\frac{2143295}{221064}$.—	

The convergents preceding the denominators, 10 of the complete quotients $/y'=126/13$ and $85038/8771$ ....(4), as they are even fractions. $\therefore$ answer the -10 of (3). To obtain other values of t' and y', take $94u^2=1$..................(5).

$$\text{(5) and } \pm188t'uvy' \quad \left.\begin{array}{l}(t'v\pm94uy')^2-94(t'u\pm vy')^2=-10\\ \text{or, } t_n'^2-94y_n'^2=-10\end{array}\right\}\text{..............(6).}$$

The smallest integral values for $v/u=2143295/221064$, but as fractional values for t' and y' can be used as shown in (3), to obtain these we solve (5).

Let $v=v'/z$ and $u=u'/z$; then $v'^2-z^2=94u'^2$ ....

Now let $u^2=pq$ and let 94=any two factors, then (7) can be made

$$\left.\begin{array}{l} v'+z=p^2 \text{ or } 2p^2 \\ v'-z=94q^2 \text{ or } 47q^2 \end{array}\right\}$$

add and subtract, etc. $v'=p^2+94q^2$ or $2p^2+47q^2$; $z=p^2-97q^2$ or $2p^2-47$
$u'=2pq$..................(8).

In the right-hand values if $p=5$ and $q=1$, $v'=97$; $z=3$; $u'=10$. Th
are an infinite number of values but these are the only ones admissible.

(7) $v=97/3$ and $u=10/3$; substituting these along with those of (4) se
ately in (6) we have $t_n'=2/3$ and 24442/3 ; and $t_n'=3946/3$ and 16493426/3 w
those in (4), will make six values for t', and now in (3) and (2) $x=0, 4, -$
260, -2714, and 175462, etc. The sign=side($2x\pm1$). $y=94$, 39, 407, 2
26313.

III. Solution by the PROPOSER.

This problem is suggested by a remark in No. 5, Vol. I.: "x^2-
$=\pm1$; this is the most difficult number under 100."

1. Find initial terms in that infinite series of rational rectangular so
where the edges of each term are in proportion as 2 : 3 : 9, within 1
the *thickness*.

Let $2x\pm1$, $3x$ and $9x$ be the edges ; then $94x^2\pm4x+1=\square=(mx\pm$
$=m^2x^2\pm2mx+1$. $x=(\pm2m\mp4)/(94-m^2)$.

Say $m=\surd(94)=9/1$, 10/1, 29/3, 97/10, 126/13, 223/23, 1241/
1464/151, etc.

When $m=$	10	29/3	97/10	223/2x	
Then $x=$	4	42	260	2714	
$2x\pm1=$	9	83	521	5427	Thickness.
$3x=$	12	126	780	8142	Width.
$9x=$	36	378	2340	24426	Length.
$\surd(94x^2\pm4x+1)=$	39	407	2521	26313	Solid diago

2. Find first term in an infinite series of rational parallelopipeds w
the dimensions of every solid are in proportion as 2 : 3 : 9, within 1 in the wi

Let $2x$, $3x\pm1$ and $9x$ represent the edges. Then $94x^2\pm6x+1$
$=(mx\pm1)^2=m^2x^2\pm2mx+1$. Whence $x=(2m\mp6)/(94-m^2)$, $m=\surd(94)=9$
29/3, 97/10, 126/13, etc.

$m=$	29/3	126/13
$x=$	24	429
$2x=$	48	858
$3x\pm1=$	73	1286
$9x=$	216	3861
Solid diagonal=	233	4159

3. Find a term in an infinite series of rational parallelopipeds where the dges are in proportion as 2 : 3 : 9, within unity in *length*.

Let $2x$, $3x$, and $9x\pm1$ be the edges. $94x^2\pm18x+1=\square=(mx\pm1)^2=m^2x^2\pm2mx+1$. $x=(2m\mp18)/(94-m^2)$. Substitute $m=1464/151$, and $x=15855$, $2x=31710$, $3x=47565$, $9x-1=142694$.

Proof: $31710^2+47565^2+142694^2=153719^2$.

4. Find some term in an infinite series of rational parallelopipeds where e dimensions come within 1 unit in the *thickness* of being in proportion 3 : 6 : 7.

Let edges be $3x\pm1$, $6x$ and $7x$. $94x^2\pm6x+1=\square=(mx\pm1)^2=m^2x^2\pm2mx$ 1. $x=(2m\mp6)/(94-m^2)$.

When $m=$	29/3	$m=$	126/33	
$x=$	24	$x=$	429	
$3x\pm1=$	144	$3x\pm1=$	1286	etc.
$6x=$	144	$6x=$	2574	
$7x=$	168	$7x=$	3003	
S. d. $=$	233	S. d. $=$	4159	

Proof: $73^2+144^2+168^2=233^2$.

5. Find some term in an infinite series of rational rectangular solids here the edges come within 1 unit in the *width* of being in the proportion of : 6 : 7. Let the edges be represented by $3x$, $6x\pm1$ and $7x$. Then $94x^2\pm12x$ $1=\square=(mx\pm1)^2=m^2x^2+2mx+1$. $x=(2m\mp12)/(94-m^2)$. When $m=\sqrt{94}$ ....1464/151. Then $x=84258$ or 357870.

$3x=252774$	or $3x=1073610$
$6x-1=505547$	$6x+1=2147221$
$7x=589806$	$7x=2505090$
Diagonal $=816911$	Diagonal $=3469679$

6. Find a term in that infinite series of rational parallelopipeds wherein e edges of every solid are within unity in the *length* of being in proportion to ch other as 3 : 6 : 7.

$(3x)^2+(6x)^2+(7x\pm1)^2=94x^2\pm14x+1=\square=(mx\pm1)^2$.

$94x\pm14=m^2x\pm2m$. $x=(2m\mp14)/(94-m^2)$. $m=\sqrt{94}$. Now when $=29/3$, $x=60$, $3x=180$, $6x=360$, $7x-1=419$.

$180^2+360^2+419^2=581^2$.

Also solved by *J. H. DRUMMOND*.

61. Proposed by H. C. WILKES, Skull Run, West Virginia.

The difference between the roots of two successive triangular square numbers, [i. e. angular numbers that are also square numbers], equals the sum of two successive inte- al numbers, the sum of whose squares will be a square number. Demonstrate. Or, if s d t be the roots of any two successive triangular number that are also square numbers, ove that $t-s=2n+1$, where $n^2\ (n+1)^2=\square$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

$$\frac{n(n+1)}{2} \text{ is a square when } n=\frac{(1+\sqrt{2})^{2m}+(1-\sqrt{2})^{2m}-2}{4}.$$

$$\therefore \pm\sqrt{\frac{n(n+1)}{2}}=\pm\left\{\frac{(1+\sqrt{2})^{2m}-(1-\sqrt{2})^{2m}}{4\sqrt{2}}\right\} \quad\ldots\ldots\ldots\ldots\ldots(1).$$

$$\pm\sqrt{\frac{n'(n'+1)}{2}}=\pm\left\{\frac{(1+\sqrt{2})^{2m+2}-(1-\sqrt{2})^{2m+2}}{4\sqrt{2}}\right\} \quad\ldots\ldots\ldots\ldots\ldots(2).$$

Taking (2)+ and (1)−, and then taking their difference, we easily get,

$$\frac{(1+\sqrt{2})^{2m+2}-(1-\sqrt{2})^{2m+2}}{4\sqrt{2}}+\frac{(1+\sqrt{2})^{2m}-(1-\sqrt{2})^{2m}}{4\sqrt{2}}=2y+1.$$

$$\therefore \frac{(1+\sqrt{2})^{2m+1}+(1-\sqrt{2})^{2m+1}}{2}=2y+1.$$

$$\therefore \left\{\frac{(1+\sqrt{2})^{2m+1}+(1-\sqrt{2})^{2m+1}}{4}-\tfrac{1}{2}\right\}^2+$$

$$\left\{\frac{(1+\sqrt{2})^{2m+1}+(1-\sqrt{2})^{2m+1}}{4}+\tfrac{1}{2}\right\}^2=y^2+(y+1)^2.$$

$$\therefore 2\left\{\frac{(1+\sqrt{2})^{2m+1}+(1-\sqrt{2})^{2m+1}}{4}\right\}^2+\tfrac{1}{2}=y^2+(y+1)^2.$$

$$\therefore \left\{\frac{(1+\sqrt{2})^{2m+1}-(1-\sqrt{2})^{2m+1}}{2\sqrt{2}}\right\}^2=y^2+(1y+1)^2.$$

In above m can have any positive integral value.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

This problem is true if we read "The sum of" instead of "The difference between." It might also be stated as follows: The difference between the roots of two successive triangular square numbers equals a number whose square is the sum of the squares of two successive integral numbers.

From Solution III of Problem 36, Vol. III., No. 3, page 82, we find that when one of the triangular square numbers is $n(n+1)/2$, the next in order, in terms of n, is $\left(2n+1+3\sqrt{\frac{n(n+1)}{2}}\right)^2$.

The *difference* of the two roots is $2n+1+2\sqrt{\frac{n(n+1)}{2}}$.

The *sum* of the two roots is $2n+1+4\sqrt{\frac{n(n+1)}{2}}$, which equals the sum of the two consecutive integral numbers, $n+2\sqrt{\frac{n(n+1)}{2}}$ and $n+1+2\sqrt{\frac{n(n+1)}{2}}$.

But $\left(n+2\sqrt{\frac{n(n+1)}{2}}\right)^2+\left(n+1+2\sqrt{\frac{n(n+1)}{2}}\right)^2=6n^2+6n+1+(8n+4)\sqrt{\frac{n(n+1)}{2}}$

hich equals the square of the *difference* of the two roots, or

$$\left(2n+1+2\sqrt{\frac{n(n+1)}{2}}\right)^2$$

Illustration.—From the series of triangular square numbers, 1^2, 6^2, 35^2, 04^2, 1189^2, etc., take 6 and 35. $35-6=29$; $35+6=41=20+21$; $20^2+21^2=29^2$.

This problem and problems No. 45, (Vol. III., No. 5, page 153), and No. 6, of Diophantine Analysis, are very closely related.

Also solved by the *PROPOSER*.

52. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbiam University, Washington, D. C.

Prove that a "magic square" of nine integral elements, whose rows, columns, and iagonals have a constant sum, is only possible when this sum is a multiple of three.

I. Solution by M. W. HASKELL, M. A., Ph. D., Associate Professor of Mathematics, University of California, Berkeley, California.

Let the magic square be

a	b	c
d	e	f
g	h	k

and let S be the constant sum.

Then $S=a+b+c=d+e+f=g+h+k=a+d+g=b+e+h=c+f+k=a+e+k=c+e+g$.

Adding these all together, we have $8S=3a+2b+3c+2d+4e+2f+3g+2h+3k=3(a+c+g+k)+2(b+e+h)+2(d+e+f)$. But the last two quantities in arenthesis are each$=S$. Hence $4S=3(a+c+g+k)$, and S is a multiple of 3.

II. Solution by —— (Paper Unsigned.)

Suppose the numbers occupying the magic square to be a, b, c, d, e, f, g, , k. Now $a+e+k=b+e+h=c+e+g=S$.

$\therefore$ $a+k\equiv k(\text{mod } 3)$, $b+h\equiv k(\text{mod } 3)$, $c+g\equiv k(\text{mod } 3)$, where $S-e\equiv k(\text{mod } 3)$.

Adding the congruences, $(a+b+c)+(g+h+k)\equiv 0(\text{mod } 3)$. Or, since $a+b+c)+(g+h+k)\equiv 0(\text{mod } 3)$, $2S\equiv 0(\text{mod } 3)$.

Multiply by 2, and divide by 3, and the result is $S\equiv 0$. Q. E. D.

III. Solution by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, uisiana.

Let the rows of the "square" be a, b, c ; x, y, z ; and l, m, n, and let the nstant sum be k. We have to show that $k/3$ is integral. We have $a+y+n=k$; $+y+m=k$; $l+y+c=k$. Add, and we have $(a+b+c)+(l+m+n)+3y=3k$, at is, $2k+3y=3k$.

$\therefore$ $3y=k$. $\therefore$ $y=k/3$. But y is integral. $\therefore$ $k/3$ is integral.

Also solved by *M. A. GRUBER* and *G. B. M. ZERR*.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

48. Proposed by P. H. PHILBRICK, C. E., Pineville, Louisiana.

A, B, C, D, and E play with dice, each throwing three, three successive times, for a stake a. A, B, and C throw; C throwing the highest, 52. What is his expectation?

I. Solution by the PROPOSER.

If D or E or both throw 52, C gets but a part of the stake. If D or E or both throw 53 or 54, C gets none of the stake.

$52=18+18+16=18+17+17$. $53=18+18+17$. $54=18+18+18$.

The chance of throwing 16 at a single throw is $\frac{6}{216}$.

The chance of throwing 17 at a single throw is $\frac{3}{216}$.

The chance of throwing 18 at a single throw is $\frac{1}{216}$.

Hence since D may throw 16 (or 18) at any one of the three throws, his chance of throwing 52 at three throws is $3(\frac{1}{216}\times\frac{1}{216}\times\frac{6}{216})+3(\frac{3}{216}\times\frac{1}{216}\times\frac{3}{216})$ $=p_1$ say. E has the same chance of reaching the same result. The chance that D (or E) will not throw 52 is $(1-p_1)$; and the chance that D or E will throw 52 and the others not is $p_1(1-p_1)$, in which case the expectation is $p_1(1-p_1)\frac{1}{2}a$.

The chance that D, and E also, will throw 52 is p_1^2, in which case their joint expectation is $p_1^2\frac{1}{3}a$. Hence the expectation of D or E or of both, coming from throwing 52 is, $2p_1(1-p_1)\frac{1}{2}a+p_1^2\frac{1}{3}a=p_1(3-p_1)\frac{1}{3}a$.

The chance of D or E throwing 53 is, $3(\frac{1}{216}\times\frac{1}{216}\times\frac{3}{216})=p_2$; and the chance that one or both will throw 53 is, $2p_2(1-p_2)+p_2^2=p_2(2-p_2$; and their joint expectation is, $p_2(2-p_2)a$.

The chance that D or E will throw 54 is $(\frac{1}{216}+\frac{1}{216}+\frac{1}{216})=p_3$); and the chance that one or both will throw 54 is, $2p_3(1-p_3)+p_3^2=p_3(2-p_3)$; and their joint expectation is, $p_3(2-p_3)a$. Hence C's expectation is,

$$\{1-\tfrac{1}{3}[p_1(3-p_1)]-p_2(2-p_2)-p_3(2-p_3)\}a=(1+325p_3^2-47p_3)a.$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, and J. SCHEFFER, A. M., Hagerstown, Maryland.

D and E may each throw 52, 53, or 54.

52 can be thrown as follows: (6, 6, 6), (6, 6, 5), (6, 6, 5); (6, 6, 6), (6, 6, 6), (6, 6, 4).

53 can be thrown as follows: (6, 6, 6), (6, 6, 6), (6, 6, 5).

54 can be thrown as follows: (6, 6, 6), (6, 6, 6), (6, 6, 6).

D's chance of throwing 52, 53, or 54 is,

$$p=\frac{9}{(216)^3}+\frac{3}{216^3}+\frac{3}{216^3}+\frac{1}{216^3}=\frac{16}{(216)^3}=\frac{2^4}{6^9}.$$

$1-p=1-2^4/6^8=$ [illegible] $=P$, chance that D will throw less than 52.

P^2=chance that D and E will both throw less.

$\therefore aP^2=C$'s expectation on the supposition that C wins and no ties.

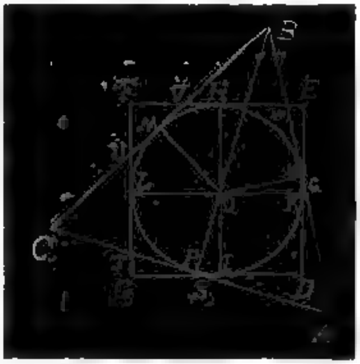
40. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, ngfield, Missouri.

A square whose side is $2a$ and an equilateral triangle whose altitude is $3a$ are fasten- ogether at their centers, but otherwise free to move. If they are thrown on a floor at lom, what is the average area common to both?

Solution by HENRY HEATON, M. Sc., Atlantic, Iowa, and the PROPOSER.

In the figure let O be the common center of the square and the triangle.

Then $OK=ON=OH=OM=OL=OP=OI=a$.



Let the triangle $KON=2\theta$.

Then $\angle NOH=\frac{1}{2}\pi-2\theta$, $\angle HOM=\frac{2}{3}\pi-$ $-2\theta)=\frac{1}{6}\pi+2\theta$, $\angle MOL=\frac{2}{3}\pi-(\frac{1}{6}\pi+2\theta)=\frac{1}{2}\pi-$ $\angle LOP=\frac{2}{3}\pi-(\frac{1}{2}\pi-2\theta)=\frac{1}{6}\pi+2\theta$, and $\angle POI$ $-(\frac{1}{6}\pi+2\theta)=\frac{1}{2}\pi-2\theta$.

Area of surface, $KONQ$, $=a^2\tan\theta$;

area of surface, $NOHW$, $=a^2\tan(\frac{1}{4}\pi-\theta)$;

area of surface, $HOMV$, $=a^2\tan(\frac{1}{12}\pi+\theta)$;

area of surface, $MOLU$, $=a^2\tan(\frac{1}{4}\pi-\theta)$;

area of surface, $KOPT$, $=a^2\tan(\frac{1}{4}\pi+\theta)$;

area of surface, $POIS$, $=a^2\tan(\frac{1}{12}\pi-\theta)$;

area of square, $OKDI$, $=a^2$.

Hence the area common to the square and triangle is

$$a^2[1+\tan\theta+\tan(\tfrac{1}{4}\pi-\theta)+\tan(\tfrac{1}{12}\pi+\theta)+\tan(\tfrac{1}{4}\pi-\theta)+\tan(\tfrac{1}{4}\pi+\theta)+\tan(\tfrac{1}{12}\pi-\theta)].$$

The positions for $\theta>\frac{1}{12}\pi$ are exact repetitions of those for $\theta<\frac{1}{12}\pi$.

Hence the required average area is

$$\int_0^{\frac{1}{12}\pi} Sd\theta \div \int_0^{\frac{1}{12}\pi} d\theta=\frac{12a^2}{\pi}\int_0^{\frac{1}{12}\pi}\Big[1+\tan\theta+\tan(\tfrac{1}{4}\pi-\theta)+\tan(\tfrac{1}{12}\pi+\theta)+$$

$$\tan(\tfrac{1}{4}\pi-\theta)+\tan(\tfrac{1}{4}\pi+\theta)+\tan(\tfrac{1}{12}\pi-\theta)\Big]d\theta=a^2\Big[1+\frac{12}{\pi}\log_e 2\Big].$$

This problem was also solved in a very excellent manner by G. B. M. Zerr.

50. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Find (1), the average length of all straight lines having a given direction, between 0 ; (2), the average length of chords drawn from one extremity of the diameter a of a -circle to all points in the semi-circumference; and (3), find the average area of all gles formed by a straight line of constant length a sliding between two straight lines ght angles.

Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

(1). Let x=length of one of the straight lines. Then the required average is $\int_0^a xdx \div \int_0^a dx = \frac{1}{2}a$.

(2). Let θ=angle between the chord and diameter. Then the length of the chord is $2a\cos\theta$, and the average length of all chords is

$$2a\int_0^{\frac{1}{2}\pi}\cos\theta d\theta \div \int_0^{\frac{1}{2}\pi} d\theta = 4a/\pi.$$

(3). Let θ=the angle between the sliding line and one of the fixed ones. The area of the triangle is $(\frac{1}{2}a^2)\sin\theta\cos\theta$. The average area of all such triangles depends upon circumstances. If the areas be taken at equal angular intervals, the required average is $A_1=\int_0^{\frac{1}{2}\pi}(\frac{1}{2}a^2)\sin\theta\cos\theta d\theta \div \int_0^{\frac{1}{2}\pi} d\theta = a^2/2\pi$.

If the areas be taken at equal intervals as measured on one of the fixed lines along which the end of the sliding line moves, the average area is

$$A_2=\int_0^{\frac{1}{2}\pi}(\frac{1}{2}a^2)\sin\theta\cos\theta d(a\sin\theta) \div \int_0^{\frac{1}{2}\pi} d(a\sin\theta) = \frac{1}{2}a^2\int_0^{\frac{1}{2}\pi}\sin\theta\cos\theta d\theta = \frac{1}{6}a^2.$$

This is but a repetiton of Problem 26, about which there has been so much controversy. In the absence of a distinct statement as to the intervals at which the areas shall be taken I see no reason for preferring either of the above solutions to the other.

The editor quotes me as saying that there is no correct solution of the problem. I must have failed in expressing myself clearly, for my position is that in such problems no solution can be considered a full one that does not discuss all possible cases.

51. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Three points are taken at random in a sphere and a plane passed through them. Find the average volume of the segment cut off from the sphere.

Solution by the PROPOSER.

Let A, B, C be the three random points, EF the diameter of the section of the sphere made by a plane through A, B, C; M the center of this section; O the center of the sphere; OG a line such that AB is parallel to the plane MOG.

Let $OE=r$, $MA=x$, $AB=y$, $AC=z$, $\angle EOM=\theta$, $\angle BAM=\varphi$, $\angle CAM=\psi$, $\angle MOG=\lambda$, the angle the plane MOG makes with some fixed plane through OG $=\rho$.

The element of the sphere at A is $r\sin\theta d\theta.2\pi xdx$; at B, $y^2dyd\varphi d\lambda$; at C, $\sin(\varphi+\psi)\sin\lambda z^2dzd\psi d\rho$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of x, 0 and $r\sin\theta$, and tripled; of φ, $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; of ψ, $-\varphi$ and $\frac{1}{2}\pi$, and doubled; of λ, 0 and π; of ρ, 0 and 2π; of y, 0 and $2x\cos\varphi$; of z, 0 and $2x\cos\psi$.

Let $r\sin\theta = x'$, $2x\cos\varphi = y'$, $2x\cos\psi = z'$, V = volume of segment.

$OM = r\cos\theta$. $\therefore$ The height of the segment is $r(1-\cos\theta) = 2r\sin^2\frac{1}{2}\theta$.

$\therefore$ $V = \frac{4}{3}\pi r^3 \sin^4\frac{1}{2}\theta(3-2\sin^2\frac{1}{2}\theta)$.

Since the whole number of ways the three points can be taken is $(\frac{4}{3}\pi r^3)^3$, required average is,

$$= \frac{6.3^3}{64\pi^3 r^9}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{-\varphi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{y'}\int_0^{z'} Vr\sin\theta d\theta 2\pi x dx$$

$$\times \sin(\varphi+\psi)d\varphi d\psi \sin\lambda d\lambda d\rho y^2 dy z^2 dz.$$

$$= \frac{27}{2\pi^2 r^6}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\varphi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{y'} V\sin\theta\sin(\varphi+\psi)\cos^3\psi\sin\lambda$$

$$\times d\theta x^4 dx d\varphi d\psi d\lambda d\rho y^2 dy$$

$$\frac{36}{r^6}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\varphi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi} V\sin\theta\sin(\varphi+\psi)\cos^3\varphi\cos^3\psi\sin\lambda x^7 d\theta dx$$

$$\times d\varphi d\psi d\lambda d\rho$$

$$\frac{144}{\pi r^6}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\varphi}^{\frac{1}{2}\pi} V\sin\theta\sin(\varphi+\psi)\cos^3\varphi\cos^3\psi x^7 d\theta dx d\varphi d\psi$$

$$\frac{18}{\pi r^6}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} V\sin\theta[3(\frac{1}{2}\pi+\varphi)\sin\varphi+2\cos\varphi+\sin^2\varphi\cos\varphi]\times\cos^3\varphi x^7 d\theta dx d\varphi$$

$$\frac{315}{3\pi r^6}\int_0^{\frac{1}{2}\pi}\int_0^{x'} V\sin\theta x^7 d\theta dx$$

$$\frac{5r^3}{32}\int_0^{\frac{1}{2}\pi}\sin^4\frac{1}{2}\theta(3-2\sin^2\frac{1}{2}\theta)\sin^9\theta d\theta = \frac{4}{3}r^3.$$

This is the average volume of the lesser segment.

$\frac{4}{3}r^3(\pi-1)$ = average volume of greater.

EDITORIALS.

The MONTHLY will not appear during the months of July and August, but the August-September number will appear about the first of September.

We are pleased to state that our valued contributor, Dr. G. B. M. Zerr, has been called to the presidency of The Russell College, Lebanon, Va. May success, as we know it will, follow him in his new field of work.

The MONTHLY is now sorely in need of funds to carry it on further. Will those of our subscribers who are in arrears remit the amount of their subscriptions at once, so that no delay may be caused through lack of funds in getting out our next issue?

The degree of Doctor of Philosophy was conferred June 9th, by the University of Pennsylvania, on Prof. Robert J. Aley, the subject of his thesis being, "Some Contributions to the Geometry of the Triangle." We congratulate Dr. Aley on having received this degree as it is not an honorary one, but was obtained by actual work done at the University during the past year.

We are sorry that we were obliged to disappoint our readers in failing to give in the May number of the MONTHLY, the first of a series of articles on Lie's Transformation groups, by Dr. Edgar Odell Lovett. Owing to some unavoidable circumstances, Dr. Lovett was unable to prepare the articles, but he assures us that he will have his first article ready for the August-September number. We shall look forward with a good deal of interest for the appearance of the next number.

It has been proposed that the number of pages of the MONTHLY be increased from 32 to 50, half the number of which shall be devoted to papers and the other half to the solutions of problems, and the price of subscription raised to $5. per year. We shall be pleased to hear from every one of our subscribers in regard to this matter, that in case the proposition meets with the necessary endorsement it may be carried into execution at the beginning of the fifth volume. We are at all times open to advice and suggestions from our readers and no pains will be spared on our part to increase the usefulness of our journal.

The University of Chicago, Summer, 1897. The following mathematical courses will be offered :—By Professor Moore : Abstract groups ; Projective geometry.—By Professor Bolza : Hyperelliptic functions ; Advanced integral calculus.—By Dr. Lovett : The geometry of Lie's transformation-groups.—By Dr. Young : [1]Conferences on mathematical pedagogy ; [1]Determinants ; Culture Calculus ; [2]Plane trigonometry.—By Mr. Slaught : Integral Calculus ; College algebra. The courses are four or five hours weekly for twelve weeks from July 1, 1897 ; the two courses marked 1 are, however, only for the first six weeks, and

:ourse marked 2 is ten hours weekly for the second six weeks. Those who ›ct to work in mathematics at the University of Chicago during the coming mer, as well as those who desire further information, are requested to com- icate with Professor Moore.

It was our intention to have appear in this issue a group of some of our ributors, but it was impossible for us to make all the necessary arrangements out delaying this number. So we have decided to have our group in the ust-September number.

We are indebted to Dr. Artemas Martin for pamphlet copies of his valu- papers on "Formulas for the Sides of Rational Plane Triangles," and Method of Finding, without Tables, the Number Corresponding to a given arithm." These papers will appear in Vol. II., No. 11 of the *Mathematical 'azine.*

We have received a copy in pamphlet form of "Transcendental Numbers," Prof. Heinrich Weber. Translated into English by Prof. W. W. Beman. rinted from the *Bulletin of the American Mathematical Society.* Thanks are Professor Beman for giving us this reproduction in English of this very in- sting and valuable paper.

Ginn & Co. announce for June a *Higher Arithmetic* by Wooster Woodruff ıan, of the University of Michigan, and David Eugene Smith, of the Michi- State Normal School. Teachers will await with much interest this new k on arithmetic by these well-known authors. The same publishers announce ɜady "*An Elementary Arithmetic,*" by William W. Speer, being the second c of this new series.

BOOKS AND PERIODICALS.

Differential Equations. By D. A. Murray, Ph. D., of the Department of hematics in Cornell University. Price $1.90. 230 pages. New York and don : Longmans, Green & Co. 1897.

This work aims to meet the needs of students of physics and engineering who wish ɪe the subject as a tool, as well as of those students who have more time to give to the ral theory and who wish to proceed to the study of the higher mathematics. For the class, the theoretical explanations have been given as briefly as is consistent with ness and in most cases the examples have been worked in full detail. In addition, chapters have been introduced dealing with geometrical and physical problems. For second class of students, notes have been inserted in the latter part of the book giving lemonstration of additional theorems and more vigorous proofs of theorems partially ed in the first part of the book. Interesting historical and biographical notes have given in proper places, and many references are made to sources where fuller explan- ıs and developments than the scope of the work allows may be found. We commend ɔook as providing an excellent introductory course in Differential Equations. J. M. C.

Analytical Geometry. By F. R. Bailey, A. M., and F. S. Woods, Ph. D., Assistant Professors of Mathematics in Massachusetts Institute of Technology. 371 pages. Boston and London: Ginn & Co. 1897.

This book is intended primarily for students in colleges and technical schools. The treatment of subjects included has been complete and rigorous. There are no important departures in method of treatment, but we notice that more space than is usual has been given to the more general form of the equations of the first and second degrees; that the equations of the conics have been derived from a single definition and then by translation of the origin equations of the second degree, wanting the xy term, are handled; and that only later the general equation of the second degree is fully discussed. In solid geometry the treatment is very satisfactory. The examples are numerous and well chosen. No use is made of determinants or calculus—a feature which many will commend and others criticise. Altogether the book is undoubtedly a good one and it should prove a useful text.

J. M. C.

Higher Algebra. By George Lilley, Ph. D., LL. D., Ex-President South Dakota College. 504 pages. Silver, Burdett & Co., New York, Boston and Chicago. 1894.

The first 400 pages are the same as the author's "Elements." As the book only professes to cover the ground required for admission to colleges and universities, this feature is not so objectionable as it would be in a work intended for college and university use. Under the chapter on "Theory of Limits," there are several features which invite attention, such as the proof of the Theory; the sum of an infinite decreasing Geometrical Series; the invention of a symbol to represent an Infinitesimal, etc. However, to our mind the author's interpretation of the for $a/0$, or 0 *as a divisor*, is objectionable, and the proof that, in general, $a/0=0$, defective. The proof as given is,

$$\frac{12}{+2}=6,\ \frac{12}{+1}=12,\ \frac{12}{0}=0,\ \frac{12}{-1}=-12,\ \frac{12}{-2}=-6, \text{ etc.},$$

where the quotient, 0, means that there is *no number of times zero* that the divisor, 0, can be subtracted from 12 and leave *zero*. It would misrepresent the author's position not to add that he invents a new symbol to represent an infinitesimal and shows that a (an infinitesimal)$=\infty$, and he would not confound the 0, arising from dividing a by *infinity*, with the *absolute zero*, nor *perhaps* the absolute zero with the zero, meaning "no number of times," in the quotient $a/0=0$. In interpreting the result, $t=a/0$, in Clairaut's problem of the Couriers, he would say, as there is no number of times zero that subtracted from a leaves *zero*, so there is no number of hours when they have been or will be together, and that the form $a/0$ indicates that *the problem is impossible*. That our readers may catch the spirit and meaning of his article, we have invited Dr. Lilley to give some elaboration to his views in a short article for the Monthly to be published in a future number. Although we do not approve some of the positions which the author has taken, still we regard the treatise on the whole as one of decided merit. The book has evidently been made for the class room and for actual use, and bears the marks of having been written by an experienced and practical teacher. We have only space to note further the demonstration for "Undetermined Coefficients," on page 419; "Pascal's Arithmetical Triangle," on page 442, which has published in the Monthly for December, 1894; and the many interesting notes on the subject of logarithms in the Appendix.

J. M. C.

The following periodicals have been received: Journal de Mathématiques Élémentaires, (1er Juin 1897); American Journal of Mathematics, (April, 1897); The Mathematical Gazette, (February, 1897); L' Intermédiare des Mathématiciens, (Mai, 1897); Miscellaneous Notes and Queries, (May, 1897); The Kansas University Quarterly, (January, 1897); The Monist, (April, 1897); Bulletin of the American Mathematical Society, (May, (1897); The Educational Times, (May, 1897), Science, (No. for June 11, 1897); The Review of Reviews, (June, 1897), The Cosmopolitan, (June, 1897); The Arena, (June, 1897).

Yours Truly
De Volson Wood.

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BIOGRAPHY.

PROFESSOR DE VOLSON WOOD.

PROFESSOR WOOD was a man of wide and enviable reputation. It had been the fortune of many generations of students to sit under his teachings, he had written books which are standard in the technical schools and among engineers, and he had been active all his life in written and spoken discussions before the several societies of which he was a member. Furthermore, he had personal qualities which impressed themselves promptly and strongly upon those who came in contact with him, and as a consequence of all these conditions he was one of the best-known professors in the United States. But beyond all that lay extraordinary ability as a mathematician and as an analyst, remarkable strength and simplicity of character, and a genius for teaching which made his reputation a good deal more than temporary or local.

Professor Wood was a man of considerable practical mechanical ability, but that ability had never been turned to very important results. His powers as a mathematician, however, have given him a permanent place in the literature of engineering, and no student of the higher mathematics of engineering can remain ignorant of the name of DeVolson Wood. But his real greatness was as teacher. In one sense perhaps that is a misfortune for a man, because he leaves no monument except in the hearts and the minds of the men who actually came under his personal influence. His fame becomes a tradition, fading away and gradually disappearing. On the other hand, is this not the very best work that a man can do in the world—the work of a really strong and sound teacher?

It would be difficult to sum up in a few words all the qualities which made Professor Wood great as a teacher, but the fundamental quality was his own downright sincerity and his faith in his own work; his mind knew only one test, and that was the truth. To him things were either right or they were wrong, and facts were facts or they were not facts, and he saw no occasion for trying to find any middle ground. But the pursuit of the truth is often enough an arid enterprise, and a man needs more than his own sincerity to get young men to follow him eagerly in that enterprise; and Professor Wood did get his students to work with alacrity, with eagerness, with enthusiasm. A strong element in this was his own rugged and wholesome enthusiasm; another was his air. His solid and robust figure, his keen eye and square jaw, his frank and ready smile—all these were part of his influence on the young men. Added to the genuineness which appeared in all his speech and all his manner was a gift of geniality. The youth who came in contact with him could not help feeling that he stood before a real man, a man strong and sound, mentally and physically; and while youth is not very analytical it is impressed by a man of such quality without knowing why it is impressed. The writer of these words, who had the fortune to sit under Professor Wood four years in civil engineering, can testify that no other teacher ever gave him such hard lessons or ever got out of him so good recitations, and yet there was no sense of hardship in it. It seemed a natural and inevitable thing to work about five times as hard for Professor Wood as for any other teacher, and this perhaps was largely a result of his own enthusiasm in the work. He had furthermore a gift of personal interest in his students. Probably a very small percentage of his pupils—and they must have been unworthy students at that—failed to feel that Professor Wood had a particular personal interest in them. It was not that he took any special trouble with any one man, but he was always able to carry a man's personality in his mind, and he seemed always to be interested in knowing something about a man's career, And so it came about that his influence on the lives of his students did not cease when they left his class-room.

Professor Wood was an active and sincere Christian gentleman, always interested in good work and always exerting a good influence in the community about him. Among a select body of students his name will be known and honored for generations to come as the name of a clear and able writer on the mathematics and mechanics of engineering; among a great body of teachers, students, engineers, and administrators he is remembered in gratitude and love as a strong and wholesome and stimulating friend." *From the Railroad Gazette of July 2* 1897.

Professor Wood was born near Smyrna, New York, on June 1, 1832, and died at Hoboken, New Jersey, June 27, 1897. He began teaching in 1849 teaching for three terms in Smyrna. In 1853, he graduated from the Albany State Normal School. During the same and the following year he was principal at Napanoch. He was assistant professor of mathematics in Albany Normal 1854-5, assistant instructor at the Renselaer Polytechnic Institute, Troy, 1855-

n which he received the degree of Civil Engineer. Hamilton College confer- the degree of Master of Arts in 1859.

At the University of Michigan he was professor from 1857 to 1872, eiving the degree of Master of Science in the second year of his professorship. rough his labors the department of civil engineering was organized. He be- me professor of mathematics and mechanics at Stevens Institute of Technology, oboken, New Jersey, in 1872, and upon the withdrawal of Prof. R. H. Thurs- n, to become president of Sibley College, Cornell, he became professor of echanical engineering, which position he was holding at the time of his death.

He was a member of the American Society of Civil Engineers from 1871 1885, also of American Association for the Advancement of Science, since '9, and its vice president in 1885. He was a member of the American Math- atical Society, and of the Society of Mechanical Engineers, and an honorary mber of the Society of Architects. He was the first president of the Society the Promotion of Engineering Education, started in Chicago at the time of World's Fair.

He was engineer of the ore-dock, Marquette, Michigan, in 1864, and in- tor of a steam rock drill and air compressor.

He contributed articles to the New York Teacher, Johnson's Cyclopædia, pleton's Cyclopædia of Mechanics, the London Philosophical Magazine, Van- strand's, The American Engineer, Michigan Journal of Education, Journal of nklin Institute, Railroad Gazette, of which his son is now one of the editors; e Mining and Engineering Journal, Science, The Mathematical Visitor, The alyst, The Annals of Mathematics, THE AMERICAN MATHEMATICAL MONTHLY, l other magazines.

He was the author of Trusses, Bridges and Roofs, published in 1872, od's Edition of Mahan's Civil Engineering, Treatise on the Resistance Materials, Elements of Analytical Mechanics, Wood's Edition of Magnus' Les- s in Elementary Mechanics, Coördinate Geometry and Quaternions, Key and pplement to Elements of Mechanics, and to the Mechanics of Fluids, Trigon- etry, Turbines, and in 1887 he published one of the greatest of his books, rmodynamics, which has entered a number of universities and gone through eral editions.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London
ematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from June-July Number.]

SCHOLION I. And this it is, that I said before in Cor. II. after XX this; obviously that no place would remain over for the hypothesis of acut gle, or Euclidean Geometry would be most exactly established, if any straights existing in the same plane, as suppose AX, BX, which the straigh meeting (the point B being assumed at a distance from the point A as gre you choose) makes with them toward the same parts of the points X two a less than two right angles, if (I say) nowhere at another place (this stan they can admit a common perpendicular.

For then these two AX, BX mutually approach each other ever mor deed either within a certain determinate limit, as in XXV of this, or withou certain limit, and therefore even to meeting, anyhow after infinite producti in this XXVII.

But it holds that in either of the aforesaid cases the destruction of th pothesis of acute angle has now been shown. Quod intendebatur.

SCHOLION II. And again this it is, that I promised at the end of Sch IV after XXI of this, as from the very terms clearly shines out.

SCHOLION III. Moreover I could wish here to be observed the diffe between this proposition and the preceding XVII. For there (recall Fig. 15) has been shown the destruction of the hypothesis of acute angle, if (the straight AB being as small as you choose) every BD erected at whatever acute angle, must at length meet in some point K the perpendicular AH produced.

But here (viceversâ) in fact is permitted the designation of however most small an acute angle at the point A, while still the sect AB to which is to be erected the indefinite perpendicular BX, may be taken of any length whatever.

[To be Continued.]

ON THE COMPLEX ROOTS OF NUMERICAL EQUATIONS OF THE THIRD AND FOURTH DEGREE.

By A. C. BURNHAM, Berlin, Germany.

The real roots of a numerical equation can, as is well known, be found to any desired degree of accuracy by Horner's method of approximation. The complex roots as well can, for cubic and biquadratic equations, be very easily found by the same method. In fact a single application of Horner's method is in these cases sufficient for determining all the roots to any desired number of decimal places, whether the roots be positive or negative, commensurable or incommensurable, real or complex.

THE CUBIC EQUATION.

Let the cubic equation

$$x^3+a_1x^2+a_2x+a_3=0 \ldots\ldots\ldots\ldots (A),$$

have the roots c, $a+bi$, $a-bi$, since one root must be real, where $i=\sqrt{-1}$ and a, b, c are real. The sums of the products of the roots one, two, and three at a time are equal respectively to $-a_1$, a_2, $-a_3$. That is

$$a+bi+a-bi+c=-a_1,$$

$$(a+bi)(a-bi)+(a+bi)c+(a-bi)c=a_2,$$

$$(a+bi)(a-bi)c=-a_3,$$

or

$$2a+c=-a_1 \ldots\ldots\ldots\ldots (1).$$

$$a^2+b^2+2ac=a_2 \ldots\ldots\ldots\ldots (2).$$

$$(a^2+b^2)c=-a_3 \ldots\ldots\ldots\ldots (3).$$

From these three equations it is not difficult to get the following:

$$8a^3+8a_1a^2+2(a_1^2+a_2)a+a_1a_2-a_3=0 \ldots\ldots\ldots\ldots \text{I}.$$

$$c=-(a_1+2a) \text{ or } a=-\tfrac{1}{2}(c+a_1) \ldots\ldots\ldots\ldots \text{II}.$$

$$\pm\sqrt{\frac{a_3-a_1a^2-2a^3}{2a+a_1}}=b=\pm\sqrt{\frac{-a_3}{c}-a^2}=\pm\sqrt{\frac{-a_3-\frac{1}{4}c(c+a_1)^2}{c}} \ldots\ldots \text{III}.$$

Equations I, II, III give all the roots to any desired degree of accuracy. One may find c from the given equation (A) by Horner's method, or a f equation I by the same method, according to which is the easier. The a or c and b are given by II and III by a mere substitution. It is, of course, immaterial whether the positive or negative value of b be taken, since, in any case, both are used. b will be imaginary only when the original equation (A) has all three roots real. It is also of no consequence which of the three values for a given by equation I be taken, but I will in no case have a greater number of real roots than the given equation (A).

EXAMPLE. Find the roots of $x^3-2x-5=0$.

The one real root c, easily found by Horner's method is, $c=2.0945+$.

We have moreover, $a_1=0$, $a_2=-2$, $a_3=-5$. Therefore, $a=-\frac{1}{2}(c+a_1)$ $=-1.0472+$, and $b=\sqrt{\frac{5}{2.0945}-(-1.0472)^2}=1.123$.

The roots therefore are $-1.0472\pm1.123\sqrt{-1}$ and 2.0945.

In this example, equation I. takes the form

$$8a^3-4a+5=0,$$

which has the one real root $a=1.0472+$. This is the same result as above.

THE BIQUADRATIC EQUATION.

Let the roots of the biquadratic equation

$$x^4+a_1x^3+a_2x^2+a_3x+a_4=0 \quad \ldots\ldots\ldots\ldots (B).$$

be $a\pm bi$, $c\pm di$. We then have as before

$$a+bi+a-bi+c+di+c-di=-a_1,$$

$$(a+bi)(a-bi)+(a+bi)(c+di)+(a+bi)(c-di)+(a-bi)(c+di)$$
$$+(a-bi)(c-di)+(c+di)(c-di)=a_2,$$

$$(a+bi)(a-bi)(c+di)+(a+bi)(a-bi)(c-di)+(a+bi)(c+di)(c-di)$$
$$+(a-bi)(c+di)(c-di)=-a_3,$$

$$(a+bi)(a-bi)(c+di)(c-di)=a_4,$$

or

$$2(a+c)=-a_1 \quad \ldots\ldots\ldots\ldots (1).$$

$$(a^2+b^2)+(c^2+d^2)+4ac=a_2 \quad \ldots\ldots\ldots\ldots (2).$$

$$2c(a^2+b^2)+2a(c^2+d^2)=-a_3 \quad \ldots\ldots\ldots\ldots (3).$$

$$(a^2+b^2)(c^2+d^2)=a_4 \quad \ldots\ldots\ldots\ldots (4).$$

From these four equations we find

$$a=-\tfrac{1}{2}(2c+a_1)\dots\dots\dots\dots\dots\dots\dots\dots (5).$$

$$c=-\tfrac{1}{2}(2a+a_1)\dots\dots\dots\dots\dots\dots\dots\dots (6).$$

$$c^2+d^2=u=\frac{2ca_2+a_3+4c^2(2c+a_1)}{4c+a_1}\dots\dots\dots\dots\dots\dots \text{I}.$$

$$a^2+b^2=t=-\frac{2au+a_3}{2c}=\frac{a_4}{u}\dots\dots\dots\dots\dots\dots\dots\dots \text{II}.$$

If we now eliminate u and a by means of 5, I and II, we have the following equation of the sixth degree for c:

$$64c^6+96a_1c^5+16(3a_1^2+2a_2)c^4+8(4a_1a_2+a_1^3)c^3$$
$$+4(a_2^2+2a_1^2a_2+a_1a_3-4a_4)c^2+2(a_1a_2^2+a_1^2a_3-4a_1a_4)c$$
$$+a_1a_2a_3-a_1^2a_4-a_3^2=0\dots\dots\dots\dots\dots\dots \text{III}.$$

From I and II we have moreover,

$$d=\pm\sqrt{u-c^2}\dots\dots\dots\dots\dots\dots\dots\dots (7).$$

$$b=\pm\sqrt{t-a^2}\dots\dots\dots\dots\dots\dots\dots\dots (8).$$

Therefore, after getting a single value of c from III by Horner's method, a, u, t, d, and b follow respectively from (5), I, II, (7), and (8) by mere substitutions, and thus a single application of Horner's method suffices to find all of the roots, no matter what their character.

If the equation (B) has no real roots, then III will have only two real roots. They are separately the values for a and c, and either can be taken for c. That is, the equation of the sixth degree giving a is the same as III giving c.

Example. Find the roots of the equation

$$x^4-6x^3+18x^2-30x+25=0.$$

In this equation $a_1=-6$, $a_2=18$, $a_3=-30$, $a_4=25$.
We have therefore as equation III,

$$4c^6-36c^5+144c^4-324c^3+425c^2-303c+90=0.$$

It is immediately seen that one is a root of this equation, therefore $c=1$, from which there follows,

from (5), $a=-\tfrac{1}{2}(2-6)=2$,

from I, $u=[2.18-30+4(2-6)]/[-2]=5$,

from II, $t=a_4/u=\frac{25}{5}=5$,

from (7), $d=\pm\sqrt{5-1}=\pm 2$,

from (8), $b=\pm\sqrt{5-4}=\pm 1$.

The roots are therefore $2\pm i$, $1\pm 2i$.

The value 2 for a satisfies the equation III as it should, and 1 and 2 are the only real roots which III possesses.

If the given equation (B) has two real roots and both are known to any desired degree of accuracy, the two other roots are very easily found. Put

$$a+bi=h$$
$$a-bi=k$$

where h and k are known. Then

$$a=\tfrac{1}{2}(h+k),$$
$$b=-\tfrac{1}{2}(h-k)i,$$
$$c=-\tfrac{1}{2}(h+k+a_1) \text{ from (5)},$$

and u is found from I and d from (7) as before. Thus the roots are all determined.

If all of the roots of (B) are real, they will be equally well given by the first method above. In this case b and d will be imaginary.

A DEVICE FOR EXTRACTING THE SQUARE ROOT OF CERTAIN SURD QUANTITIES.

By ROBERT J. ALEY, A. M., Ph. D., Professor of Mathematics, University of Indiana, Bloomington, Indiana.

$ABMN$ is a square. OL is an arm revolving freely about O. This arm beyond C is divided into equal parts at E, x, y, z, etc.

To determine the character of the divisions made on FP by the points of division on OL as OL revolves. Call the side of the square AB, $2a$; BC, b; CE, c; and CD, x.

Then $OC=\sqrt{a^2+(a+b)^2}$.

$$GC=2\sqrt{a^2+(a+b)^2}+c.$$

$FC=2(a+b)+x.$

From the properties of two intersecting chords we have,

$$x\{x+2(a+b)\}=c\{c+2\sqrt{a^2+(a+b)^2}\}$$

$$x^2+2(a+c)x+(a+b)^2=a^2+2ab+b^2+c^2+2c\sqrt{a^2+(a+b)^2}$$

$$x+(a+b)=\sqrt{(a+b)^2+c^2+2c\sqrt{a^2+(a+b)^2}}.$$

Suppose that we examine the results when integral values are given to e constants.

Put $a=c=1$, $b=0$. (Let c take successively the values 1, 2, 3, 4, ... c.)

Then $x+1=\sqrt{2+2\sqrt{2}}$,

$$x+1=\sqrt{5+4\sqrt{2}},$$

$$x+1=\sqrt{10+6\sqrt{2}},$$

$$x+1=\sqrt{17+8\sqrt{2}}, \text{ etc.},$$

d the law of the series is readily seen.

Put $a=c=b=1$, and let c vary as before.

$$x+2=\sqrt{5+2\sqrt{5}},$$

$$x+2=\sqrt{8+4\sqrt{5}},$$

$$x+2=\sqrt{13+6\sqrt{5}},$$

$$x+2=\sqrt{20+8\sqrt{5}}, \text{ etc.}$$

The law is again evident.

Put $a=1$, $b=2$, and let c vary.

$$x+3=\sqrt{10+2\sqrt{10}},$$

$$x+8=\sqrt{13+4\sqrt{10}},$$

$$x+8=\sqrt{18+6\sqrt{10}},$$

$$x+8=\sqrt{25+8\sqrt{10}}, \text{ etc.}$$

The law is again evident.

Put $a=1$, $b=3$, and let c vary.

$$x+4=\sqrt{17+2\sqrt{17}},$$

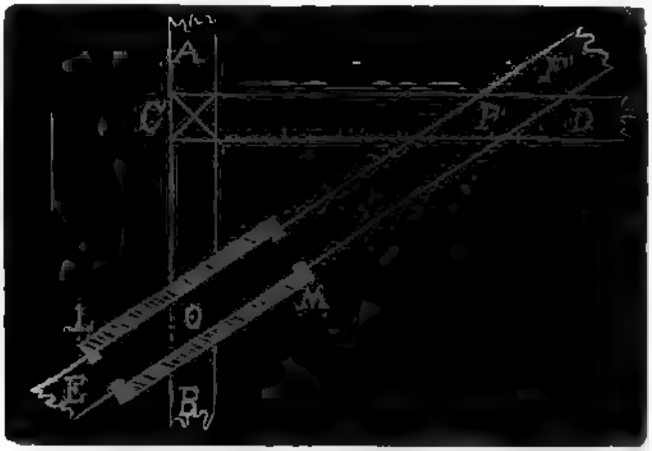
$$x+4=\sqrt{20+4\sqrt{17}},$$

$$x+4=\sqrt{25+6\sqrt{17}}.$$

These examples show how the various series will be found.

From the previous considerations we at once have the data for the construction of a simple mechanical device for the extraction of roots of certain surd quantities.

AB is an upright so arranged that *CD* will slide up and down always parallel to itself. It is accurately marked to scale so that *CD* may be set at any de-



sired a. *FE* works in a slide *LM* which is free to rotate about *O*. It is accurately ruled to scale from *P* to *F*. By sliding it in *LM*, *P* may be set at any desired $a+c$. *CD* is ruled to scale and is also provided with a diagonal scale, so that by the use of dividers, results may be read to hundredths. When the instrument is set at any chosen a and b, all the roots for that set may be read off at once.

Tables may be easily constructed. A few samples are here given.

The a's are read in the vertical columns, the b's horizontally, and in the squares the c's take successively the values 1, 2, 3, etc. But three terms are given in each square, enough to make the law perfectly evident.

TABLE FOR INTEGRAL VALUES OF a, b, AND c.

$b=$	0	1	2	3	4	5	6	7	8
$a=$ 1	$\sqrt{2+2\sqrt{2}}$ $\sqrt{5+4\sqrt{2}}$ $\sqrt{10+6\sqrt{2}}$	$\sqrt{5+2\sqrt{5}}$ $\sqrt{8+4\sqrt{5}}$ $\sqrt{13+6\sqrt{5}}$	$\sqrt{10+2\sqrt{10}}$ $\sqrt{13+4\sqrt{10}}$ $\sqrt{18+6\sqrt{10}}$	$\sqrt{17+2\sqrt{17}}$ $\sqrt{20+4\sqrt{17}}$ $\sqrt{25+6\sqrt{17}}$	$\sqrt{26+2\sqrt{26}}$ $\sqrt{29+4\sqrt{26}}$ $\sqrt{34+6\sqrt{26}}$	$\sqrt{37+2\sqrt{37}}$ $\sqrt{40+4\sqrt{37}}$ $\sqrt{45+6\sqrt{37}}$	$\sqrt{50+2\sqrt{50}}$ $\sqrt{53+4\sqrt{50}}$ $\sqrt{58+6\sqrt{50}}$	$\sqrt{65+2\sqrt{65}}$ $\sqrt{68+4\sqrt{65}}$ $\sqrt{73+6\sqrt{65}}$	$\sqrt{82+2\sqrt{82}}$ $\sqrt{85+4\sqrt{82}}$ $\sqrt{90+6\sqrt{82}}$
2	$\sqrt{5+2\sqrt{8}}$ $\sqrt{8+4\sqrt{8}}$ $\sqrt{13+6\sqrt{8}}$	$\sqrt{10+2\sqrt{13}}$ $\sqrt{13+4\sqrt{13}}$ $\sqrt{18+6\sqrt{13}}$	$\sqrt{17+2\sqrt{20}}$ $\sqrt{20+4\sqrt{20}}$ $\sqrt{25+6\sqrt{20}}$	$\sqrt{26+2\sqrt{29}}$ $\sqrt{29+4\sqrt{29}}$ $\sqrt{34+6\sqrt{29}}$	$\sqrt{37+2\sqrt{40}}$ $\sqrt{40+4\sqrt{40}}$ $\sqrt{45+6\sqrt{40}}$	$\sqrt{50+2\sqrt{53}}$ $\sqrt{53+4\sqrt{53}}$ $\sqrt{58+6\sqrt{53}}$	$\sqrt{65+2\sqrt{68}}$ $\sqrt{68+4\sqrt{68}}$ $\sqrt{73+6\sqrt{68}}$	$\sqrt{82+2\sqrt{85}}$ $\sqrt{85+4\sqrt{85}}$ $\sqrt{90+6\sqrt{85}}$	
3	$\sqrt{10+2\sqrt{18}}$ $\sqrt{13+4\sqrt{18}}$ $\sqrt{18+6\sqrt{18}}$	$\sqrt{17+2\sqrt{25}}$ $\sqrt{20+4\sqrt{25}}$ $\sqrt{25+6\sqrt{25}}$	$\sqrt{26+2\sqrt{34}}$ $\sqrt{29+4\sqrt{34}}$ $\sqrt{34+6\sqrt{34}}$	$\sqrt{37+2\sqrt{45}}$ $\sqrt{40+4\sqrt{45}}$ $\sqrt{45+6\sqrt{45}}$	$\sqrt{50+2\sqrt{58}}$ $\sqrt{53+4\sqrt{58}}$ $\sqrt{58+6\sqrt{58}}$	$\sqrt{65+2\sqrt{73}}$ $\sqrt{68+4\sqrt{73}}$ $\sqrt{73+6\sqrt{73}}$	$\sqrt{82+2\sqrt{90}}$ $\sqrt{85+4\sqrt{90}}$ $\sqrt{90+6\sqrt{90}}$		
4	$\sqrt{17+2\sqrt{32}}$ $\sqrt{20+4\sqrt{32}}$ $\sqrt{25+6\sqrt{32}}$	$\sqrt{26+2\sqrt{41}}$ $\sqrt{29+4\sqrt{41}}$ $\sqrt{34+6\sqrt{41}}$	$\sqrt{37+2\sqrt{52}}$ $\sqrt{40+4\sqrt{52}}$ $\sqrt{45+6\sqrt{52}}$	$\sqrt{50+2\sqrt{65}}$ $\sqrt{53+4\sqrt{65}}$ $\sqrt{58+6\sqrt{65}}$	$\sqrt{65+2\sqrt{80}}$ $\sqrt{68+4\sqrt{80}}$ $\sqrt{73+6\sqrt{80}}$	$\sqrt{82+2\sqrt{97}}$ $\sqrt{85+4\sqrt{97}}$ $\sqrt{90+6\sqrt{97}}$			
5	$\sqrt{26+2\sqrt{50}}$ $\sqrt{29+4\sqrt{50}}$ $\sqrt{34+6\sqrt{50}}$	$\sqrt{37+2\sqrt{61}}$ $\sqrt{40+4\sqrt{61}}$ $\sqrt{45+6\sqrt{61}}$	$\sqrt{50+2\sqrt{74}}$ $\sqrt{53+4\sqrt{74}}$ $\sqrt{58+6\sqrt{74}}$	$\sqrt{65+2\sqrt{89}}$ $\sqrt{68+4\sqrt{89}}$ $\sqrt{73+6\sqrt{89}}$	$\sqrt{82+2\sqrt{106}}$ $\sqrt{85+4\sqrt{106}}$ $\sqrt{90+6\sqrt{106}}$				
6	$\sqrt{37+2\sqrt{72}}$ $\sqrt{40+4\sqrt{72}}$ $\sqrt{45+6\sqrt{72}}$	$\sqrt{50+2\sqrt{85}}$ $\sqrt{53+4\sqrt{85}}$ $\sqrt{58+6\sqrt{85}}$	$\sqrt{65+2\sqrt{100}}$ $\sqrt{68+4\sqrt{100}}$ $\sqrt{73+6\sqrt{100}}$	$\sqrt{82+2\sqrt{117}}$ $\sqrt{85+4\sqrt{117}}$ $\sqrt{90+6\sqrt{117}}$					
7	$\sqrt{50+2\sqrt{98}}$ $\sqrt{53+4\sqrt{98}}$ $\sqrt{58+6\sqrt{98}}$	$\sqrt{65+2\sqrt{113}}$ $\sqrt{68+4\sqrt{113}}$ $\sqrt{73+6\sqrt{113}}$	$\sqrt{82+2\sqrt{130}}$ $\sqrt{85+4\sqrt{130}}$ $\sqrt{90+6\sqrt{130}}$						
8	$\sqrt{65+2\sqrt{128}}$ $\sqrt{68+4\sqrt{128}}$ $\sqrt{73+6\sqrt{128}}$	$\sqrt{82+2\sqrt{145}}$ $\sqrt{85+4\sqrt{145}}$ $\sqrt{90+6\sqrt{145}}$							
9	$\sqrt{82+2\sqrt{162}}$ $\sqrt{85+4\sqrt{162}}$								

FRACTIONAL VALUES OF a AND b.

$a=$ \ $b=$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$
$\frac{1}{2}$	$\sqrt{2+\sqrt{5}}$ $\sqrt{5+2\sqrt{5}}$ $\sqrt{10+3\sqrt{5}}$	$\sqrt{5+\sqrt{17}}$ $\sqrt{8+2\sqrt{17}}$ $\sqrt{13+3\sqrt{17}}$	$\sqrt{10+\sqrt{37}}$ $\sqrt{13+2\sqrt{37}}$ $\sqrt{18+3\sqrt{37}}$	$\sqrt{17+\sqrt{65}}$ $\sqrt{20+2\sqrt{65}}$ $\sqrt{25+3\sqrt{65}}$	$\sqrt{26+\sqrt{101}}$ $\sqrt{29+2\sqrt{101}}$ $\sqrt{34+3\sqrt{101}}$
$\frac{3}{2}$	$\sqrt{5+\sqrt{25}}$ $\sqrt{8+2\sqrt{25}}$ $\sqrt{13+3\sqrt{25}}$	$\sqrt{10+\sqrt{45}}$ $\sqrt{13+2\sqrt{45}}$ $\sqrt{18+3\sqrt{45}}$	$\sqrt{17+\sqrt{73}}$ $\sqrt{20+2\sqrt{73}}$ $\sqrt{25+3\sqrt{73}}$	$\sqrt{26+\sqrt{109}}$ $\sqrt{29+2\sqrt{109}}$ $\sqrt{34+3\sqrt{109}}$	
$\frac{5}{2}$	$\sqrt{10+\sqrt{61}}$ $\sqrt{13+2\sqrt{61}}$ $\sqrt{18+3\sqrt{61}}$	$\sqrt{17+\sqrt{89}}$ $\sqrt{20+2\sqrt{89}}$ $\sqrt{25+3\sqrt{89}}$	$\sqrt{26+\sqrt{125}}$ $\sqrt{29+2\sqrt{125}}$ $\sqrt{34+3\sqrt{125}}$		
$\frac{7}{2}$	$\sqrt{17+\sqrt{113}}$ $\sqrt{20+2\sqrt{113}}$ $\sqrt{25+3\sqrt{113}}$	$\sqrt{26+\sqrt{149}}$ $\sqrt{29+2\sqrt{149}}$ $\sqrt{34+3\sqrt{149}}$			
$\frac{9}{2}$	$\sqrt{26+\sqrt{181}}$ $\sqrt{29+2\sqrt{181}}$ $\sqrt{34+3\sqrt{181}}$				

SIMPLE SURD VALUES OF a AND b.

$a=$ \ $b=$	0	$\sqrt{5}$	$2\sqrt{5}$	$3\sqrt{5}$
$\sqrt{5}$	$\sqrt{6+2\sqrt{10}}$ $\sqrt{9+4\sqrt{10}}$ $\sqrt{14+6\sqrt{10}}$	$\sqrt{21+2\sqrt{25}}$ $\sqrt{24+4\sqrt{25}}$ $\sqrt{29+6\sqrt{25}}$	$\sqrt{46+2\sqrt{50}}$ $\sqrt{49+4\sqrt{50}}$ $\sqrt{54+6\sqrt{50}}$	$\sqrt{81+2\sqrt{85}}$ $\sqrt{84+4\sqrt{85}}$ $\sqrt{89+6\sqrt{85}}$
$2\sqrt{5}$	$\sqrt{21+2\sqrt{40}}$ $\sqrt{24+4\sqrt{40}}$ $\sqrt{29+6\sqrt{40}}$	$\sqrt{46+2\sqrt{65}}$ $\sqrt{49+4\sqrt{65}}$ $\sqrt{54+6\sqrt{65}}$	$\sqrt{81+2\sqrt{100}}$ $\sqrt{84+4\sqrt{100}}$ $\sqrt{89+6\sqrt{100}}$	
$3\sqrt{5}$	$\sqrt{46+2\sqrt{90}}$ $\sqrt{49+4\sqrt{90}}$ $\sqrt{54+6\sqrt{90}}$	$\sqrt{81+2\sqrt{125}}$ $\sqrt{84+4\sqrt{125}}$ $\sqrt{89+6\sqrt{125}}$		
$4\sqrt{5}$	$\sqrt{81+2\sqrt{160}}$ $\sqrt{84+4\sqrt{160}}$ $\sqrt{89+6\sqrt{160}}$			

SIMPLE SURD VALUES OF a AND b.

$a=$ \ $b=$	0	$\sqrt{2}$	$2\sqrt{2}$	$3\sqrt{2}$
$\sqrt{2}$	$\sqrt{3+2\sqrt{4}}$ $\sqrt{6+4\sqrt{4}}$ $\sqrt{11+6\sqrt{4}}$	$\sqrt{9+2\sqrt{10}}$ $\sqrt{12+4\sqrt{10}}$ $\sqrt{17+6\sqrt{10}}$	$\sqrt{19+2\sqrt{20}}$ $\sqrt{22+4\sqrt{20}}$ $\sqrt{25+6\sqrt{20}}$	$\sqrt{33+2\sqrt{34}}$ $\sqrt{36+4\sqrt{34}}$ $\sqrt{41+6\sqrt{34}}$
$2\sqrt{2}$	$\sqrt{9+2\sqrt{16}}$ $\sqrt{12+4\sqrt{16}}$ $\sqrt{17+6\sqrt{16}}$	$\sqrt{19+2\sqrt{26}}$ $\sqrt{22+4\sqrt{26}}$ $\sqrt{25+6\sqrt{26}}$	$\sqrt{33+2\sqrt{40}}$ $\sqrt{36+4\sqrt{40}}$ $\sqrt{41+6\sqrt{40}}$	
$3\sqrt{2}$	$\sqrt{19+2\sqrt{36}}$ $\sqrt{22+4\sqrt{36}}$ $\sqrt{27+6\sqrt{36}}$	$\sqrt{33+2\sqrt{50}}$ $\sqrt{36+4\sqrt{50}}$ $\sqrt{41+6\sqrt{50}}$		
$4\sqrt{2}$	$\sqrt{33+2\sqrt{64}}$ $\sqrt{36+4\sqrt{64}}$			

SIMPLE SURD VALUES OF a AND b.

$a=$ \ $b=$	0	$\sqrt{3}$	$2\sqrt{3}$	$3\sqrt{3}$
$\sqrt{3}$	$\sqrt{4+2\sqrt{6}}$ $\sqrt{7+4\sqrt{6}}$ $\sqrt{12+6\sqrt{6}}$	$\sqrt{13+2\sqrt{15}}$ $\sqrt{16+4\sqrt{15}}$ $\sqrt{21+6\sqrt{15}}$	$\sqrt{28+2\sqrt{30}}$ $\sqrt{31+4\sqrt{30}}$ $\sqrt{36+6\sqrt{30}}$	$\sqrt{49+2\sqrt{51}}$ $\sqrt{52+4\sqrt{51}}$ $\sqrt{57+6\sqrt{51}}$
$2\sqrt{3}$	$\sqrt{13+2\sqrt{24}}$ $\sqrt{16+4\sqrt{24}}$ $\sqrt{21+6\sqrt{24}}$	$\sqrt{28+2\sqrt{39}}$ $\sqrt{31+4\sqrt{39}}$ $\sqrt{36+6\sqrt{39}}$	$\sqrt{49+2\sqrt{60}}$ $\sqrt{52+4\sqrt{60}}$ $\sqrt{57+6\sqrt{60}}$	
$3\sqrt{3}$	$\sqrt{28+2\sqrt{54}}$ $\sqrt{31+4\sqrt{54}}$ $\sqrt{36+6\sqrt{54}}$	$\sqrt{49+2\sqrt{75}}$ $\sqrt{52+4\sqrt{75}}$ $\sqrt{57+6\sqrt{75}}$		
$4\sqrt{3}$	$\sqrt{49+2\sqrt{96}}$ $\sqrt{51+4\sqrt{96}}$			

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

81. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

How far will a body fall in the first second on the sun, the density of the sun being .25 times that of the earth and its diameter 866400 miles?

Solution by G. B. M. ZERR, A. M., Ph. D., President of Russell College, Lebanon, Va.; and E. W. MORRELL, A. M., Professor of Mathematics, Montpelier Seminary, Montpelier, Vt.

Let G=gravity on sun, g=gravity on earth.

D=density of sun, d=density of earth.

R=radius of sun, r=radius of earth.

Then $G:g=DR:dr$. $\therefore G=gDR/dr$.

Now $g=32.2$, $D=.25d=\frac{1}{4}d$, $R=109.5r$.

$$\therefore G=\frac{32.2\times109.5}{4}=881.475.$$

$\frac{1}{2}G=440.7875$ feet, the distance a body will fall the first second.

82. Proposed by CHAS. C. CROSS, Laytonsville, Md.

Two men, A and B, started from the same point at the same time; A traveled southeast for 10 hours and at the rate of 10 miles per hour, and B due south for the same time, going 6 miles per hour; they then turned and traveled directly towards each other at the same rates respectively, till they met. How far did each man travel?

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; G. B. M. ZERR, A. M., Ph. D., President of Russell College, Lebanon, Va.; P. S. BERG, Larimore, N. D.; and H. C. WILKES, Skull Run, Va.

Let C be the starting point; CA=the distance A traveled southeast, and CB=the distance B traveled south. Then $CA=100$ miles, and $CB=60$ miles. Now draw AD perpendicular to CB produced to D. As $\angle D$=a right angle, and $\angle C=45°$, then $CD=AD$.

Then $2\overline{AD}^2=\overline{AC}^2=100^2$; whence $AD=50\sqrt{2}$, and $BD=50\sqrt{2}-60$. $\therefore$ From the right triangle ADB,

$$AB=\sqrt{(50\sqrt{2})^2+(50\sqrt{2}-60)^2}=\sqrt{13600-6000\sqrt{2}}$$

$$=71.517261+ \text{ miles}.$$

A and B together travel 16 miles per hour, and the time required, until they meet in traveling AB, is $\frac{1}{16}$ of AB=4.469828+ hours. Therefore, A traveled 44.69828+ miles and B, 26.81897+ miles of the distance AB.

$\therefore$ The total distance traveled by A is 144.69828+ miles, and by B, 86.81897+ miles.

This problem was also solved by *F. R. HONEY*, *C. A. JONES*, and *E. W. MORRELL*.

Solutions of problem 80 were received from G. B. M. Zerr, P. S. Berg, E. W. Morrell, F. R. Honey, and H. C. Wilkes.

NOTE. Hon. Josiah H. Drummond says, in reference to problem 78: "How can you make $966\times3+569=3905$? The question is erroneously enunciated or erroneously solved, or both."

If we assume that the problem is correctly stated, then certainly the published solution is not the solution of the problem. The following is an algebraic statement of the problem as proposed:

Let x=number of cows. Then $3x+569$=number of horses. Let y=number of sheep. Then $4y-126$=number of cows. Hence, $x=4y-126$, $3x=12y-378$, and $3x+569=12y-378+569=12y+191$, the number of horses expressed in terms of the number of sheep. Hence, y, the number of sheep, $+4y-126$, the number cows, $+12y+191$, the number of horses, or $17y+65$=total number=5169. Solving this equation, we do not obtain integral results. If 9 were changed to 5, then y would be integral, and the problem possible. We failed to find this problem in Brooks' Higher Arithmetic. EDITOR.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

72. Proposed by CHAS. C. CROSS, Laytonsville, Md.

Prove that $\frac{2\sqrt{2+\sqrt{3}}}{4+\sqrt{6}-\sqrt{2}}=\sqrt{6}-\sqrt{2}+\sqrt{3}-2$, when reduced to its lowest terms.

I. Solution by JOSIAH H. DRUMMOND, Portland, Maine.

$$\frac{2\sqrt{2+\sqrt{3}}}{4+\sqrt{6}-\sqrt{2}}=\frac{\sqrt{2}\sqrt{4+2\sqrt{3}}}{4+\sqrt{2}(\sqrt{3}-1)},\ =\frac{1+\sqrt{3}}{2\sqrt{2}+\sqrt{3}-1},$$

$$=\frac{(1+\sqrt{3})(2\sqrt{2}-\sqrt{3}+1)}{2(2+\sqrt{3})},\ =\frac{(2-\sqrt{3})(1+\sqrt{3})(2\sqrt{2}-\sqrt{3}+1)}{2},$$

$$=\frac{(\sqrt{3}-1)(2\sqrt{2}-\sqrt{3}+1)}{2},\ =\sqrt{6}-\sqrt{2}+\sqrt{3}-2.$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Russell College, Lebanon, Va.; and P. S. BERG, Principal of Schools, Larimore, N. D.

$$\frac{2\sqrt{2+\sqrt{3}}}{4+\sqrt{6}-\sqrt{2}}=\frac{\sqrt{6}+\sqrt{2}}{4+\sqrt{6}-\sqrt{2}},\ =\frac{(\sqrt{6}+\sqrt{2})(4+\sqrt{6}+\sqrt{2})}{(4+\sqrt{6})^2-2},$$

$$=\frac{\sqrt{6}+\sqrt{3}+\sqrt{2}+2}{5+2\sqrt{6}},\ =\frac{(\sqrt{6}+\sqrt{3}+\sqrt{2}+2)(5-2\sqrt{6})}{25-24},$$

$$=\sqrt{6}-\sqrt{2}+\sqrt{3}-2.$$

Also solved by *COOPER D. SCHMITT* and the *PROPOSER*.

78. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, Russell College, banon, Va.

Find the worth of each of five persons, A, B, C, D, and E, knowing, 1st, that when 's worth is added to a times what B, C, D, and E are worth, it is equal to m; 2nd, when 's worth is added to b times what A, C, D, and E are worth, it is equal to n; 3rd, when 's worth is added to c times what A, B, D, and E are worth, it is equal to p; 4th, when 's worth is added to d times what A, B, C, and E are worth, it is equal to q; 5th, when 's worth is added to e times what A, B, C, and D are worth, it is equal to r.

I. Solution by the PROPOSER.

Let x, y, z, u, v be the worth of A, B, C, D, and E, respectively. Then

$$\begin{aligned}
&x+a(y+z+u+v)=m,\\
&y+b(x+z+u+v)=n,\\
&z+c(x+y+u+v)=p,\\
&u+d(x+y+z+v)=q,\\
&v+e(x+y+z+u)=r.
\end{aligned}$$

Let $x+y+z+u+v=s$; then $x+a(s-x)=m$.

$x=(m-as)/(1-a)$..........(1). Similarly, $y=(n-bs)/(1-b)$........(2),

$=(p-cs)/(1-c)$................(3), $u=(q-ds)/(1-d)$................(4),

$v=(r-es)/(1-e)$................(5).

Adding (1), (2), (3), (4), and (5), we get

$$=\frac{m-as}{1-a}+\frac{n-bs}{1-b}+\frac{p-cs}{1-c}+\frac{q-ds}{1-d}+\frac{r-es}{1-e}.$$

$$\therefore s=\frac{\left\{\frac{m}{1-a}+\frac{n}{1-b}+\frac{p}{1-c}+\frac{q}{1-d}+\frac{r}{1-e}\right\}}{\left\{1+\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}+\frac{d}{1-d}+\frac{e}{1-e}\right\}}.$$

This value of s in (1), (2), (3), (4), (5) gives x, y, z, u, v.

II. Solution by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxv Tenn.; and Professor CHAS. C. CROSS, Laytonsville, Md.

By the conditions we have at once the five equations ; letting x, y, z, u, $=A$, B, C, D, E, and F's shares, respectively :

$$x+ay+az+au+at=m,$$
$$bx+y+by+bu+bt=n,$$
$$cx+cy+z+cu+ct=p,$$
$$dx+dy+dz+u+dt=q,$$
$$ex+ev+ez+eu+t=r.$$

Hence by Determinants,

$$x=\begin{vmatrix} m & a & a & a & a \\ n & 1 & b & b & b \\ p & c & 1 & c & c \\ q & d & d & 1 & d \\ r & e & e & e & 1 \end{vmatrix} \div \begin{vmatrix} 1 & a & a & a & a \\ b & 1 & b & b & b \\ c & c & 1 & c & c \\ d & d & d & 1 & d \\ e & e & e & e & 1 \end{vmatrix},$$

$$y=\begin{vmatrix} 1 & m & a & a & a \\ b & n & b & b & b \\ c & p & 1 & c & c \\ d & q & d & 1 & d \\ e & r & e & e & 1 \end{vmatrix} \div \begin{vmatrix} 1 & a & a & a & a \\ b & 1 & b & b & b \\ c & c & 1 & c & c \\ d & d & d & 1 & d \\ e & e & e & e & 1 \end{vmatrix},$$

and so with z, u, and t, each determinant possessing 120 terms, when expand

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to his

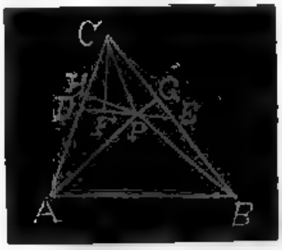
SOLUTIONS OF PROBLEMS.

76. Proposed by L. B. FRAKER, Bowling Green, Ohio.

Lines run from a point, P, within a triangular piece of land to the angles A, B, C are 91, 102, and 80 rods, respectively; and a line 78 rods in length passing through point, P, and terminating in the sides AC and BC cuts off 3024 square rods adjacent to gle C. Required the dimensions of the land.

Solution by CHAS. C. CROSS, Laytonsville, Md.

Let ABC be the required triangle. $AP=a=91$ rods, $BP=b=102$ rods, $CP=c=80$ rods, $DE=d=78$ rods, the line drawn through P and cutting off 3024 square rods adjacent to the angle C, $CD=x$, $CE=y$, $PE=z$, and area of triangle $DEC=k$.



Draw the perpendiculars PG and PH, from the point P to the sides BC and AC respectively, and draw CF perpendicular to DE. Then

$$x^2=y^2+d^2\pm 2d\times FE, \text{ whence}$$

$$FE=\frac{y^2+d^2-x^2}{2d}. \quad \text{Also } FE=\pm\sqrt{y^2-\frac{4k^2}{d^2}}.$$

$$\text{Hence, } \mp\frac{y^2+d^2-x^2}{2d}=\pm\sqrt{y^2-\frac{4k^2}{d^2}},$$

whence $-(x^2-y^2)^2+2d^2(x^2+y^2)=16k^2+d^4$, an equation containing two unknown quantities. Hence since no other conditions are given by which x or y can be found, it follows that the problem is indeterminate.

By trial, we find that $x=90$ and $y=84$ satisfies the above equation.

Hence, these values furnish a solution, in positive integers, of the problem. Then

$$PF=\sqrt{c^2-\frac{4k^2}{d^2}}=19\tfrac{9}{13} \text{ rods, and } FE=\sqrt{y^2-\frac{4k^2}{d^2}}=32\tfrac{4}{13} \text{ rods.}$$

Hence, $z=PF+FE=52$ rods.

$$BG=\frac{b^2+BC^2-c^2}{2BC} \text{ and } CG=\frac{c^2+y^2-z^2}{2y}. \quad \text{But } BG+CG=BC. \quad \text{Hence,}$$

$$BC=\frac{c^2+y^2-z^2\pm\sqrt{4(b^2-c^2)y^2+(c^2+y^2-z^2)^2}}{2y}=154 \text{ rods.}$$

By similar reasoning with the triangles APC and DPC, we find that $AC=165$ rods.

$$\cos ACB=(90^2+84^2-18^2)\div 2.90.84=\tfrac{6}{7}.$$

$$AB=\sqrt{AC^2+BC^2-2AC.BC\times\cos ACB}=143 \text{ rods.}$$

Hence, the dimensions of the field are $AB=143$ rods, $BC=154$ rods, and $AC=165$ rods.

This problem was proved indeterminate by A. H. Bell.

77. Proposed by CHAS. C. CROSS, Laytonsville, Md.

A line is drawn perpendicular to BC, of the triangle ABC, whose sides are BC $CA=b$, and $AB=c$, through A to D, a distance d, (d being equal to or greater than a- from D a line is drawn to E, a distance e, (e being equal to or greater than $a+b+c$) on extended. Required the area of the ellipse which is isogonal conjugate to the straight DE with respect to the triangle ABC.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President of Russell College, Lebanon, Va.

Using trilinear coördinates and letting F be the point where $AD=d$ c BC, we get $DF=(d-b\sin C)$, $EF=\sqrt{e^2-(d-b\sin C)^2}=f$.

$\therefore$ The coördinates of D, E are respectively,

$$\{-(d-b\sin C),\ e\cos C,\ e\cos B\} \text{ and}$$
$$\{0,\ -(f-b\cos C)\sin C,\ -(f+c\cos B)\sin B\}.$$

Let $l=e\{(f-b\cos C)\sin C\cos C-(f+c\cos B)\sin B\cos B\}$,

$m=-(d-b\sin C)(f+c\cos B)\sin B$, and

$n=(d-b\sin C)(f-b\cos C)\sin C$.

Then $l\alpha+m\beta+n\gamma=0$, is the equation to DE, and $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$, the equation to the ellipse isogonal conjugate to DE.

Let B be the origin, BC, BA the axes of (x, y).

Then $\alpha=y\sin B$, $\gamma=x\sin B$, $\beta=(ac\sin B-a\alpha-c\gamma)/b$.

$\therefore$ $\beta=\sin B(ac-ay-cx)/b$.

Substituting these values of α, β, γ the equation to the ellipse become

$$clx^2+any^2+(al+cn-bm)xy-aclx-acny=0.$$

Let $\Delta=\dfrac{a^2c^2ln(2-cn-ae)}{4acln-(al+cn-bm)^2}$ be the discriminant of the ellipse.

The two values of z in the equation,

$$z^2+\frac{16(cl+an)\Delta}{\{4acln-(al+cn-bm)^2\}^2}z-\frac{64\Delta^2}{\{4acln-(al+cn-bm)^2\}^3}=0.$$

give the values of the squares of the semi-axes.

$\therefore$ Area of ellipse

$$=\frac{8\pi\Delta}{\{4acln-(al+cn-bm)^2\}^{\frac{3}{2}}}=\frac{8\pi a^2c^2ln(2-cn-al)}{\{4acln-(al+cn-bm)^2\}^{\frac{5}{2}}}.$$

II. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

The coördinates of D are $(d, -\beta_1, -\gamma_1)$, and of E, $(0, \beta_2, -\gamma_2)$, and we en have,

$$=(abc)/(4\Delta^2)\{a(\beta_1+\beta_2)(\gamma_2-\gamma_1)+bd(\gamma_1-\gamma_2)+cd(\beta_1+\beta_2)\} \ldots\ldots(1).$$

The equation to the perpendicular to BC through A is

$$\gamma\cos C-\alpha\cos A=0 \ldots\ldots(2),$$

id this, passing through D, gives

$$\gamma_1\cos C+d\cos A=0 \ldots\ldots(3).$$

We have the constant relation

$$a\alpha+b\beta+c\gamma=2\Delta \ldots\ldots(4),$$

d this being satisfied by the coördinates of D and E,

$$ad-b\beta_1-c\gamma_1=2\Delta \ldots\ldots(5).$$

$$b\beta_2+c\gamma_2=2\Delta \ldots\ldots(6).$$

The equation to DE is

$$\alpha(\beta_1\gamma_2+\beta_2\gamma_1)+\beta d\gamma_2+\gamma d\beta_2=0 \ldots\ldots(7),$$

e isogonal conjugate of which is

$$\beta\gamma(\beta_1\gamma_2+\beta_2\gamma_1)+\alpha\gamma d\gamma_2+\alpha\beta d\beta_2=0 \ldots\ldots(8),$$

iich by the problem is an ellipse.

The area of (8) is expressed by

$$=2\pi\Delta abc\left\{\begin{vmatrix} 0, & \tfrac{1}{2}d\beta_2, & \tfrac{1}{2}d\gamma_2 \\ \tfrac{1}{2}d\beta_2, & 0, & \tfrac{1}{2}(\beta_1\gamma_2+\beta_2\gamma_1) \\ \tfrac{1}{2}d\gamma_2, & \tfrac{1}{2}(\beta_1\gamma_2+\beta_2\gamma_1), & 0 \end{vmatrix} \div \begin{vmatrix} 0, & \tfrac{1}{2}d\beta_2, & \tfrac{1}{2}d\gamma_2, & -a \\ \tfrac{1}{2}d\beta_2, & 0, & \tfrac{1}{2}(\beta_1\gamma_2+\beta_2\gamma_1), & -b \\ \tfrac{1}{2}d\gamma_2, & \tfrac{1}{2}(\beta_1\gamma_2+\beta_2\gamma_1), & 0, & -c \\ a, & b, & c, & 0 \end{vmatrix}^{\frac{3}{2}}\right\} \ldots\ldots(9).$$

β_1, γ_1 are determined by (3) and (5), and then β_2 and γ_2 from (1) and (6), ing K in terms of d and elements of the triangle of reference.

It is not obvious how much of a reduction (9) admits, and I have not at-npted any.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

59. Proposed by MOSES COBB STEVENS, M. A., Department of Mathematics, Purdue University, Lafayette, Ind.

Solve $n\dfrac{d^2y}{dx^2}(x^2+y^2)^{\frac{1}{2}}=\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$.

[From *Forsyth's Differential Equations.*]

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.

Let $y/x=z$, $d^2y/dx^2=y/x$; then the given equation becomes

$$nv\sqrt{1+z^2}=\{1+p^2\}^{\frac{3}{2}},$$

$$\text{or } v=\frac{\{1+p^2\}^{\frac{3}{2}}}{n\sqrt{1+z^2}},=(p-z)\frac{dp}{dz}\ \ldots\ldots\ldots\ldots\ldots\ldots(A).$$

Assuming $p=(z-t)/(1+zt)$, (A) reduces to

$$\frac{t(dt/dz)}{[1+t^2][t+(1/n)\sqrt{1+t^2}]}-\frac{1}{1+z^2}=0\ \ldots\ldots\ldots\ldots\ldots(B),$$

in which the variables are separated.

II. Solution by G. B. M. ZERR, M. A., Ph D., President and Professor of Mathematics, Russell College, Lebanon, Va.

Let $x=r\cos\theta$, $y=r\sin\theta$, then the equation becomes,

$$nr^3+2nr\left(\frac{dr}{d\theta}\right)^2-nr^2\frac{d^2r}{d\theta^2}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}\ \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let $\dfrac{dr}{d\theta}=p$, so that $\dfrac{d^2r}{d\theta^2}=p\dfrac{dp}{dr}$:

$\therefore$ (1) becomes, $nr^3+2nrp^2-nr^2p\dfrac{dp}{dr}=(r^2+p^2)^{\frac{3}{2}}$.

Let $p=r/v$, so that $\dfrac{dp}{dr}=\dfrac{1}{v}-\dfrac{r}{v^2}\dfrac{dv}{dr}$;

$\therefore$ (2) becomes, $nv(1+v^2)dr+nrdv=(1+v^2)^{\frac{3}{2}}\,dr$.

$$\therefore\ dr/r = \frac{n\,dv}{(1+v^2)\{(1+v^2)^{\frac{1}{2}} - nv\}}.$$

$$\therefore\ \log r = \log\left\{\frac{(1+v^2)^{\frac{1}{2}}}{(1+v^2)^{\frac{1}{2}} - nv}\right\} + \log A.$$

$$\therefore\ r = \frac{A(1+v^2)^{\frac{1}{2}}}{(1+v^2)^{\frac{1}{2}} - nv},\ = \frac{A(r^2+p^2)^{\frac{1}{2}}}{(r^2+p^2)^{\frac{1}{2}} - nr}.$$

$$\therefore\ r\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}} - nr^2 = A\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}$$

$$\therefore\ d\theta = \pm\frac{(r-A)dr}{r\sqrt{n^2r^2-(r-A)^2}},\ = \pm\frac{(r-A)dr}{nr^2\sqrt{1-\{(r-A)/(nr)\}^2}}.$$

$$\therefore\ \theta = \pm\frac{1}{\sqrt{n^2-1}}\log[(n^2-1)r + A + \sqrt{(n^2-1)\{r^2n^2-(r-A)^2\}}]$$

$$\mp\sin^{-1}\left(\frac{r-A}{nr}\right) + B.$$

$$\therefore\ \tan^{-1}(y/x) \pm \sin^{-1}\left(\frac{\sqrt{x^2+y^2}-A}{n\sqrt{x^2+y^2}}\right) - B,\ = \pm\frac{1}{\sqrt{n^2-1}}\log[(n^2-1)\sqrt{x^2+y^2}$$

$$+A+\sqrt{(n^2-1)\{n^2(x^2+y^2)-(\sqrt{x^2+y^2}-A)^2\}}\,].$$

re A and B are constants of integration.

60. Proposed by SETH PRATT, C. E., Assyria, Mich.

To remove $(1/a)$th of the volume of a sphere of a given radius by a conical hole, se axis is the axis of the sphere, and whose vertex is at the surface of the sphere. Re- ed the height of the cone and the diameter of its base.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let O be the center of the sphere, ABC the cone, $AO=r$, $DE=x$, $AD=y$; ı the volume of spherical segment

$$ABE = \frac{\pi x^2}{3}(3r-x),$$

that of the cone

$$ABC = \frac{\pi y^2(2r-x)}{3};$$

ondition, therefore,

$\frac{\pi x^2}{3}(3r-x)+\frac{\pi y^2}{3}(2r-x)=(1/a).\frac{4}{3}\pi r^3$. Substituting $y^2=2rx-x^2$, we obtain the final equation, $x^2-4rx=-(4r^2/a)$, whence $x=2r\{1-[1-(1/a)]^{\frac{1}{2}}\}$.

$\therefore$ Height of cone $2r-x=2r[1-(1/a)]^{\frac{1}{2}}$, and $y=2r\{[1-(1/a)]^{\frac{1}{2}}-[1-(1/a)]\}^{\frac{1}{2}}$.

II. Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Mass.

Figure shows section of sphere through axis, with $AEBF$ as section of hole.

Let $h=AH=$height of cone; $b=FH=$radius of base; $dA=ACD=$elements of area AEB; $x(=DK)$ is perpendicular to AB; $\theta=\angle BAD$.

Then $dA=\frac{1}{2}\overline{AD}^2d\theta=2r^2\cos^2\theta d\theta$.

Center of gravity of ADC is at distance $2x/3$ from AB. The element of volume found by revolving ADC about AB, or $dV=2\pi\times(2x/3)\times2r^2\cos^2\theta d\theta$.

But $x=AD\sin\theta=2r\cos\theta\sin\theta$.

$\therefore dV=(16\pi/3)r^3\cos^3\theta\sin\theta d\theta$;

$$\cos\angle EAH=\frac{AH}{AE}=\frac{h}{\sqrt{2rh}}=\sqrt{\frac{h}{2r}}.$$

$$\therefore \text{Volume } AEBF=\frac{16\pi r^3}{3}\int_0^{\cos^{-1}\sqrt{(h/2r)}}\cos^3\theta\sin\theta d\theta,=(\pi r/3)(4r^2-h^2),$$

which equals $(1/a)$th of volume of sphere, or $4\pi r^3/3a$.

$\therefore h=2r\sqrt{[1-(1/a)]}$.

$$\text{Diameter}=2b=2\sqrt{h(2r-h)}=4r\sqrt{\sqrt{1-(1/a)}[1-\sqrt{1-(1/a)}]}.$$

Volume $AEBF$ can be as easily found by geometry without the use of calculus.

III. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, Russell College, Lebanon, Va.

Let $AO=r$, $DO=y$, $DB=x$. Volume of cone $ABC=\frac{1}{3}\pi(r+y)x^2$; volume of segment $BDCE=\frac{1}{3}\pi(r-y)^2(2r+y)$.

$\therefore \frac{1}{3}\pi(r+y)x^2+\frac{1}{3}\pi(r-y)^2(2r+y)=(4\pi r^3/3a)$, but $x^2=r^2-y^2$.

$\therefore (r+y)(r^2-y^2)+(r-y)^2(2r+y)=(4r^3/a)$.

$\therefore y^2+2ry=(3ar^2-4r^2)/a$.

$$\therefore y=\pm2r\sqrt{\frac{a-1}{a}}-r.$$

The plus sign alone is admissible. $\therefore\ y=2r\sqrt{\frac{a-1}{a}}-r$.

$\therefore$ Altitude $=r+y$, $=2r\sqrt{\frac{a-1}{a}}$; $2x$=diameter of base$=2\sqrt{r^2-y^2}$.

$$2x=2r\sqrt{\sqrt{\frac{a-1}{a}}-\frac{a-1}{a}},\ =2r\sqrt[4]{\frac{a-1}{a}}\sqrt{\frac{\sqrt{a}-\sqrt{a-1}}{\sqrt{a}}}.$$

MECHANICS.

...by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

...posed by J. SCHEFFER, A. M., Hagerstown, Md.

...plane quadrilateral $ABCD$ in the vertical wall of a cistern, filled with water, has ...r vertices A, B, C, D at the distances 10 feet, 4 feet, 5 feet, and 7 feet respectively, ... the surface of the water. The projections of AB, BC, CD upon the surface are ...ectively 2 feet, 3 feet, and 1 foot. Find the pressure of the water upon the quadrilat... and the position of the center of mean pressure.

...Solution by J. C. NAGLE, A. M., M. C. E., Professor of Civil Engineering in the State Agricultural and ...nical College, College Station, Texas, and the PROPOSER.

In the figure, $AE=10$, $BF=4$, $CG=5$, $DH=7$, $EF=2$, $FG=3$, $GH=1$. ...coördinates of the vertices A, B, C, and D with ...ect to EH and EA as axes are respectively, (10, 0), ...), (5, 5), and (6, 7).

Hence, the area of $ABCD$

$$=\tfrac{1}{2}\left[\begin{vmatrix}x_1 & x_2\\ y_1 & y_2\end{vmatrix}+\begin{vmatrix}x_2 & x_3\\ y_2 & y_3\end{vmatrix}+\begin{vmatrix}x_3 & x_4\\ y_3 & y_4\end{vmatrix}\right]=17\tfrac{1}{2};$$

...of triangle $ABC=10$; and area of $ACD=7\tfrac{1}{2}$.

Distance of center of gravity of $\triangle ABC$ from ...$\tfrac{1}{3}(10+4+5)=6\tfrac{1}{3}$, and distance of center of gravity of $\triangle ACD$ from EH ...$+4+7)=7\tfrac{1}{3}$. Denoting the distance of center of gravity of $ABCD$ by z, ...$17\tfrac{1}{2}z=10\times6\tfrac{1}{3}+7\tfrac{1}{2}\times7\tfrac{1}{3}$. Hence, $z=6\tfrac{16}{21}$. Hence, pressure of water up... ...$D=17\tfrac{1}{2}\times6\tfrac{16}{21}w=\frac{355}{2}w$, w denoting the weight of a cubic foot of water. ...$w=62\tfrac{1}{2}$ pounds, we find the pressure to be $11093\tfrac{3}{4}$ pounds.

...Let AD represent the surface of the water, $ABCD$ a rectangle, BCE ...t triangle, $AE=a$, $CD=b$, $AD=c$. Then, omitting w, the moment of the pressure upon $ABCD$ with respect to $AD=c\int_0^b x^2dx=\frac{1}{3}b^3c$, and the mom

the pressure upon $BCE=\frac{c}{a-b}\int_0^{a-b}(a-b-x)(x+b)^2dx=\frac{1}{12}c(a-b)(a^2+2ab-$

Adding, we find the moment of the pressure upon the trapezoid with respect to $AD=\frac{1}{12}c(a+b)(a^2+b^2)$

For $ABFE$, $a=10$, $b=4$, $c=2$; $\therefore$ Moment$=270\frac{2}{3}$.

For $FBCG$, $a=4$, $b=5$, $c=3$; $\therefore$ Moment$=92\frac{1}{4}$.

For $GCDH$, $a=5$, $b=7$, $c=1$; $\therefore$ Moment$=74$.

For $EADH$, $a=10$, $b=7$, $c=6$; $\therefore$ Moment$=1266\frac{1}{2}$.

Adding the moments of the first three and then subtracting the sum from the moment of the fourth, we get the moment of $ABCD=829\frac{7}{12}$. Therefore, distance of the center of pressure of $ABCD$ from $EH=829\frac{7}{12}\div$ [illegible] $=7$ [illegible].

Let $AECD$ represent a trapezoid with right angles at A and D, A surface of the water, OP a perpendicular, an a perpendicular to OD; $AE=a$, $CD=b$, A $OA=h$, $MN=y$, $AM=x$.

$\therefore$ Moment of pressure upon $AECD$ wi spect to OP

$$=\frac{1}{2}\int_0^c y^2(x+h)dx, \text{ where } y=\frac{c}{a-b}(a-b-x).$$

Substituting, we get for the moment of A with respect to AD the expression $\frac{1}{24}c[c(a^2+2ab+3b^2)+4h(a^2+ab+b^2)]$.

For $ABFE$, $a=10$, $b=4$, $c=2$, $h=0$; $\therefore$ Moment$=38$.

For $BCGF$, $a=4$, $b=5$, $c=3$, $h=2$; $\therefore$ Moment$=110$ [illegible].

For $CDHG$, $a=5$, $b=7$, $c=1$, $h=5$; $\therefore$ Moment$=100$ [illegible].

For $ADHE$, $a=10$, $b=7$, $c=6$, $h=0$; $\therefore$ Moment$=580$ [illegible].

Subtracting the sum of the first three from the last, we find fo moment of $ABCD$ with respect to AE, 330 [illegible].

$\therefore$ Distance of the center of pressure from $AE=330$ [illegible] $\div$ [illegible] $=2$ [illegible].

And thus the position of the center of pressure is fully determined.

51. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training Philadelphia, Pennsylvania.

"Swift of foot was Hiawatha.
He could shoot an arrow from him
And run forward with such fleetness
That the arrow fell behind him!
Strong of arm was Hiawatha;
He could shoot ten arrows upward
Shoot them with such strength and swiftness
That the tenth had left the bowstring
Ere the first to earth had fallen." *Longfellow.*

Assuming Hiawatha to have been able to shoot an arrow every second and t aimed when not shooting vertically so that the arrow might have the longest range was Hiawatha's time in a hundred yards?

I. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss.

An arrow rises $4\frac{1}{2}$ seconds when shot vertically, and therefore, the initial city which Hiawatha is able to impart to an arrow is $\frac{9}{2}g$ feet per second.

The angle of elevation for the longest range is 45°, and therefore, horizontal component of the velocity of the arrow is $\frac{1}{4}(9\sqrt{2})g$. This being watha's speed, his time for 100 yards is a very little less than 3 seconds.

In the above it has been assumed that Hiawatha ran the whole distance at iform rate. The range is much more than a hundred yards.

II. Solution by S. ELMER SLOCUM, Union College, Schenectady, N. Y.; J. P. BURDETT, Class '97, Dick- College, Carlisle, Penn.; and E. W. MORRELL, A. M., Professor of Mathematics, Montpelier Seminary, elier, Vt.

Let t=time of flight when the arrows are shot vertically upward, and u be nitial velocity. Then $t=2u/g$, and $u=\frac{1}{2}gt=144$ feet per second.

The range of a projectile is $u^2\sin2\theta/g$, and since the greatest value n2θ is 1, the maximum range is u^2/g.

$\therefore$ Range$=u^2/g=648$ feet. Time of flight for projectile is $2u\sin\theta/g$ 363 seconds.

$\therefore$ Velocity$=648\div6.363=101.8+$ feet per second.

Time for 100 yards$=(300\div101.8+)=2.94$ seconds.

AVERAGE AND PROBABILITY.

onducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

12. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, field, Missouri.

A straight line of length a is divided into three parts by two points taken at random; he chance that no part is greater than b. [From *Hall and Knight's Higher Algebra.*]

I. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

There are two cases. I, when $b>\frac{1}{3}a$ and $<\frac{1}{2}a$, and II, when $b>\frac{1}{2}a$ $<a$.

Case I. Let AB represent the line a. P be the position of the first point, and $AP=x$. Lay off PC and BD each$=b$. the favorable positions for the second t lie between C and D. $DC=x+2b-a$. The limits of x are $a-2b$ and b.

Hence the required chance is $P_1=\dfrac{1}{a^2}\displaystyle\int_{a-2b}^{b}(x+2b-a)dx=\dfrac{(3b-a)^2}{2a^2}$.

Case II. In this case the limits of x are 0 and $a-b$.

Hence, $P_2=\int_0^{a-b}(x+2b-a)dx=\frac{(3b-a)(a-b)}{2a^2}$.

Corollary. When $b=\frac{1}{2}a$, $P_1=P_2=\frac{1}{4}$.

II. Solution by J. O. MAHONEY, B. E., M. S., Graduate Fellow and Assistant in Mathematics, University, Nashville, Tenn.

Let AB be the straight line of length a, and let the random poin be at distances x, y from A, so that $AP=x$, $AQ=y$, and $PQ=a-x-y$. orable cases we must have $x<b$, $y<b$, and $a-x-y<b$; and in possib $x+y<a$.

Construct the right-angled triangle ABC where $AB=AC=a$. With A as origin and AC and AB as axes c the lines MN, LH, and RS, whose equations are $y=b$, $x=b$, and $x+y=a$ spectively. (1) When $b>\frac{1}{2}a$, the favorable cases will be restricted to the area $MNHLSR$ in Fig. 1, and the required chance is $1-3[(a-b)/a]^2$. (2) When $b<\frac{1}{2}a$, the favorable cases will be restricted to the area 123 in Fig. 2. This is a right-angled isosceles triagle a side of which is $AM-SL=b-(a-2b)=3b-a$. Therefore, the required chance is $[(3b-a)/a]^2$.

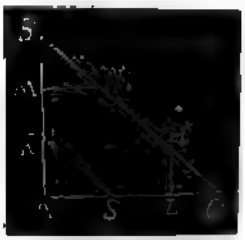


Fig. 1.

Fig. 2.

III. Solution by G. B. M. ZERR, M. A., Ph D., President and Professor of Mathematics, Russell C anso, Va.

Let $ABCD$ be a square side a, and take $AE=CF=b$. The coördi a point taken at random in $ABCD$ are the distances of two such points from one end of the line.

Without restriction the point might fall anywhere upon ABC, but the condition confines it to the triangle EBF.

$$\therefore p=\frac{EBF}{ABC}=\frac{(a-b)^2}{a^2}=\left(\frac{a-b}{a}\right)^2.$$

$$\text{Otherwise } p=\frac{\int_b^a\int_a^{y-b}dydx}{\int_0^a\int_0^y dydx}=\left(\frac{a-b}{a}\right)^2.$$

Professors Scheffer and Zerr should have received credit in the last number of the Mo solving problem 50, and Professor Henry Heaton should have received credit for solving prob

No solution of problem 58 has yet been received.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

46. Proposed by EDWARD R. ROBBINS, Master in Mathematics and Physics, Lawrenceville School, Lawrenceville, New Jersey.

Required several numbers each of which, when divided by 10 leaves a remainder 9; by 9 leaves 8; by 8 leaves 7; by 7 leaves 6; and so on. Also find the least such number which, when divided by 28 leaves 27; by 27 leaves 26; by 26 leaves 25; by 25 leaves 24, *et sic ad unum*.

I. Solution by M. W. HASKELL, A. M., Ph. D., Department of Mathematics, University of California, Berkeley, Cal.; NELSON L. RORAY, Professor of Mathematics in South Jersey Institute, Bridgeton, N. J.; A. H. BELL, Hillsboro, Ill., and H. C. WILKES, Skull Run, W. Va.

The problem can be also stated as follows: Required several numbers of which, divided by 10, 9, 8, and so on, leaves a remainder (-1).

If then L be the least common multiple of 10, 9, 8 and so on, all numbers of the form $kL-1$, where k is any integer, will have the required character.

Now the least common multiple of 10, 9, 8, ...2, 1 is 2520. The required numbers are then $(2520k-1)$ *e. g.*, 2519, 5039, 7559, 10079, etc.

The second problem is solved in exactly the same way. The least common multiple of 28, 27, 26, ...2, 1 is 80313433200. So the required number is one less, or 80313433199.

II. Solution by the PROPOSER.

One less than the product of any number of factors will be divisible by ly of the factors, or products of any or all of them, with a remainder one less an the divisor. Because $ab^2c^2d^2-1$ divided by acd gives b^2cd^2-1 for quotient and $acd-1$ for remainder. Thus, the different factors occurring in natural numbers 1, 2, 3, etc., to 10, are (1.2.2.2.3.3.5.7), one less than product of which is 2519, which leaves remainders less by unity than the divisors when divided by numbers 1, 2, 3, ...10. All multiples (diminished one) of the continued product of these factors will satisfy the same demands the problem, to-wit: 7559, 10079, 12599, etc., etc., *ad libitum*.

The factors occuring in numbers 1......28 are (1 2.2.2.2.3.3 3.5 5.7.11.17.19.23) and one less than their continued product gives 80313433199, the ımber required.

NOTE. Of course the same numbers will accomodate 5 and 6; 9 and 10; and 12; 13, 14, and 15; 17 and 18; 19, 20, 21, and 22; 23 and 24; 25 and 26; 27 and 28; and so on.

III. Solution by JOSIAH H. DRUMMOND, Portland, Maine.

I. $10a+9$ answers the first condition; multiply this by 9 and add 8, and we have $90a+89$; proceeding in the same manner we finally have $3,628,800a+3,628,799$, in which a may be zero or any number.

II. Or, in the process as above, we may leave out factors of numbers already used and we reach the result $2520a+2519$, in which a may be zero or any number; if $a=$zero, we have 2519, the smallest number that will answer the conditions of the first question.

III. It is manifest that if we take 1 from a number divisible by all the given divisors, the remainder when divided by those divisors will always leave a remainder one less than the divisor. Hence the least common multiple of the given divisors, less 1, is the number required. Hence, omitting the common factors in the second part of the question, we have $28.27.26.25.23.22.19.17....1=80,313,433,199$, the number required.

IV. **Solution by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, La.**

Let $10x_{10}+9=$the number, also, $9x_9+8$; $8x_8+7$; $7x_7+6$; $6x_6+5$; and so on to $2x_2+1=$the number.

$\therefore 9x_9+8=10x_{10}+9$. $\therefore x_9=x_{10}+\frac{1}{9}(x_{10}+1)$.

But x_9 and x_{10} are both integral.

$\therefore (x_{10}+1)/9=m$ an integer. $\therefore x_{10}=9m-1=(90m/10)-1$.

The value of x_9 from the above equation is $(90m/9)-1$.

Similarly for the other values, the expression $x_n=(90m/n)-1$, giving one of the values for each value of n from 10 to 2. But since all these values are to be integral, $90m$ must be a multiple of each of the natural numbers from 2 to 10 inclusive. This requires m to be $4\times7=28$, or some multiple of 28. If $m=28$, $x_{10}=251$.

$\therefore 10x_{10}+9=2519=$one of the numbers.

Taking $m=$the multiples of 28, we get other numbers, 5039, 7559, 10,079. Still other numbers can be obtained by taking the higher multiples of 28 for m.

A similar solution gives for second statement, the number 80,313,433,199.

V. **Solution by O. W. ANTHONY, M. Sc., Columbian University, 1702 S Street, Washington, D. C.**

The problem in question may be generalized thus: Find a number such that if it be divided by a particular number or any number less than this number the remainder will be one less than the divisor.

Let x be the required number. It is evident, if k and $k+l$ be any two numbers less than the first divisor in question, the following conditions must be satisfied:

$$x/k=u_1+(k-1)/k\text{......(1)}.\qquad k/(k+l)=u_2+(k+l-1)/(k+l)\text{.....(2)},$$

$$\text{or } x-ku_1+k-1\text{..............(3)},\quad\text{and } x=(k+l)u_2+k+l-1\text{...........(4)}.$$

Take the value of x given in (3) and substitute it for u_2 in (4). Then

$x=(k+l)[ku_1+k-1]+k+l-1\ldots\ldots(5)$, which may be reduced to the following form: $x=k[(k+l)u_1+k+l-1]+k-1\ldots\ldots(6)$.

Thus (5) and (6), which are identical, contain both the forms (3) and (4). Thus if we substitute in the manner indicated the result will contain two original forms. Some special forms required by the problem in question are:

$x=2u_1+1\ \ldots(1)$; $x=3u_2+2\ldots.(2)$; $x=4u_3+3\ldots.(3)$; $x=5u_4+4\ \ldots(4)$;

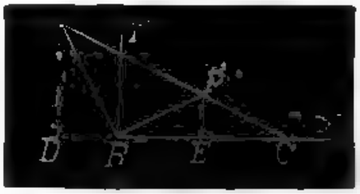
&c., etc. Substitute (1) in (2) in the manner indicated above and we have $x=6u_1+5$. This includes (1) and (2). Substitute this in (3); the result is $x=24u_1+23$. This includes (1), (2), and (3) by the previous demonstration. Continuing this we have as a result $x=\underline{|k}\,u_1+\underline{|k}-1=\underline{|k}(u_1+1)-1\ldots\ldots(A)$. This contains forms (1), (2), (3), (4), etc., and is the general form of number required. The examples cited are special applications of this general form. Thus $x=\underline{|8}(u_1+1)-1$ contains all the numbers required in the first part of the problem, and, letting $u_1=0$, and $k=25$, we have $x=\underline{|25}-1$, the number required in the last part of the problem.

46. Proposed by A. H. HOLMES, Box 963, Brunswick, Maine.

The base BC of the triangle ABC is $2c$, the sum of the two sides, AB and BC, is $2a$. BP is always perpendicular to AB and cuts AC in P. What is the locus of the point P?

I. Solution by GEORGE LILLEY, Ph. D., LL. D., 394 Hall Street, Portland Ore.

Take BC for the axis of x; let P be (x, y); draw AD at right angles to BC, produced; and PE at right angles to BC.



Area ABP+area PBC=area ABC, or

$(a-c)\sqrt{x^2+y^2}+cy=c\times AD\ \ldots\ldots\ \ldots\ldots(1)$.

Triangles ABD and BPE are similar.

Hence, $AD=[2y(a-c)]/\sqrt{x^2+y^2}\ldots\ldots(2)$.

From (1) and (2), $(a-c)(x^2+y^2)+cy\sqrt{x^2+y^2}=2cy(a-c)$.

$\therefore\ c^2y^2(x^2+y^2)=(a-c)^2(2cy-x^2-y^2)$, for the required locus.

If $\angle ABC$ be an acute angle, y must be taken negatively. Then, area ABC +area BPC=area ABP, or

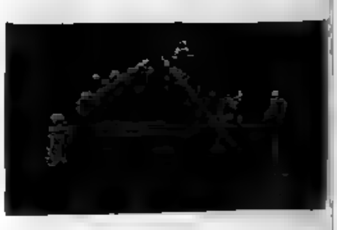
$$c\times AD+c(-y)=(a-c)\sqrt{x^2+y^2}\ \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$\text{and } AD=[2x(a-c)]/\sqrt{x^2+y^2}\ \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

From (3) and (4), $c^2y^2(x^2+y^2)=(a-c)^2(2cx-x^2-y^2)^2$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Lebanon, Va.

Let B be the origin, Bx the initial line, $BP=r$, $\angle CBP=\theta$, $BC=2c$, $AB=2(a-c)$.



Then $\cos B=\sin\theta$.

$AC=\sqrt{8c^2-8ac+4a^2-8c(a-c)\sin\theta}$.

$CP=\sqrt{4c^2-4rc\cos\theta+r^2}$.

$r^2+4(a-c)^2=AP^2=(AC+CP)^2$.

Substituting and reducing we get for the locus

$$4c^2(r\cos\theta+2a\sin\theta-2c-2c\sin\theta)^2$$

$$=(4c^2-4rc\cos\theta+r^2)(\{8c^2-8ac+4a^2-8c(a-c)\sin\theta\}.$$

Also solved by *A. H. BELL*

47. Proposed by S. HART WRIGHT, A. M., Ph. D., Penn Yan, New York.

In longitude 75 degrees west of Greenwich, latitude 43 degrees, 30 minutes north on January 1, 1895, at 3 o'clock A. M., local time. What points of the ecliptic were then rising, setting and on the meridian? Any other necessary data may be taken from an ephemeris.

Solution by the PROPOSER.

January 1, 1895, 3 A. M., in local mean time, at the station, is December 31, 1894, 15th hour astronomical time. And 15h.+5h., the longitude=20h., mean solar time=20h. 3m. 17.1295s. of sidereal time. To this add from ephemeris sidereal time of mean noon at Greenwich, 18h. 39m, 36.83s., and we have 14h. 42m. 52.9595s.$=h$, the sidereal time at station. The vernal equinox is then h hours *west* of the meridian of station, or 24h.$-h$ *east* of it, and therefore $24-(h+6)=$3b. 17m. 7.0405s.$=49°\ 16'\ 45.6''=a$, east of, and below the east point of the horizon of the station.

Let $NQSBN$ be the horizon of station, $EIAC$ a portion of the ecliptic, $QOBC$ a portion of the equator, C the place of the vernal equinox, A the rising point of the ecliptic, I the point then on the meridian, and E the setting point, P the autumnal equinox, and B the point east of the horizon.

Then $BC=a$, the angle BCA, IPO=obliquity of ecliptic$=23°\ 27'\ 19''$ per ephemeris for the date. The angle $ABC=90°+$the latitude of station$=133°\ 30'$. In the spherical triangle ABC, we have, therefore, the angles B and C given and the side $a=49°\ 16'\ 45.6''$ to find the side $AC=b$. By spherical trigonometry, $b=73°\ 45'\ 15''$. In the right spherical triangle IPO, right angled at O, we have $h-12$ hours=2h. 42m. 52.9595s.$=PO=40°\ 43'\ 14.4''$. By spherical trigonometry, $PI=43°\ 10'\ 36''$. Hence the rising point is $360°-b=286°\ 14'\ 45''$ of the ecliptic, and as great circles intersect in opposite points, E will be 180° less than A, or 106° 14′ 45″, and $180°+PI=223°$ 10′ 36″, the longitude of the point passing the meridian.

The senseless divinations of Astrology, are almost entirely based upon finding the three points of the ecliptic required in this problem, for the moment of birth, at a given place.

Also solved by *EDMUND FISH*, Hillsboro, Ill.

48. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Penn.

In case of *mischance*, with what force would the cow, weighing $w=700$ pounds, jumping over the moon, have struck Her Lunar Majesty in the face?

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, Russell College, Lebanon, Va.

Let m=mass of cow on moon, $g'=\frac{1}{6}g$=gravity on moon, r=2163 miles=radius of moon, a=238840 miles=distance from earth to moon, A=momentum $=mv$, E=kinetic energy$=\frac{1}{2}mv^2$.

$$\text{Then } v^2=2g'r\left(\frac{a-r}{a}\right),\quad m=\frac{700}{6g'}.$$

$$\therefore\ A=\frac{350}{3}\sqrt{\frac{2r}{ag'}(a-r)},=\frac{700}{3}\sqrt{\frac{3r}{ag}(a-r)},$$

$$=\frac{700}{3}\sqrt{\frac{6489\times5280\times236677}{238840\times32.2}}=239595.79 \text{ foot-pounds.}$$

$E=(350r/3a)(a-r)=1320841350.762$ foot-pounds.
The value of A is the force required.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

83. Proposed by the late REV. G. W. BATES, A. M., Pastor of M. E. Church, Dresden City, Ohio.

A has three notes; the first and second, $1000 each, and the third $467; all dated April 1, 1884. The first is due April 1, 1888, second, April 1, 1889, and the third, April 1, 1890, and each bearing interest at 6%. What must B pay for the three notes September 21, 1896 that the investment will bring him 8% compound interest?

[NOTE—The above problem was the result of an actual business transaction.]

84. Proposed by SYLVESTER ROBBINS, North Branch Depot, N. J.

Show how to find sides, integral, fractional, and irrational for twenty-four triangles, each one containing 880 square yards.

85. Proposed by E. W. MORRELL, A. M., Professor of Mathematics, Montpelier Seminary, Montpelie

In turning a one-horse chaise within a ring of a certain diameter, it was obse that the outer wheel made two turns, while the inner wheel made but one. The wh were each 4 feet high; and supposing them fixed at the distance of 5 feet on the axlet what was the circumference of the track described by the outer wheel? From *Greenl National Arithmetic.*

86. Proposed by EDGAR H. JOHNSON, Professor of Mathematics, Emroy College. Oxford, Ga.

$$\tfrac{1}{7}=.\dot{1}4285\dot{7}\ ;\ \tfrac{1}{11}=.\dot{0}\dot{9}\ ;\ \tfrac{1}{13}=.\dot{0}7692\dot{3}\ ;\ \tfrac{1}{17}=.\dot{0}58823529411764\dot{7}.$$

Observe that if the numbers forming the first half of the repetend be added resp ively to the numbers forming the second half of the repetend, the sum is in every cas What is the general law of which these are special cases?

GEOMETRY.

80. Proposed by J. C. GREGG, Superintendent of Schools, Brazil, Ind.

One circle touches another internally, and a third circle whose radius is a mean p portional between their radii passes through the point of contact. Prove that the ot intersections of the third circle with the first two are in a line parallel to the comm tangent of the first two. [From *Phillips and Fisher's Geometry.*]

81. Proposed by CHAS. C. CROSS, Laytonsville, Md.

A circle is drawn bisecting the lines joining the points of contact of the inscri circles with the sides produced. Another circle is drawn passing through the center the circles drawn tangent externally to the in-circle and internally to the sides of the angle. Prove that the centers of these two circles, the incenter and the circumcenter collinear.

82. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Atronomy, Pacific Col Santa Rosa, Cal.

If the extremities of the base of a triangle be joined by straight lines to the ex ior angles of squares constructed upon its two sides, the superior pair of lines t drawn intersect at right angles; the inferior pair intersect at a point in a line drawn f the vertical angle perpendicular to the base.

MECHANICS.

58. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, Un sity, Miss.

An endless uniform chain is hung over two small smooth pegs in the same hori tal line. Show that, when it is in a position of equilibrium, the ratio of the distance tween the vertices of the two catenaries to half the length of the chain is the tangen half the angle of inclination of the portions near the pegs. [From *Ruth's Analytical ics. Mathematical Trifos, 1855.*]

59. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

Find the radius of sphere of given specific gravity which will rest just immers a fluid whose density varies as its depth.

60. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

What must be the ratio of the two legs of a uniform and heavy right triangle pended from the center of the inscribed circle, if this triangle will rest with the sh leg in a horizontal position?

AVARAGE AND PROBABILITY.

57. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, Russell College, ibanon, Va.

A chord is drawd through two points taken at random in the surface of a circle. If second chord be drawn through two other points taken at random in the surface, find ıe chance that the quadrilateral formed by joining the extremeties of the two chords ill contain the center of the circle.

58. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

From a point on the surface of a circle two lines are drawn to the circumference. Required the average area that may be cut from the circle in this way if the lines are upposed to be drawn at equal angular intervals.

Query I. How does this differ from problem 32?

Query II. Is *sector* the proper word to use for the surface thus cut off?

Query III. It is absolutely correct to use the word *random* in average problems?

NOTES.

THE INTERNATIONAL MATHEMATICAL CONGRESS.

The meeting at Zurich, August 9th–11th, of the International Congress of lathematicians was in every way a success. More than two hundred members ›ok part. America sent seven representatives, including, however, three Camridge graduates, now transplanted to Pennsylvania, Professors Harkness, Mor›y and Charlotte Scott. The greatest mathematician in the world, Sophus Lie, ·as not expected; and the greatest French mathematician, Poincaré, though own for a speech, did not come; but the actual program was particularly rich nd interesting.

It is very noteworthy that the Congress was divided into five sections: 1) Arithmetic and Algebra; (2) Analysis, and Theory of Functions; (3) Geomtry; (4) Mechanics and Mathematical Physics; (5) History and Bibliography.

The program of the first section contained the only title in English: "On 'asigraphy, its present state and the pasigraphic movement in Italy," by Ernst ichroeder, of Karlsruhe, author of "Algebra der Logik."

The second section contained a title from Z. de Galdeano, whose heroic fforts gave Spain a Journal of Mathematics, now unfortunately dead in the deadence of that beautiful, priest-ridden land.

The program of the third section, the only one consecrated wholly to a ingle title, Geometry, contained two titles on the non-Euclidean geometry.

Burali: Les postulats pour la geométrie d'Euclide et de Lobatschewsky.

Andrade: La statique non euclidienne et diverses formes mécaniques du postulatm d'Euclide.

In Section IV. Stodola treated an important subject, "Die Beziehungen er Technik zur Mathematik."

In the fifth section Eneström gave an important discussion of bibliography, a point where the Congress can and will render aid of fundamental importance.

In the first general assembly Rudio spoke on the aim and organization of international mathematical congresses.

It was determined that the next Congress should take place at Paris in 1900, under the auspices of the Société Mathématique de France.

As aims were specified : (1) to promote personal relations between mathematicians of different lands ; (2) to give, in reports or conferences, an aperçu of the actual state of the divers branches of mathematics, and to treat questions of recognized importance ; (3) to deliberate on the problems and organization of future congresses ; (4) to treat questions of bibliography, of terminology, etc., on subjects where an *entente international* appears necessary.

Rudio mentioned the yearly issue of an address-book of all the mathematicians of the world with indication of their specialties ; also of a biographic dictionary of living mathematicians with portraits ; also of a literary journal for mathematics.

At the second general assembly Peano gave a conference : "Logica matematica"; and Felix Klein a conference on teaching higher mathematics.

Three important resolutions were introduced by Vasiliev, of Kazan; Laisant, of Paris, and G. Cantor, of Halle, constituting : (1) a commission for preparation of general reports ; (2) a standing bibliographic and terminology commission ; (3) a commission to give the congress a permanent character by archives, libraries, stations for correspondence, editing or publishing noteworthy works, etc.

Surely this Congress has proven that it came only in the fullness of time, and that the world moves !

George Bruce Halsted.

Austin, Texas.

EDITORIALS.

Dr. O. E. Lovett has been called to Princeton University as Assistant Professor of Mathematics.

Dr. George Lilley, LL. D., has been elected to the Chair of Mathematics in the State University of Oregon.

A portrait of a group of five of our contributors will appear soon. We were unable to complete the arrangements for this number.

Dr. L. E. Dickson, who spent last year at the Universities of Göttingen and Paris, has been elected Assistant Professor of Mathematics in the University of California.

Miss Mary F. Winston, Ph. D., has been elected Professor of Mathematics at the Kansas State Agriculturist College, Manhattan, Kansas.

Prof. E. D. Roe, Jr., Assistant Professor of Mathematics in Oberlin College, is taking a two years course in mathematics, in Göttingen, Germany.

Professor D. A. Lehman, the past year Professor of Mathematics in the College of the Pacific, has been called to the Chair of Mathematics in the Balwin University, Berea, Ohio.

The biography of Professor J. J. Sylvester which appeared in the June-July number of the MONTHLY has been translated in Russian and published by Professor Vasiliev, the great Russian Mathematician.

We regret to record the death of one of our valued contributors, De Volson Wood, Professor of Mechanical Engineering at the Steven Institute of Technology, Hoboken, N. J., on June 27, at the age of sixty-five years. We take pleasure in giving our readers a short account of his life in this issue.

We are pleased to state that we have in our hands Dr. Lovett's first article on Sophus Lie's Transformation Groups, which will surely appear in our next issue. It is Dr. Lovett's purpose to make the series of articles very elementary at first and thus bring this most important subject within the comprehension of the most of our readers. These articles alone will be worth many times the price of subscription to the MONTHLY.

BOOKS AND PERIODICALS.

The Non-Regular Transitive Substitution Groups whose Order is the Product of Three Unequal Prime Numbers. Reprint of a paper in *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich.* By Dr. G. A. Miller, Paris, France. 6 pages. B. F. F.

A History of the United States. By Allen C. Thomas, A. M., Professor of History in Haverford College, Penn. 8vo. cloth and leather back. 418 and lxxiv pages. Boston: D. C. Heath & Co.

This is the best school history of the United States that has yet been published. B. F. F.

The Tutorial Statics. By William Briggs and G. H. Bryan. 260 pages. Price, $1.00. London: W. B. Clive. New York: Hinds and Noble.

The plan of this work is good and the execution satisfactory. With the exception of some looseness of statement in certain paragraphs, the work is well written and should prove serviceable for class use. There are many valuable hints, explanations and alternative proofs, and a large selection of examples, throughout the text. An excellent summary of results follows each chapter. J. M. C.

Grammar School Arithmetic by Grades. Edited by Eliakim Hastings Moore, Ph. D., Head Professor of Mathematics, The University of Chicago. 8vo. cloth. 352 pages. Price, 60 cents. Chicago: American Book Co.

Some of the prominent features of this work are, the accurate definitions of terms according to modern usage, the use throughout of the inductive or laboratory method, the numerous well selected problems, and the entire absence of rules. The treatment of arithmetic as given in this book is a definite departure from the old ruts, and we believe that the timely appearance of this work will go far towards correcting many of the vicious and unwholesome methods pursued in many schools. B. F. F.

Elementary Text-Book of Physics. By Prof. Wm. A. Anthony, formerly of Cornell University, and Prof. Cyrus F. Brackett, of Princeton University. Revised by Prof. William Frances Magie, of Princeton. Eighth edition, revised. 8vo. cloth. 512 pages. Price, $3.00. New York and London: John Wiley & Sons.

This work deserves especial praise for the direct and logical manner in which it discusses the fundamental principles of Physics. The pictorial representations of apparatus are purposely omitted as are also the illustrations of the fundamental principles by detailed description of special methods of experimentation and of devices necessary for their applications in the arts, and thus space is saved for the discussion of important principles.

The work is admirably adapted to those schools and colleges having a large collection of apparatus, but for those that have but few pieces of apparatus, the absence of pictorial representations in a text book would in many cases leave the student without any ideas at all as to their construction. B. F. F.

Theory of Physics. By Joseph S. Ames, Ph. D., Associate Professor of Physics and Sub-Director of the Physical Laboratory in Johns Hopkins University. Crown 8vo. cloth. 514 pages. Price, $1.60; by mail, $1.75. New York: Harper and Brothers.

"To present successfully the subject of Physics to a class of students, three things seem to me as necessary: a text-book, a course of experimental demonstrations and lectures, accompanied by recitations, and a series of laboratory experiments, mainly quantitative, to be performed by the students themselves under the direction of instructors. I place "text-book" first, because for many reasons I believe it to be the most important of the three. None but advanced students can be trusted to take accurate and sufficient notes of lectures; and a text-book which states the theory of the subject in a clear and logical manner so that recitations can be held on it, seems to me to be absolutely essential." *Preface.*

This work which has just recently been issued discusses in a most satisfactory manner, the latest discoveries made in Physics. The doctrines of energy are stated with the utmost clearness and are made the framework for a consecutive treatment of Physics as a whole. The strong points in favor of this book are too numerous to mention in the limited space at our disposal. B. F. F.

The New Arithmetic. Part Part One for Teachers. By William W. Speer, Assistant Superintendent of Schools, Chicago. 154 pages. Boston and London: Ginn & Co. 1897.

This book is one of a series now in press. Some rather radical departures are proposed. The author thinks that the study of Arithmetic should be advanced from the science of *number* to that of the *definite relations of quantity*. The book gets the idea of *technical measurement* in early. Simple ratios are made the key to the solution of all problems.

The quotations in support of the theory of the book it seems to us are carried to excess. We doubt if the representation of *cents* by *lines*, p. 118, leads to clear ideas of relative values, and the "guessing" exercise on page 42 seems rather ludicrous. Notwithstanding minor objections the book is undoubtedly one of many excellencies, and the appearance of the other books of the series will be awaited with more than usual interest. J. M. C.

Mathematical Questions and Solutions. From the "Educational Times," with an Appendix. Edited by W. J. C. Miller, B. A. Vol. LXVI. 128 pages. Francis Hodgson, 89 Farringdon Street, E. C., London.

This valuable reprint contains solutions of 145 interesting problems. The price is 5s. 3d., postpaid. J. M. C.

Descriptive Geometry. Straight Line and Curves. By William J. Meyers, Professor of Mathematics in the State Agricultural College of Colorado, Fort Collins, Colo. Pages, 66 and several pages of excellent Plates. Printed by the Author.

The author has aimed to strike a mean between an abstract and difficult treatment and a diffuse and easy one. The method is based on the authors experience in his class room. The book is well supplied with suitable exercises, and deserves careful examination on the part of teachers who have occasion to use an elementary text on this subject. J. M. C.

Introduction to Infinite Series. By William F. Osgood, Ph. D., Assistant Professor of Mathematics in Harvard University. 71 pages. Cambridge: Published by Harvard University. 1897.

This little book deals with an important topic. The presentation aims to acquaint the student with the nature and use of these series and to introduce him to the theory in such a way that at each step he sees precisely the question at issue. As aids to this end the work gives a variety of illustrations of applications of these series to computations in pure and applied mathematics, a full and careful exposition of the meaning and scope of the more difficult theorems, and the use of diagrams and graphical illustrations in the proofs. We have read these chapters with much interest and heartily commend the book to our readers as a valuable supplement to the treatment given in the usual text-books on the Differential and Integral Calculus. J. M. C.

Intermediate Algebra. Uniyersity Tutorial Series. By William Briggs, M. A., F. C. S., F. R. A. S., and G. H. Bryan, Sc. D., F. R. S. 375 pages. Price, $1.00. London: W. B. Clive. New York Depot: Hinds and Noble.

This is a work of more than ordinary merit. It is based on the treatise of Radhakrishnan, with such alterations and additions as were necessary to render it suitable to the wants of English and American students. The simple properties of Inequalities are treated at an early stage, the important properties of Zero and Infinity are adequately presented, and the theory of Quadratic expressions and Maxima and Minima are fully discussed. The chapters on Logarithms, Interest and Annuties are excellent in every detail. J. M. C.

Elementary and Constructional Geometry. By Edgar H. Nichols, A. B., of the Brown and Nichols School, Cambridge, Mass. Pages 138. New York: Longmans, Green & Co.

This book is very carefully written and is admirably adapted for the place it is designed to fill. The author uses the words *symparallel* and *antiparallel* for parallel lines that have the same and the opposite directions, respectively. A proper use of the blank pages at the end of the book for a summary of facts, definitions, and principles will add greatly to the usefulness of the book. J. M. C.

The Science of Mechanics. A Critical and Historical Exposition of Its Principles. By Dr. Ernst Mach, Professor of Physics in the University of Prague. Translated from the Second German Edition by Thomas J. McCormack. With two hundred and fifty cuts and illustrations. Half morocco, gilt top, marginal analysis, exhaustive index. Price, $2.50. Chicago: The Open Court Publishing Co.

This is one of the most readable works on Mechanics that has yet come to our notice. The rigorous and rigid mathematical reasoning is interspersed by many interesting historical facts concerning the application and development of the principles under consideration, as well as giving some pleasing accounts of the first discoveries of these principles. The work is in every way worthy the highest patronage, and no difference what text-book on Mechanics may be adopted for class use, Dr. Mach's book ought to be in use in every class to supplement the work of the regular course. The book is beautifully printed and handsomely bound. B. F. F.

Elementary Mathematical Astronomy. With Examples and Examination Papers. By C. W. C. Barlow, M. A., B. Sc., Gold Medalist in Mathematics at London M. A.; Sixth Wrangler, and First Class First Division Part II. Mathematical Tripos, Cambridge, and G. H. Bryan, M. A., Sc. D., F. R. S., Smith's Prizeman, Fellow of St. Peter's College, Cambridge; Joint Author's of "Coördinate Geometry." 16mo. cloth. 442 pages. Price, $1.50. London: W. B. Clive, University Correspondence College Press; and New York: Hinds and Noble.

Nothing but words of praise can be said of this work. A somewhat careful examination leads us to pronounce it the best in the particular field it is designed to cover. The book gives a most excellent description of the methods by which the structure of Scientific Astronomy has been built up with a very small amount of mathematical knowledge. The book should be the delight of every student of Astronomy. The arrangement is good, the diagrams clear and accurate, and the whole treatment excellent. B. F. F.

The Open Court. A Monthly Magazine devoted to the Science of Religion, the Religion of Science, and the Extention of the Religious Parliament Idea. Edited by Dr. Paul Carus; T. J. McCormack, Assistant Editor; E. C. Hegeler. and Mary Carus, Associate Editors. Price, $1.00 per year in advance. The Open Court Publishing Co., Chicago, Ill.

Among the articles in the August number are the following: The Religion of Islam, by Hyacinthe Loyson; History of the People of Israel, from the Beginning of the Destruction of Jerusalem, by Dr. C. H. Corniell, Professor of Old Testament History in the University of Königsberg; and the Evolution of Evolution, by Dr. Moncure D. Conway.

B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hudson.

The Mathematical Gazette. Edited by F. S. Macauley, St. Paul's School, West Kensington, London. Issued three times a year, viz : in February, June, and October. Price, one shilling, net.

The June number contains an article on Spherical Geometry: I. Orthogonal Projection, by Prof. Alfred Lodge, M. A.; II. Stereographic Projection, by P. J. Heawood, M. A. Also Notes, Mathematical Notes, Examination Questions and Problems, Solutions, and Reviews and Notices. In "Notes" is an extended notice of Dr. Halsted's article on the "Non-Euclidean Geometry" which appeared in the March number of the MONTHLY.

B. F. F.

The Monist. A Quarterly Magazine devoted to the Philosophy of Science. Edited by Dr. Paul Carus ; T. J. McCormack, Assistant Editor ; E. E. Hegeler, and Mary Carus, Associate Editors. Price, $2.00 per year in advance. Single number, 50 cents. The Open Court Publishing Co., Chicago, Ill.

The following articles appeared in the January, 1897, number: The Logic of Relatives, by Chas. S. Peirce; Man as a Member of Society, Introduction, by Dr. P. Topinard; The Philosophy of Budhism, by Dr. Paul Carus; Panlogism, by E. Douglas Fawcett; The International Scientific Catalogue, and the Decimal System of Classification, by Thomas J. McCormack; and Literary Correspondence—France, by Lucien Arréat. B. F. F.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single Number, 25 cents. The American Monthly Review of Reviews Co., 13 Astor Place, New York City.

We are pleased to note that since our last issue this valuable magazine has changed its name to *The American Monthly Review of Reviews*, a more significant title than its former one.

The September number has a good deal to say about the Andrews incident and Brown University—not so much, as the editor remarks, on account of the personal interests involved in the case, as because of the far-reaching principles affecting academic life and liberty which have become matters at issue. A fair-minded and judicious estimate of President Andrews' services to Brown is given by a writer fully conversant with the facts, and the protest of the faculty is printed in full. The editorial comments on the awkwardness and needlessness of the situation are piquant and to the point.

Among the contributed articles in the September number are sketches of the three members of the new Nicaragua Canal Commission—Admiral Walker, Capt. O. M. Carter, Corps of Engineers, U. S. A., and Prof. Lewis M. Haupt. These sketches are illustrated with portraits, and serve to convey an idea of the peculiar qualifications possessed by these gentlemen for the task to which they have been appointed by President McKinley.

B. F. F.

The Arena. An Illustrated Monthly Magazine. Edited by John Clark Ridpath, LL. D. Price, $2.50 per year in advance. Single number, 25 cents. Boston : The Arena Co.

Every true American citizen should read Dr. John Clark Ridpath's splendid paper, "The Cry of the Poor," and his "Open Letter" to President E. B. Andrews, which appear in the September number of the *Arena*. In them the Doctor has drawn a picture that appeals to every man and woman in our land who has God-given rights and privileges which, owing to the intervention of plutocratic influences, they are not allowed to enjoy.

"Why," asks the Doctor, "should the voice of the poor ever be heard rising like a wail from plantation, hamlet, and cityful ? Why should there be seen standing at th door of the homes of the American people the gaunt spectre—Want ?" "And why," he again asks, "should we allow the voice of our teachers to be smothered by plutocratic powers ?" There may be those who sanction the conduct of Brown University in expelling Professor Andrews, but it is very evident that the editor of the *Arena* and the author of "The Bond and the Dollar" and "The True Inwardness of Wall Street" does not.

Among the other papers are "The Concentration of Wealth, its Cause and Results: Part I," by Herman E. Taubeneck; "The Multiple Standard for Money," by Eltweed Pomeroy; "The Future of the Democratic Party: A Reply," by David Overmyer; "The Author of 'The Messiah'," by B. O. Fowler; "Anticipating the Unearned Increment," by I. W. Hart; "Studies in Ultimate Society:" I. "A New Interpretation of Life," by Laurence Gronlund; II. "Individualism vs Altruism," by K. T. Takahashi; "General Weyler's Campaign," by Crittenden Marriott; the "Plaza of the Poets," "Book Beviews," and "The Editor's Evening," make up this bright and instructive number.

CORRECTIONS AND REVISIONS OF THE ARTICLE

"ON THE CIRCULAR POINTS AT INFINITY,"

MAY MONTHLY, PP. 132—145.

(P.=page ; l. κ=κth line from above ; lb. κ=κth line from below.)

P. 132, l. 1 of the article, read Coördinate for Coordinate ; l. 2, Cartesian for Cartesion. P. 133, (4) and (4)′ for (A) and (A)′; l. 19—21, finish parenthesis ; l. 23, = for —. P. 134, l. 2, vanishes for vanisnes ; interchange lines 14 and 15. P. 135, l. 4, bring "all true" down to l. 6 ; l. 12, add "and" after "infinity"; l. 19 and 23, coördinates for coöordinates ; l. 25, coördinates for coödinates. P. 136, l. 4, add exponent 2 to numerator ; l. 6, ρ^2 here taken equal to 1, might have been retained in the numerator. If retained, (21) p. 140 would contain ρ^4 instead of ρ^2, but this would have no effect on the final result (22). Whether ρ^2 is retained or not, (14) would have to be made homogeneous in all the coördinates involved, as well as (21), for practical uses, since this is required of all such equations. (14) can be made homogeneous by the use of the solution of (4). l. 9, $+\sin\alpha_1\sin\alpha_2$ for $-\sin\alpha_1\sin\alpha_2$; $-\kappa_1\kappa_2\cos C$ for $\kappa_1\kappa_2\cos C$; lb. 6, c for C. P. 137, l. 16, r for γ. P. 138, lb. 4, $x_2'^2$ for $x_1'^2$; lb. 5, r for γ ; lb. 8, $\cos C$ for $\cos B$; lb. 9, $x_1'^2$ for x_1^2. P. 139, l. 5, κ_3^2 for x_2^2 and for κ_2 ; l. 12, $x_3'x_1$ for x_2x_1. P. 140, lb. 2, $(x'\xi\xi')^2$ for $(x'\xi\xi)^2$. P. 141, l. 14, x' for x ; l. 17, $\kappa_2^2u_2^2+\kappa_3^2u_3^2$ for $\kappa_2u_2^2+\kappa_3u_3^2$; lb. 1, $x_2'^2$ for x_2' ; in foot note, "Nicht-Euklidische Geometrie" for "Nicht-Euclidsche Geometry." P. 142, l. 9, $-iA$ for $-iB$; l. 11, *two lines* for *the lines* ; l. 13 and 14, x and y might be interchanged, though this is not necessary ; the other angle between the two lines would be given ; l. 15, The double ratio of these is : Taking them in the order named, using etc.; l. 18, s for 5 ; l. 19, $+s\lambda'$ for $+s\lambda$. P. 144, l. 11, $\tan\phi$ for ϕ ; slopes for tangents would be better ; l. 12, it is necessary and sufficient that the purely imaginary part of x should become indefinitely great ; l. 18, the German word "quadrupel" is here appropriated ; l. 23, ± 1 for $\pm l$; l. 27, is for in ; in foot note, * for †. P. 145, l. 4, $\Sigma xx.\Sigma x'x'$ for $\Sigma xx'.\Sigma x'x$ in numerator and denominator ; $(\Sigma xx')^2$ for $\Sigma xx'$ under radical in denominator ; l. 5, two points [illegible] points ; l. 7, Σxx for $\Sigma xx'$.

G. B. M. ZERR WILLIAM E. HEAL HON. JOSIAH H. DRUMMOND
JOHN M. ARNOLD O. W. ANTHONY

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SOPHUS LIE'S TRANFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

By EDGAR ODELL LOVETT, Princeton University.

I.

1. Without entering unnecessarily into definitions which will occur more properly later, the following paragraph may serve for purposes of orientation. Among the most important notions of modern pure mathematics are the idea of a *group* and its associated notions *transformation*, *substitution*, *invariant* and *differential invariant*. Groups fall naturally and historically into two classes, *discontinuous* and *continuous*. The former are usually called *substitution groups* and are not infrequently referred to as GALOIS' groups ; the latter are known as *continuous transformation groups* and may with propriety be called LIE groups. Substitution groups find their greatest usefulness in the theory of algebraic equations. with a limited range of application to geometry ; transformation groups play a similar rôle in the theory of differential equations, with a wide application to geometry and mechanics. The idea of a substitution group in its modern signification and in its relation to the theory of algebraic equations is due to GALOIS ; LIE, after having modified and extended the idea of a substitution group, introduced the new notion into the domain of analysis and geometry and thus created his theory of transformation groups.

The great fruitfulness and remarkable simplicity of LIE's theories are their most striking characteristics. Because of their manifold applications, a thorough and systematic study of the fundamental properties of continuous groups is co-tain to yield the reader a liberal education in mathematics ; in addition to a knowledge of the technicalities of the theory of groups, there is gained at the same time a properly proportioned perspective of the many fields of the science brought into one domain. The group idea is a unifying principle which tends to reduce the various and in many cases apparently heterogeneous subjects of mathematics into a homogeneous body of doctrine.

It is the purpose of these notes to present some of the more elementary theorems of LIE's theories and to call attention to a few of the many applications to geometry and differential equations. The material has been drawn from the numerous published treatises* and memoirs of LIE and from his lectures delivered at the University of Leipsic in 1895 and '96. In order to an intelligent perusal of the sequel no more is required than a familiarity with the facts and processes of elementary mathematics including the simpler operations of the differential and integral calculus.

2. The simplest LIE groups are those of one parameter ; however, before proceeding to the fundamental theorems of the theory of groups of one parameter a few examples already familiar to the reader of analytical geometry as transformations of coördinates will be useful in introducing the notions.

For the sake of simplicity let the study be made in the plane. Consider the plane as a manifoldness of points, *i. e.* as a space whose space element is a point. Since it takes two independent coördinates to fix the position of a point in the plane we may say that there are ∞^2 points† in the plane or, what amounts to the same thing, that the plane is a two-dimensional space if the point is its space element. Consider the ensemble of all points of the plane and suppose that this aggregate be moved a given distance in a given direction. By this *translation* every point in the plane will be carried into the position of one of the others. In order to represent this analytically, let us suppose that the x-axis of a Cartesian coördinate system lies in the direction of the translation and that the distance through which all the points of the plane are moved is a, then the point (x, y) is carried over into the point

$$x_1 = x + a, \ y_1 = y.$$

The segment a can be given all values from $-\infty$ to $+\infty$, and if a be varied in this manner we obtain ∞^1 translations in one and the same direction or in its opposite direction.

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*A list of these treatises is to be found in the June (1897) number of the Bulletin of the American Mathematical Society or in Teubner's catalogue. The reader who desires to prosecute the study of the subject further than the scope of these notes will find the following order of attack on Lie's published works the most satisfactory: 1° Lectures on Differential Equations with Known Infinitesimal Transformations; 2° Lectures on Continuous Groups; 3° Geometry of Contact Transformations; 4° Theory of Transformation Groups, the three volumes of this treatise in their order.

†This notation is very convenient. Its general form is—If a configuration depends on n independent parameters, of which none is superfluous, the configuration assumes ∞^n positions if the parameters are allowed to vary from $-\infty$ to $+\infty$. So, for example, there are ∞^1 points on a line, ∞^2 in the plane, ∞^3 in space, since the position of the point depends on one, two, or three parameters, respectively. Similarly there are ∞^3 circles in the plane, ∞^4 straight lines in space, ∞^5 conics in the plane, and so on. The symbol ∞^n in this connection is read "n-ply infinite number of."

Suppose now that two of these translations be carried out in succession, the first through the distance a changes the point (x, y) into the point

$$x_1 = x + a, \ y_1 = y,$$

and the second through the distance a_1 carries the new point (x_1, y_1) into the position

$$x_2 = x_1 + a_1, \ y_2 = y_1,$$

which together with (x, y) and (x_1, y_1) lies on a parallel to the x-axis. Now it is clear geometrically, that the passage from the initial position (x, y) to the final position (x_2, y_2) can be effected by a single translation through the distance $a + a_1$, and in fact simultaneously for all points of the plane. This also appears analytically from the fact that the elimination of the intermediate position (x_1, y_1) from the above equation gives

$$x_2 = x + a + a_1, \ y_2 = y.$$

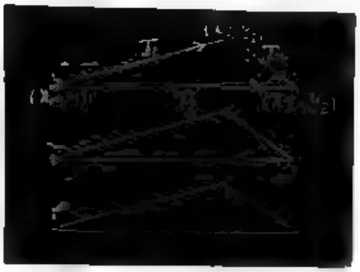
This very simple result may be formulated in the following manner:

The successive application of two translations of the family of ∞^1 *translations*

$$x_1 = x + a, \ y_1 = y$$

is equivalent to a translation belonging to the same family.

For this reason the family is called a *group of translations*. It contains one arbitrary parameter a and hence ∞^1 translations; accordingly it is said to be a *one-parameter group*.



3. So far the translations have been limited in direction; let us now consider all translations in the plane. As in the preceding case let all the points of the plane be moved through the same distance a and in the same direction α; if a and α be given all possible values we obtain a family of ∞^2 translations which includes the preceding family as a particular case. Any one of the translations of the family changes the point (x, y) into the point

$$x_1 = x + a, \ y_1 = y + b,$$

where a and b are two arbitrary values but remain the same for all points of the plane. If a translation, T_1, carry the point (x, y) into the position of the point (x_1, y_1), and a second translation, T_2, carry the point (x_1, y_1) over to (x_2, y_2), it is clear geometrically that the point (x, y) could have been carried directly to the position (x_2, y_2) by a single translation, T_3. The length and direction of this third translation T_3, equivalent to the successive application of T_1 and T_2, is found by constructing the third side of the triangle formed by the translations T_1 and T_2, or in kinematical parlance, by taking the geometric sum of T_1 and T_2. This result appears analytically by eliminating x_1, y_1 from the equations representing the translations

$$T_1, \quad x_1=x+a, \ y_1=y+b\,;$$

$$T_2, \quad x_2=x_1+a_1, \ y_2=y_1+b_1\,;$$

this elimination yields the equation

$$T_3, \quad x_2=x+a+a_1, \ y_2=y+b+b_1,$$

which is of the same form as the equations representing T_1 and T_2 and hence belongs to the same family as T_1 and T_2; therefore we conclude that

The successive application of any two translations of the family of all translations of the plane

$$x_1=x+a, \ y_1=y+b$$

is equivalent to a single translation belonging to the same family.

Because of the possession of this remarkable property* the family of all translations of the plane is called a *group of translations*. The *group* contains two arbitrary constants a and b, *i. e.* it has ∞^2 different translations; for this reason the group of all translations in the plane is called a *two-parameter* group.

4. In order to present simple concrete examples illustrative of several other fundamental notions let us return to the family of all translations parallel to the x-axis

$$x_1=x+a, \ y_1=y\,; \tag{1}$$

among these ∞^1 translations there is one to be noted, namely that one for which

*It is easy to see that this property of the equivalence of the successive applications of any two transformations of a family of transformations to a third transformation belonging to the same family is a remarkable one, peculiar to certain families, and not common to all. For example, the equations,

$$x_1=a-x, \quad y_1=y,$$

represent a family of ∞^1 transformations, which may be readily constructed geometrically, but a transformation S_1 changing (x, y) into

$$x_1=a-x, \quad y_1=y,$$

followed by S_2, changing (x_1, y_1) into

$$x_2=a_1-x_1, \quad y_2=y,$$

produces, by the elimination of (x_1, y_1) from these equations, the equations

$$x_2=(a_1-a)+x, \quad y_2=y,$$

which represents the transformation S_3 equivalent to the successive application of S_1 and S_2. But the x of the original family is equal to a constant minus the old x, while in S_3 the new x is equal to a constant plus the old x, hence S_3 does not belong to the same family as S_1 and S_2. The ∞^1 transformations represented by the above equation then do not form a LIE group.

$a=0$, *i. e.* a translation through the distance zero. By this translation all points of the plane remain at rest, and strictly speaking the term translation is no longer allowable. If, for the sake of continuity, the term translation is to be retained as applicable to this case also, then the translation by which every point is changed into itself is called the *identical translation*. It is to be further remarked that for every translation of this group there is a translation of the group which, carried out after the former, cancels its effect. Thus the successive application of the translations corresponding to $+a$ and to $-a$ respectively is equivalent to the translation $a-a=0$, that is, to the identical transformation. For this reason the two translations are said to be inverse.

If we put a equal to an infinitely small constant ∂t, we obtain an *infinitesimal translation*, which gives to all points of the plane only an infinitely small motion

$$x_1=x+\partial t, \quad y_1=y.$$

By this translation the coördinates x, y receive infinitely small increments

$$\partial x=\partial t, \quad \partial y=0,$$

and if the *infinitesimal translation* be carried out n times successively, the point (x, y) is changed into

$$x_1=x+n\partial t, \quad y_1=y\ ;$$

if the infinitesimal translation be repeated an infinite number of times, or, what comes to the same thing, if n becomes infinite, then $n\partial t$ is equal to some finite quantity a and we have again a finite translation

$$x_1=x+a, \quad y_1=y.$$

We shall find later on that a one-parameter group contains but one infinitesimal transformation.

Suppose that we operate on a definite point (x_0, y_0) with all translations of the one-parameter group (1) ; the point will take ∞^1 different positions :

$$x=x_0+a, \quad y=y_0,$$

the aggregate of which is a parallel to the x-axis. This line, the locus of all the points into which a definite point is changed by operating on it with all the translations of the group, is called the *path curve of the point*, or *path curve of the one-parameter group*. There are altogether ∞^1 path curves of the group (1) consisting of the family of straight lines parallel to the x-axis.

Any translation of the group carries any one of the path curves, as a whole, forward in its own direction a distance a ; *i. e.* the path curve as a whole remains at rest. The path curves are invariant by all the translations of the group. In addition to the line at infinity and the path curves, there is no other *invariant curve* by this one-parameter group, *i. e.* no other curve all of whose points are changed into points of the same curve by all the transformations of the group.

If a function of (x, y), $F(x, y)$ is to be invariant by the group

$$x_1 = x + a, \quad y_1 = y,$$

we must have $F(x_1, y_1) \equiv F(x+a, y) = F(x, y)$, for all values of a. In order to determine the function F we need only to take an infinitesimal value for a, and carry out the infinitesimal translation $x_1 = x + \partial t$, $y_1 = y$.* Taylor's series gives

$$F(x, y) + \frac{a}{1}\frac{\partial F(x, y)}{\partial x} + \frac{a^2}{1.2}\frac{\partial^2 F(x, y)}{\partial x^2} + \ldots\ldots = F(x, y),$$

or cancelling $F(x, y)$ from each side and neglecting terms of the second order

$$\frac{\partial F(x, y)}{\partial x} = 0,$$

that is, F does not contain x and is a function of y alone. Hence every function $F(y)$ is an *invariant function* by the one-parameter group (1). An *invariant function* equated to a constant gives an *invariant equation*, which represents one or more *path curves* of the group.

The reader may find it interesting to verify the group property for the following families and to determine the path curves and forms of invariant functions:

1° Rotations about a fixed point $\begin{cases} x_1 = x\cos\alpha - y\sin\alpha, \\ y_1 = x\sin\alpha + y\cos\alpha\,; \end{cases}$

2° The affine transformations $x_1 = ma$, $y_1 = y$;

3° The perspective transformations $x_1 = ax$, $y_1 = ay$;

4° The transformations $x_1 = ax$, $y_1 = y/a$;

5° The group of all Euclidean motions in ordinary space

$$x_1 = a_1 x + a_2 y + a_3 z + a_0,$$

$$y_1 = b_1 x + b_2 y + b_3 z + b_0,$$

$$z_1 = c_1 x + c_2 y + c_3 z + c_0.$$

The University of Chicago, 10 September, 1897.

[To be Continued.]

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*In order that a function, equation or curve be invariant by all of the finite transformations of a one-parameter group, it is necessary and sufficient that the function, equation or curve be invariant by the infinitesimal transformation of the group. This theorem will be proved in the sequel.

ON A SOLUTION OF THE GENERAL BIQUADRATIC EQUATION.

By A. C. BURNHAM, Professor of Mathematics, University of Illinois, Urbana, Illinois.

Very often in mathematical work does one wish to write out without waste of time the value of the unknown in a given biquadratic equation. Nowhere in text-books or mathematical writings do I find the solution to a biquadratic given in such form that one by merely substituting in a formula may get the roots. I have found the formula here given convenient and I do not know that the formula or this particular method of getting the result has ever before been published.

Let the general biquadratic be

$$x^4+a_1x^3+a_2x^2+a_3x+a_4=0,$$

and let the roots be a, b, c, d. Then follow, as is well known,

$$a+b+c+d=-a_1,$$
$$ab+ac+ad+bc+bd+cd=a_2,$$
$$abc+abd+acd+bcd=-a_3,$$
$$abcd=a_4.$$

Now let

$$\left.\begin{aligned} z_1&=ab+cd \\ z_2&=ac+bd \\ z_3&=ad+bc \end{aligned}\right\} \text{.............................I.}$$

Then it follows that,

$$z_1+z_2+z_3=a_2,$$
$$z_1z_2+z_1z_3+z_2z_3=(ab+cd)(ac+bd)+(\quad)(\quad)+(\quad)(\quad)$$
$$=a^2bc+ab^2d+c^2ad+cbd^2+\ldots\ldots+\ldots\ldots$$
$$=\Sigma a^2bc=a_1a_3-4a_4,$$

and
$$z_1z_2z_3=(ab+cd)(ac+bd)(ad+bc)$$
$$=\Sigma a^3bcd+\Sigma a^2b^2c^2$$
$$=a_3^2+a_1^2a_4-4a_2a_4.$$

Then z_1, z_2, z_3 are therefore the roots of the reducing cubic:

$$z^3-a_2z^2+(a_1a_3-4a_4)z-(a_3^2+a_1^2a_4-4a_2a_4)=0 \text{ .... II.}$$

Now from I we have

$$z_1^2 = a^2b^2 + c^2d^2 + 2abcd$$
$$= a^2b^2 + c^2d^2 + 2a_4,$$
$$\therefore z_1^2 - 4a_4 = a^2b^2 + c^2d^2 - 2abcd = (ab - cd)^2,$$
$$\therefore \sqrt{z_1^2 - 4a_4} = ab - cd \quad \ldots\ldots (a),$$
$$\text{but } z_1 = ab + cd \quad \ldots\ldots (b).$$

Therefore by adding (a) and (b),

$$ab = \tfrac{1}{2}\{z_1 + \sqrt{z_1^2 - 4a_4}\} \quad \ldots\ldots (c),$$

and by subtracting (a) from (b) we have

$$cd = \tfrac{1}{2}(z_1 - \sqrt{z_1^2 - 4a_4}) \quad \ldots\ldots (d).$$

In the same manner we get

$$ac = \tfrac{1}{2}(z_2 + \sqrt{z_2^2 - 4a_4}) \quad \ldots\ldots (e),$$
$$bd = \tfrac{1}{2}(z_2 - \sqrt{z_2^2 - 4a_4}) \quad \ldots\ldots (f),$$
$$ad = \tfrac{1}{2}(z_3 + \sqrt{z_3^2 - 4a_4}) \quad \ldots\ldots (g),$$
$$bc = \tfrac{1}{2}(z_3 - \sqrt{z_3^2 - 4a_4}) \quad \ldots\ldots (h).$$

But $ab + ac + ad = a(b + c + d)$

$$= (-a_1 - a)a, \text{ since } b + c + d = -a_1 - a$$
$$= -a^2 - a_1 a.$$

Also $ab + ac + ad = \tfrac{1}{2}\{z_1 + z_2 + z_3 + \sqrt{z_1^2 - 4a_4} + \sqrt{z_2^2 - 4a_4} + \sqrt{z_3^2 - 4a_4}$ from (c), (e), and (g). Therefore,

$$a^2 + a_1 a + \tfrac{1}{2}\{a_2 + \sqrt{z_1^2 - 4a_4} + \sqrt{z_2^2 - 4a_4} + \sqrt{z_3^2 - 4a_4}\} = 0,$$

which is a biquadratic equation giving the value of one root a, *i. e.*

$$a = \frac{-a_1 \pm \sqrt{a_2 - 2\{a_2 + \sqrt{z_1^2 - 4a_4} + \sqrt{z_2^2 - 4a_4} + \sqrt{z_3^2 - 4a_4}\}}}{2} \quad \ldots\ldots$$

The four roots are, therefore,

$$\left.\begin{matrix} a \\ b \\ c \\ d \end{matrix}\right\} = \tfrac{1}{2}\left\{-a_1 \pm \sqrt{a_1^2 - 2\{a_2 \pm \sqrt{z_1^2 - 4a_4} \pm \sqrt{z_2^2 - 4a_4} \pm \sqrt{z_3^2 - 4a_4}}\right. \quad \ldots\ldots \text{III},$$

where the sequence of signs under the main radical, as can be seen from formulæ (c) to (h), is

for a,	+	+	+
for b,	+	−	−
for c,	−	+	−
for d,	−	−	+

For the z_1, z_2, z_3 in this solution III must be substituted the roots of the
ic II.

EXAMPLE. As an example take the biquadratic

$$x^4-x^3-7x^2+x+6=0.$$

Here we have,

$$a_1=-1,\quad a_3=1,$$
$$a_2=-7,\quad a_4=6,$$

ı which the cubic becomes $z^3+7z^2-25z-175=0$, of which the roots are 5,
and −5. Thus the roots of the biquadratic are

$$\tfrac{1}{4}\{1\pm\sqrt{1-2\{-7\pm1\pm5\pm1\}},$$

, −1, −2, 3, which are seen to be correct.

Care must be exercised that the proper sign before the main radical is taken.

Urbana, Ill., October 9, 1897.

EQUATION OF PAYMENTS.

J. A. CALDERHEAD, A. B., Professor of Mathematics, Curry University, Pittsburg, Pennsylvania.

Let it be required to find the equated time of two payments, P and P_1, at the end of t and t_1 years respectively, and r being the rate of interest.

Represent the equated time by x when $t>t_1$.

I. BY SIMPLE INTEREST.

1st Method. The discount on P for $(t-x)$ years must equal the interest P_1 for $(x-t_1)$ years.

$\dfrac{P(t-x)r}{1+(t-x)r}$=discount on P due $(t-x)$ years hence.

$P_1(x-t_1)r$=interest on P_1 for $(x-t_1)$ years.

$$\therefore\ \frac{P(t-x)r}{1+(t-x)r}=P_1(x-t_1)r.$$

$$\therefore\ x=\frac{1}{P_1 r}(P+P_1+P_1rt+P_1rt_1$$

$$\pm\sqrt{P^2+P_1^2+P_1^2r^2t^2+P_1^2r^2t_1^2+2PP_1+PP_1rt+2PP_1rt_1+P_1^2rt)}\ \ldots\ldots$$

2nd Method.

$\frac{P}{1+rt}$=present worth of P due t years hence.

$\frac{P_1}{1+rt_1}$=present worth of P_1 due t_1 years hence.

$\frac{P+P_1}{1+rx}$=present worth of $P+P_1$ due x years hence.

$\therefore$ Suppose $\frac{P}{1+rt}+\frac{P_1}{1+rt_1}=\frac{P+P_1}{1+rx}$;

then $x=\frac{Pt+P_1t_1+Prtt_1+P_1rtt_1}{P+P_1+Prt_1+P_1rt}$ ....

Since (2) differs from (1), the sum of the present worths of P and P_1 in t and t_1 years respectively, at simple interest, is not equal to the p worth of $P+P_1$ due at the equated time ; hence, the second method is not when we compute by simple interest.

II. By Compound Interest.

1st Method.

$P\left[1-\frac{1}{(1+r)^{t-x}}\right]$=discount on P for $(t-x)$ years.

$P_1[(1+r)^{x-t_1}-1]$=interest on P_1 for $(x-t_1)$ years.

$\therefore P\left[1-\frac{1}{(1+r)^{t-x}}\right]=P_1[(1+r)^{x-t_1}-1]$.

$$\therefore\ x=\frac{\log(P+P_1)[(1+r)^t(1+r)^{t_1}]-\log[P(1+r)^{t_1}+P_1(1+r)^t]}{\log(1+r)}\ \ldots$$

2nd Method.

$\frac{P}{(1+r)^t}$=present worth of P due t years hence.

$\frac{P_1}{(1+r)^{t_1}}$=present worth of P_1 due t_1 years hence.

$\frac{P+P_1}{(1+r)^x}$=present worth of $P+P_1$ due x years hence.

$\therefore$ Suppose $\frac{P}{(1+r)^t}+\frac{P_1}{(1+r)^{t_1}}=\frac{P+P_1}{(1+r)^x}$.

Then $x = \frac{\log(P+P_1)[(1+r)^t(1+r)^{t_1} - \log[P(1+r)^{t_1} + P_1(1+r)^t]}{\log(1+r)}$ ....(4).

But (4) and (3) being identical, either method may be used when com-
nd interest is considered. The first, or correct, method by simple interest
mes very complicated when more than two payments are considered ; yet
n we recall the fact that equation of payments is a subject of no practical im-
ance, making approximate methods less desirous, it matters little how com-
ated the method may be if it is correct in theory.

The following method, which is found in most arithmetics is very often much better than a good guess. A review of the solution will, at once, show erroneousness of the method.

$P(t-x)r$=interest on P for $(t-x)$ years.

$P_1(x-t_1)r$=interest on P_1 for $(x-t_1)$ years.

$P(t-x)r = P_1(x-t_1)r$.

$\therefore x = \frac{Pt+P_1t_1}{P+P_1}$.

III. By Annual Interest.

$\frac{Pr[(t-x)+\frac{1}{2}r(t-x)(t-x-1)]}{1+r[(t-x)+\frac{1}{2}r(t-x)(t-x-1)]}$=discount on P for $(t-x)$ years.

$P_1r[(x-t_1)+\frac{1}{2}r(x-t_1)(x-t_1-1)]$=interest on P_1 for $(x-t_1)$ years.

$$\frac{Pr[(t-x)+\frac{1}{2}r(t-x)(t-x-1)]}{1+r[(t-x)+\frac{1}{2}r(t-x)(t-x-1)]} = P_1r[(x-t_1)+\frac{1}{2}r(x-t_1)(x-t_1-1)] \quad .....(5).$$

From (5) x, the equated time, can be found.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

EORGE BRUCE HALSTED. A. M., (Princeton) ; Ph. D., (Johns Hopkins) ; Member of the London Mathematical Society ; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the August-September Number.]

Proposition XXVIII. *If two straights AX, BX (produced from any-sized ight AB toward the same parts, the first under an acute angle, and the other pendicularly) mutually approach each other ever more without any certain limit, at their infinite production ; I say all angles (Fig. 33.) at any points L, H, D X, from which are dropped to the straight BX perpendiculars LK, HK, DK,* *first will all be obtuse toward the parts of the point A, secondly will be ever l more distant from this point A, and finally the angles more and more distan this same point A ever more without any certain limit approach to equality right angle.*

Demonstratur. The first part follows indeed from Corollary I to sition XIII. The second part however is proved thus. For the two ang gether at *LK* toward the base *AB* are greater (from Corollary to Proposition XVI.) than the two internal and opposite angles together at *HK* toward the same base *AB*.

Fig. 33.

But the angles at each point *K* toward the base *AB* are equal to each other, as being right. Therefore the obtuse angle at *L* toward the base *AB* is greater than the obtuse angle at *H* toward the same base *AB*.

In like manner is shown that the aforesaid obtuse angle at *H* is greater than the obtuse angle at the point *D*.

And thus ever, proceeding toward the points *X*.

Finally the third part requires a longer disquisition. If therefore be done, let there be assigned (Fig. 34.) a certain angle *MNC*, than which ways greater, or anyhow not less, the excess of any aforesaid obtuse angles above a right angle. It f (from Proposition XXI.) that the sides *NM*, *NC* co hending that angle *MNC* can be so produced that th pendicular *MC* from a certain point *M* of *MN* let fall *NC* may be greater (even in the hypothesis of acute than any assigned finite length, as for instance the said base *AB*.

Fig. 34.

This standing; assume in *BX* (Fig. 35.) a c *BT* equal to *CN*, and erect from the point *T* toward *A* perpendicular *TS*, which obviously (from Scholion to Proposition XXIV.) meets *AX* in a certain point *S*. Then from the point *S* let fall to *AB* the perpendicular *SQ*.

Fig. 35.

This falls (because of Euclid I. 17.) toward the parts of the acute angle between the points *A* and *B*. Again, acute will be the angle *QST* in the quadrilateral *QSTB*, since the remaining three angles are right; else (against Proposition V. and Proposition VI.) we come upon the hypothesis either of right angle or of obtuse angle.

Hence the straight *SQ* will be greater (from Corollary I. to Proposition III.) than the straight *BT*, *N* ; and again the angle *ASQ* will be greater than the excess by which the se angle *AST* exceeds a right angle, and thus greater than the angle *MNC*. v therefore a certain *SF* cutting *AQ* in *F* and making with *SA* an angle equal 'NC. Then from the point *A* draw to *SF* produced the perpendicular *AO*. point *O* falls (from Euclid I. 17.) below the point *F*, since the angle *AFS* Euclid I. 16.) is obtuse.

Finally, however ; since *FS* is greater (by Euclid I. 19.) than *QS* and so h greater than *BT* or *CN*, assume in *FS* the piece *IS* equal to *CN*, and from point *I* erect to *FS* the perpendicular *IR* meeting *AS* in the point *R*.

But the point *R* falls between the points *A* and *S* : for if it fell on any t of *AF*, we would have in the same triangle (against Euclid I. 17.) two an- greater than two right angles, since the angle at the point *F* toward the parts e point *A* has already been shown obtuse.

After so much preparation thus I conclude. Since in the quadrilateral 'R the angles at the points *O* and *I* are right, and the angle at the point *A* Euclid I. 17.) is acute because of the right angle *AOS*, and again the angle (by Euclid I. 16.) is obtuse, since the angle *RIS* is right : the consequence ly is (by Corollary II. to Proposition III.) that the side *AO* is greater than side *IR*.

But (*OQ* joined) the side *AQ* is greater (by Euclid I. 19.) than the side because of the obtuse angle at *O*, since the angle *AOS* was made right.

Therefore the straight *AQ* will be much greater than the straight *IR*, or Euclid I. 26.) than the straight *MC*, and so much greater than the straight the part than the whole ; which is absurd.

Therefore it is not possible to assign any one angle *MNC*, than which al- s is greater, or anyhow not less, the excess of each of the aforesaid obtuse es above a right angle.

Wherefore those obtuse angles, more and more distant from this point *A*, more without any certain limit approach to equality with a right angle.

Quod erat postremo loco demonstrandum.

Corollary. But this standing, which in the last case was demonstrated, anifestly follows that those straights *AX*, *BX*, produced infinitely will final- ave, either in two distinct points, or in one same point *X* infinitely distant, mmon perpendicular.

But again, that this common perpendicular cannot be had in two distinct ts flows manifestly from this, because otherwise (by Corollary II. to Propo- n XXIII.) those straights would thence begin mutually to separate, and so meet each other at an infinite distance ; so that also (against the express sup- tion) they would not mutually approach each other without any certain limit more toward those parts.

So they must have the common perpendicular in one same point *X* infin- distant.

[To be Continued.]

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEO

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. S
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[Continued from June-July Number.]

XLVI. Fig. 27.
ABLN is equivalent to *ABMK* is equivalent to *ACIK*.
NLFH=*ABPO* is equivalent to *BEDC*.
∴ *ABFH* is equivalent to *ACIK*+*BEDC*.

XLVII. Fig. 27.
ABLN is equivalent to *ACIK*.
NLPO is equivalent to *STER* is equivalent to *MTERC*+*QFD*.
OPIH is equivalent to *REFH* is equivalent to *REFQ*+*MBT*.
∴ *ABFH* is equivalent to *ACIK*+*BEDC*.

XLVIII. Fig. 27.
AVUH is equivalent to 2*ACH* is equivalent to *ACIK*.
VBFU is equivalent to 2*CBF* is equivalent to *BEDC*.
∴ *ABFH* is equivalent to *ACIK*+*BEDC*.
Wipper.

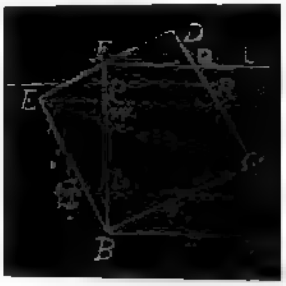


Fig. 27.

XLIX. Fig. 27.
ABWX, the half of *ABFH*, is equivalent to *ABC*+*CBW*+*CXA*.
But *ABC*=*BEF* (is equivalent to *BWE*+*AXK*).
∴ *ABWX* is equivalent to *CBE*+*CAK*.
∴ *ABFH* is equivalent to *ACIK*+*BEDC*.

L. Fig. 27.
Byz=*FDQ*. *AzyC*=*AJIK*. *ARH*=*BEF*. *HRQ*=*ACJ*.
∴ *ABFH* is equivalent to *ACIK*+*BEDC*.

LI. Fig. 27.
ABC=*BEF*. *CRa*=*FDQ*. *HRQ*=*IKG*. *HJCa* is equivalent to
∴ *ABFH* is equivalent to *ACIK*+*BEDC*.

That *HJCa* is equivalent to *IGAJ* is evident for the following re
△*ACH* is equivalent to △*ACI*, having the same base, and equal altitude
Hence, subtracting △*ACJ*, which is common to both, we have △C
equivalent to △*AJI*.
∴ *HJCa* is equivalent to *IGAJ*.

LII. Fig. 28.

ABC=*BEF*. *HRQ*=*ACJ*. *ARH*=*HKA* is equivalent to *AKIJ*+*FDQ*.

∴. *ABFH* is equivalent to *ACIK*+*BEDC*.

LIII. Fig. 28.

AMNH is equivalent to *ACLH* is equivalent *ACIK*.

So, *MBFN* is equivalent to *BEDC*.

∴. *ABFH* is equivalent to *ACIK*+*BEDC*.

Wipper.

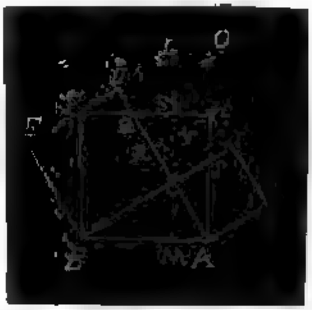


Fig. 28.

LIV. Fig. 28.

CLOJ is equivalent to *CLHA* is equivalent *ACIK*.

BFLC is equivalent to *BEDC*.

But *ABFH* is equivalent to *BFOJ*.

∴. *ABFH* is equivalent to *ACIK*+*BEDC*.

Hoffmann, 1800.

LV. Fig. 28.

ABFH+*BEF*+*FLH*+*HKA* is equivalent to *ACIK*+*BEDC*+*ABC*+*CIL* *LD*.

∴. *ABFH* is equivalent to *ACIK*+*BEDC*.

LVI. Fig. 28.

ABC=*BEF*. *ICD*=*AKH* is equivalent to *AKIJ*+*FDQ*.

SVH=*SQD*, and *VHT*=*IJT*.

∴. By properly combining and substituting, *ABFH* is equivalent to *ACIK* *BEDC*.

LVII. Fig. 28.

RDLH=*ACIK*. *ARH*=*BEF*. *ABC*=*HFL*.

∴. *ABFH* is equivalent to *ACIK*+*BEDC*.

[To be Continued.]

EUCLIDEAN GEOMETRY WITHOUT DISPUTED AXIOMS.

By G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, New Hampshire.

(*a*)

Proposition I. *If two straight lines in the same plane be perpendicular to same straight line they are parallel.*

Prove by Axiom 11, and I, 27.*

*These and the subsequent numbers refer to the Book and Proposition in Todhunter's Euclid.

(b)

PROPOSITION II. *From or through a given point in a straight line only one perpendicular to that line can be drawn in the same plane.*

PROOF. If there could be two, there would be two unequal right angles which is impossible by Axiom 11.

(c)

PROPOSITION III. *If two parallel straight lines be joined by a common perpendicular, any straight line which bisects the perpendicular and meets the two parallels is itself bisected by the perpendicular.*

Let *AB* be a straight line. Take any point in it as *C* and erect the perpendicular *CD* (I, XI). At *D* erect the perpendicular *DE* (I, 11) and extend it to *F* (Postulate 2). Then *FE* is parallel to *AB* (*a*).

Now bisect *DC* in *H*, (I, 10), take any point in *AC* as *K* and join *KH* (Postulate I). On *DE* cut off *DN* equal to *KC*, (I, 2), and join *HN*, (Postulate 1). Therefore the two triangles *KCH* and *DHN* are equal to each other (I, IV). Therefore *KH* equals *HN*. Again, since the two triangles *KCH* and *DHN* are equal, the angle *DHN* equals the angle *KHC*, being homologous angles. The angles *KHC* and *KDH* are together equal to two right angles (I, 13). Therefore since the angle *DHN* equals the angle *KHC*, the angles *DHN* and *KHD* are together equal to two right angles, and therefore *KH* and *HN* form one and the same straight line (I, 14). Therefore, since *K* is any point in *AB*, any straight line which bisects the perpendicular joining two parallel straight lines is bisected by the perpendicular.

COROLLARY. If two parallel straight lines be joined by a common perpendicular, any straight line meeting the parallels and bisecting the perpendicular cuts off equal distances on the parallels on opposite sides of the perpendicular.

(d)

PROPOSITION IV. *If a straight line is perpendicular to one of two parallel lines it is perpendicular to the other also.*

PROOF. Let *CD* be a straight line. Then from any point in it as *H* draw *HK* perpendicular to *CD*, and in the same manner draw *AB* perpendicular to *KH* (I, 11). Then *AB* and *CD* are parallel (*a*). Take any point in one of the parallels as *P* in *CD* and suppose *PQ* be drawn perpendicular to *CD*. Then will *PQ* be perpendicular to *AB* also. For cut off *HO*=*HP* (I, 2), bisect *HK* at *N* (I, 10), and draw *PS* and *OR* through *N*. Then *NO*=*NP* (I, 4). But *SN*=*NP* and *NO*=*NR* (*c*). Therefore *NS* =*NO*=*NP*=*NR* (Axiom 1). Therefore, similarly, *OH*=*HP*=*KR*=*SK* (*c*, Corollary). With *N* as a center and *NO* as a radius describe a circle (Postulate 3). The circumference of this circle will obviously pass through the points *O*, *P*, *R*, and *S*. Draw *PR*. The angle *NHO* is greater than the an-

NPH (I, 16), therefore the angle *NHP* is greater than the angle *NPH*, and efore *NP* is greater than *NH* (I, 19). Therefore the circumference of this le will intersect the two parallel lines in the points *O*, *P*, *R*, and *S*. The an- *OPR* is a right angle (III, 31), and therefore *RP* is perpendicular to *CD*. *QP* is by hypothesis perpendicular to *CD*, therefore *PQ* and *PR* cannot form separate lines (*b*). Therefore *PQ*, if properly drawn must be identical with

But the angle *SRP* is a right angle (III, 31) and therefore *PQ* is perpen- lar to *AB*. Q. E. D.

(*e*)

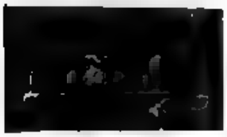
Proposition V. *If the vertex of an angle subtended by the diameter of a e is between the center and circumference, the angle is greater than a right an- and if the vertex is without the circle the angle is less than a right angle.*

Proof. Let *AKH* be a circle, *AB* a diameter at circle, and let it subtend the two angles *ACB* *D*, the vertex of the former being within, and the latter without, the circle. Extend *AC* to umference at point *H*, and join *HB* and *KA* late 1). Therefore the angles *H* and *AKB* are angles (III, 31). Therefore the angle *ACB* is ter than angle *H* and angle *D* is less than angle *B* (I, 16).

(*f*)

Proposition VI. *If two parallel straight lines be joined by two common endiculars, these two perpendiculars are equal to each other.*

Proof. Let *AB* and *CD* be two parallel straight lines and let *NH* and be perpendicular to *CD*, then are they also perpendicular to *AB* (*d*). Join and *HP* (Postulate 1). Bisect *HP* (I, 10), then with the middle point of *HP* center and one-half *HP* as a radius describe a circle (Postulate 3). The



circumference of this circle will obviously pass through the points *H* and *P*. It must also pass through *N* and *K*, otherwise the angles *HNP* and *HKP* would not be right angles (*e*). Again, bisect *NK* (I, 10) and with its middle point as a center and one-half *NK* as a rad- lescribe another circle (Postulate 3). The circumference of this circle will pass through the points *N*, *K*, *P*, and *H* for the same reason as the above. Therefore these circumferences will coincide with one another (III,

Therefore there can be but one center point which being in both the lines and *HP* must be at the point of intersection *O*. Therefore the two triangles *I* and *POK* are equal to each other (I, 4), and therefore *NH* equals *PK*.

Q. E. D.

Corollary. The intercepts on two parallel straight lines by two common endiculars are equal to each other.

For, the triangles *NOP* and *HOK* are equal to each other (I, 4). There- *NP* is equal to *HK*, being homologous sides of two equal triangles.

(*g*)

PROPOSITION VII. *If a straight line fall on two parallel straight l make the alternate angles equal to one another, etc.* (I, 29).

PROOF. Let the straight line OR fall on the two parallel straigh AB and CD, meeting them in points H and K respectively. Then the BHK and CKH shall be equal to one another.

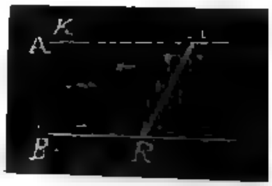
From H draw HP perpendicular to CD, and from K draw KN perpendicular to AB (I, 12). Then HP is also perpendicular to AB and KN is also perpendicular to CD (*d*). Therefore HP equals NK (*f*), and HN equals PK (*f* Corollary). Therefore the two triangles HPK and HNK are equal to each other (I, 8), and therefore the angle NHK equals the angle HKP, being homo angles of two equal triangles. Q. E. D.

PROPOSITION VIII. *The sum of the angles of every plane triangle i to two right angles.*

PROOF. Let ABC be any plane triangle, then the sum of the ang

B, and C is equal to two right angles. Through its vertices as C draw DH parallel to AB (I, 31). the angles A and DCA are equal to one another (are also the angles B and HCB for the same reaso the sum of the angles DCA, ACB, and BCH is to two right angles (I, 13). Therefore the sum of the angles, A, B, an must equal two right angles. Q. E. D.

PROPOSITION IX. *Through a given point without a given straight li one line can be drawn parallel to the given line.*



PROOF. Let BC be a straight line and H a point without. Draw AD through H parallel to BC (I, 31). Then no other line can be drawn through H parallel to BC. If possible suppose KN drawn through H parallel to BC. Then since the angles KHR and AHR are each equal to the angle HRC (*g*), they are equal to each other (Axiom 1), a part to the which is impossible. Therefore KN cannot be parallel to BC. Q. E. D.

PROPOSITION X. *If a straight line fall on two parallel straight lines, t of the two interior angles on the same side of that line shall be equal to two angles.*

PROOF. Let the straight line ON fall o two parallel straight lines AB and CD. The sum of the two angles AKH and CHK is equal t right angles. For, the sum of the two angles and KHD is equal to two right angles (I, 13) a angle AKH equals the angle KHD (*g*). Therefore, substituting the latter f former we have the sum of the two angles AKH and CHK equal to two angles. Q. E. D.

PROPOSITION XI. *If a straight line meet two straight lines so as to make two interior angles on the same side of it taken together less than two right an-s, these straight lines, being continually produced, shall at length meet on that e on which are the angles which are less than two right angles.* Euclid, Axiom 12.

PROOF. Let the straight line FH meet the two straight lines AB and CD, king the two angles BFH and FHD together less than two right angles, then nd CD shall meet, if continually produced, on that side of FH towards B . Since the angles BFH and AFH are together e two right angles, they must be greater than the f the two angles BFH and FHD. Therefore, the e AFH must be greater than the angle FHD. nce, draw the line ON through H making the angle N equal to the angle AFH (I, 23). Then ON is par-l to AB (I, 27). Therefore CD cannot be parallel to (i), and therefore CD and AB must meet if sufficiently produced. Since the n of the angles AFH and FHO equals two right angles (j), the sum of the les AFH and FHC must be greater than two right angles. Therefore AB and cannot meet on that side of FH toward A and C for then we should have iangle the sum of whose angles would be greater than two right angles which mpossible by (h). Therefore they must meet on that side of FH toward B l D. Q. E. D.

RO, INFINITESIMALS, INFINITY, AND THE FUNDAMENTAL SYMBOL OF INDETERMINATION.

By GEORGE LILLEY, Ph. D., Professor of Mathematics, State University, Washington.

The following is an outline of the method I use in explaining to the stud-in algebra how zero is used as a multiplier and a divisor, and how infinitesi-ls and infinity are used as divisors; also, interpretations of the results ained by their use.

If we multiply a by a number that decreases by 1 each time beginning h any number, as +4, and continue the multiplication until −4 is reached, e product will decrease by a. Thus,

a	a	a	a	a	a	a	a
$+4$	$+3$	$+1$	*zero*	-1	-2	-3	-4
$+4a$	$+3a$	$+a$,	zero,	$-a$	$-2a$	$-3a$	$-4a$,

where zero is onstant number and obtained by subtracting *any* number from itself, as, a=⊕, ⊕ representing *absolute* zero.

Evidently a multiplied by zero is one a less than a multiplied by +1, or

$a \times ⓞ = ⓞ$; also, a multiplied by -1 is one a less than a multiplied by zero, $a \times -1 = -a$. Similarly $a \times -2 = -2a$, $a \times -3 = -3a$, etc.

Hence, *If a constant number be multiplied by zero, the product is zero.*

Division may be defined as the process of finding how many ti the divisor can be subtracted from the dividend and leave zero.

Dividing 12 by a number that decreases by unity each time, beginn with $+3$, we have

$$\frac{12}{+3}=4,\ \frac{12}{+2}=6,\ \frac{12}{+1}=12,\ \left(\frac{12}{\text{zero}}\right),\ \frac{12}{-1}=-12,\ \frac{12}{-2}=-6,\ \frac{12}{-3}=-4;\ \text{et}$$

The quotient 4 means that only 3 times $+4$ can be subtracted from 12 leave zero; and so on for the other quotients.

Since the divisor decreases by unity, the divisor one less than 1 is ze The divisors less than zero are -1, -2, -3, etc., respectively. Then, the q tient, when zero becomes the divisor, must be between the quotients given taking $+1$ and -1 as divisors, or between $+12$ and -12.

Then $\frac{12}{\text{zero}}$ or $\frac{12}{ⓞ} = \Theta$, where Θ represents no number of times. T is, there is *no number of times zero* that the divisor, zero, can be subtracted fr 12 and *leave absolutely nothing.*

Since negative numbers are less than zero, ⓞ is not the least divisor 12, or of any other number. If $\frac{12}{ⓞ}$ = infinity, or the *largest possible number,* divided by -1 can not give -12 for a quotient. If $\frac{12}{-1} = -12$, $\frac{12}{\text{zero}}$ or - can not give infinity for a quotient, for the divisor, -1, is one less than divisor ⓞ.

Hence, in general, $\frac{a}{ⓞ} = \Theta$.

For the quotient, Θ, means that there is no number of times zero that divisor, ⓞ, can be subtracted from a and leave zero.

Hence, *If a constant number be divided by zero, the quotient is no num of times.*

It is a consequence of confounding the 0, arising from dividing a by *in ity*, with the absolute zero, that so much confusion has been created in the cussions on this subject. All absolute zeros are constants. The other 0's, u in these discussions, are infinitesimals and variables, and may be less than ⓞ

Since an infinitesimal can be subtracted from a an infinite number of ti and leave zero; therefore, $\frac{a}{◎} = \infty$, where ◎ represents an infinitesimal.

That is, *If a constant number be divided by an infinitesimal, the quotien infinity.*

Suppose, for illustration, we divide a by a number that diminishes ı each time, beginning with one ; we will have the series

$$\frac{a}{1}=a,\ \frac{a}{\frac{1}{10}}=10a,\ \frac{a}{\frac{1}{100}}=100a,\ \frac{a}{\frac{1}{1000}}=1000a,\ \frac{a}{\frac{1}{10000}}=10000a,\ \ldots\ldots$$

Evidently, by continuing the series indefinitely, the divisor becomes *less* ı any assignable number however small, and the value of the quotient 'eases without limit and becomes *greater* than any assignable number howr great.

Hence, *If a constant number be divided by a decreasing variable, as the var- ˈe becomes too small to be expressed, the quotient becomes too large to be expressed.*

Since infinity can be subtracted from a the infinitesimal part of once and re zero ; therefore, $\frac{a}{\infty}=$◎.

That is, *If a constant number be divided by infinity, the quotient is nitesimal.*

Suppose the divisor, in the above illustration, increases each time, begin- g with 1 ; we will have the series

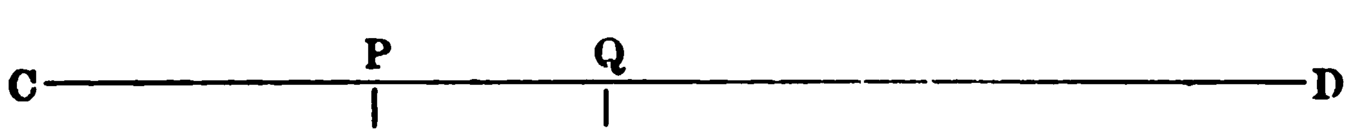
$$\frac{a}{1}=a,\ \frac{a}{10}=.1a,\ \frac{a}{100}=.01a,\ \frac{a}{1000}=.001a,\ \frac{a}{10000}=.0001a,\ \ldots\ldots$$

Evidently, by continuing the series indefinitely, the divisor becomes ater than any assignable number, however great, and the value of the quotient reases without limit and becomes *less* than any assignable number, however ıll.

Hence, *If a constant number be divided by an increasing variable, as the iable becomes too great to be expressed, the quotient becomes too small to expressed.*

This subject is also illustrated in interpreting the results obtained by as- ıing different values for the rates of travel and the distance apart, in Clair- 's problem, of the Couriers.

"Two Couriers, A and B, travel in the same direction, *CD*, at the rates m and n s an hour, respectively. If at any time, say 12 o'clock, A is at *P*, and B is at *Q*, a s from *P*, at what time and at what place are they together ?"



Let t=the number of hours traveled, after 12 o'clock, to the place where vertakes B, and d=the number of miles travelled by A in t hours ; or the ıber of miles from *P* to the place where A overtakes B.

Since the number of miles travelled by each, after 12 o'clock, equals the multiplied by the number of hours, we have

$$d=mt,\ \text{and}\ d-a=nt.$$

Solving these equations, we have

$$t=\frac{a}{m-n},\quad d=\frac{am}{m-n}.$$

We will now examine these values under different conditions.

1. If $m>n$.

This condition makes the values of t and d *positive*. That is, the C iers are together *after* 12 o'clock, and at some place to the *right* of P.

If $m<n$.

This condition makes the values of t and d *negative*. That is, the C iers are together *before* 12 o'clock, and to the *left* of P. This interpretation responds with the conditions made. For, if m is less than n, A travels n slowly than B, and it follows that they must have been together be 12 o'clock, and before they could have advanced as far as P.

3. If $m=n$, or $m-n=$zero.

Then $t=\frac{a}{\text{①}}$, and $d=\frac{am}{\text{①}}$.

As there is no number of times zèro that subtracted from a leaves z there is no number of hours when they have been or will be together. Furt more, as every number of times zero subtracted from a leaves a; tha $a-v\times\text{①}=a$, where v represents any number whatever, they are always same distance apart.

Hence, *A result in the form* $\frac{a}{\text{①}}$ *indicates that the problem is impossib*

This interpretation corresponds with the supposition made. For, if equal to n, the Couriers travel at the same rate, and since they were a n apart at 12 o'clock, it is evident they never could have been, and never will together.

4. If $a=$zero, and $m>n$, or $m<n$.

Then $t=\frac{\text{①}}{m-n}$, and $d=\frac{\text{①}}{m-n}$.

They are together at the start, as shown by $a=$zero; but, as there is number of times $m-n$ that subtracted from zero, will leave zero, they can n be together again.

Furthermore, the longer the time, the greater or less will $m-n$ be; he they will be constantly diverging. For example,

In 1 hour, zero$-(m-n)=n-m$, distance apart;
In 2 hours, zero$-2(m-n)=2n-2m$, distance apart;
In 3 hours, zero$-3(m-n)=3n-3m$, distance apart, etc.

Hence, $t=\frac{\text{①}}{m-n}$ indicates that they will be together in zero hours aft o'clock, but never after or before. For zero$-\text{①}\times(m-n)=$zero, is the value for t that will satisfy the conditions.

Similarly, $d=\frac{0}{m-n}$ means no distance from P, and shows that they were
ed together by the conditions of the problem, a=zero, but for *all other time*
roblem is impossible.

5. If a=zero, and $m=n$.

Then $t=\frac{0}{0}$, and $d=\frac{0}{0}$.

As any number of zeros subtracted from zero gives zero; that is, $-v\times 0$=zero, where v represents any number whatever, they are together
l times; for t=*any number*.

Hence, $t=\frac{0}{0}$ means all conceivable times, and $d=\frac{0}{0}$ means all con-
ble distances, and are indeterminate, not being one, but every value.

Therefore, *A result* $\frac{0}{0}$ *indicates that the problem is indeterminate.*

The form, $\frac{0}{0}$, is a *symbol of indetermination*, and does not indicate that
olution can be found, but that too many can be determined. The indeter-
ation consists in the fact that any one of the infinite solutions will answer
as well as any other.

January 11, 1897.

EDITORIALS.

This issue of the MONTHLY was delayed on account of securing sorts for
Lovett's article.

Prof. E. L. Brown, formerly of the Capital University, Columbus, Ohio,
ow a member of the Faculty of the Department of Mathematics of the Colo-
State University.

The articles on "Euclidean Geometry Without Disputed Axioms," and
o, Infinitesimals, Infinity, and the Fundamental Symbol of Indetermina-
" are published at the request of the authors. They invite criticism
eir respective articles, and, if there is any defect in the reasoning by which
arrived at their conclusions, they desire to have the same pointed out.

BOOKS AND PERIODICALS.

A Text-Book of Light. With Numerous Diagrams and Examples. By R.
ace Stewart, D. Sc., London, Author of "An Elementary Text-Book of Heat
Light," "An Elementary Text-Book of Magnetism and Electricity," etc.

Third Edition. 8vo Cloth, 208 pages. Price, 3*s*. 6*d*. London: W. B. Cle
University Correspondence College Press. New York: Hinds & Noble,
Cooper Institute.

This little treatise on Light is clearly, neatly, and accurately written. The auth
as a teacher and writer needs no introduction. His works all bear evidence of a master
the subject under consideration. This book in the hands of the student will enable h
to read with interest and profit the investigations in this most fascinating phenomenon
nature. B. F. F.

On the Transitive Substitution Groups that are Simply Isomorphic to
Symmetric or Alternating Group of Degree Six. By Dr. G. A. Miller.

This is a reprint of a paper read before the American Philosophical Society May
1897, and published in the proceedings of that Society. B. F. F.

New Principles of Geometry with Complete Theory of Parallels.
Nicolái Ivánovish Lobachévski. Translated from the Russian by Dr. Geo
Bruce Halsted. Volume fifth of the Neomonic Series.

The publication of the translation of this little pamphlet of 26 pages promised
Dr. Halsted at the Mathematical Congress of the World's Columbian Exposition was
layed for a personal visit to Kazan, the home of Lobachévski, and Maros Vásárhely,
home of Bolyai.

In this pamphlet is some interesting matter for the student of Non-Euclidean Geo
etry, and it should be read by every teacher of geometry. Several of our readers h
written to us saying that they could see no sense in the Non-Euclidean Geometry.
say to these, read this little pamphlet, then with the light you get from it turn back
read the first and all subsequent articles of Dr. Halsted's which have appeared in
MONTHLY during the last four years, then turn to other sources for information on
subject. After having done this, you will find that there is a Non-Euclidean Geomet
that its argument is as rigorous as the Euclidean, and that its deductions are equally
teresting. B. F. F.

A Text-Book of Physics. Largely Experimental, including the Harv
College "Descriptive List of Elementary Exercises in Physics" By Edwin
Hall, Ph. D., Professor of Physics in Harvard College, and Joseph Y. Berge
A. M., Instructor in the Harvard Summer School of Physics, and Junior Mast
in English High School, Boston. Revised and Enlarged. 8vo Cloth. 5
pages. New York: Henry Holt & Co.

This book needs no introduction to the public. The first edition has proved its u
fulness in all experimental courses in Physics. The second edition is even an improv
ment over the first. B. F. F.

Ordinary Differential Equations. An Elementary Text-Book with an In
troduction to Lie's Theory of the Group of One Parameter. By James Morr
Page, Ph. D., University of Leipzig, Fellow by Courtesy Johns Hopkins Un
versity, Adjunct Professor of Pure Mathematics in University of Virgini
8vo Cloth. 226 pages. New York and London: The Macmillan Co.

This is the best elementary exposition of the subject of Ordinary Differential Equ
tions with which we are acquainted. It differs from the older text-books upon the subj
in one important respect, namely, in the method of treatment. Instead of giving theor
of integration for certain classes of Differential Equations, as for instance, the Homog

ıs or Linear Differential Equations as is done in the older works on the subject, the :hor has followed the method of Professor Lie. In 1870, Lie showed that it is possible subordinate all the older methods or theories of integration to a general method. By method of Lie it is possible to derive all of the older theories from a common source l at the same time build a broader foundation for the general theory of Differential uations. The simple, elegant, and clear presentation of the subject in this work makes ›ossible for a student who has ambition and a fair knowledge of Analytical Geometry l Calculus to master this book without an instructor. B. F. F.

On the Primitive Substitution Groups of Degree Fifteen. By Dr. G. A. ller. Pamphlet, 12 pages.

This paper is an extract from the Proceedings of the London Mathematical Society, l. XXVIII., and is along Dr. Miller's favorite line of investigation. B. F. F.

Higher Arithmetic. By Wooster Woodruff Beman, Professor of Mathe-.tics in the University of Michigan, and David Eugene Smith, Professor of ıthematics in the Michigan State Normal School. 12mo. Cloth and Leather ck. 194 pages. Price, 80 cents. Chicago: Ginn & Co.

Among the many valuable features of this work are the elimination of the traional problems which have become the common property of nearly every arithmetic)lished during the last quarter century; the introduction, instead of the traditional blems, of simple problems arising in the study of elementary physics, as, for example, y problems in Electrical Measurements, problems coming under the application Boyle's Law, the law of Falling Bodies, Specific Gravity, etc.; the treatment of the tric System in the first part of the book (page 59) and the frequent use made of it in subsequent part; the introduction of the common graphic methods of representing tistics; and the complete omission of rules. The entire omission of rules is a very comn feature of the arithmetics which are published at the present time and of those ich have been published during the last three or four years. It is my belief that all es that can not be established easily by the deductive method of reasoning should be in *good print* in the arithmetics. This is especially the case with the rules in Mensuron. The student of arithmetic is in general not competent to follow the argument ich establishes the rule for finding the area of a triangle when three sides are given. t it is better for the student that he commit this rule to memory, though he does not ›w how it is established, than to be ignorant of its existence and the means by which area of triangles are computed when the sides are given. It seems to me that any thmetic treating the subject of mensuration ought to give the rules for finding the sure and volume of the three round bodies, the area of parallelograms, circles, and trians, the triangles having the base and altitude given or the three sides. To these might added the rules for finding the surface and volume of prisms and pyramids. Aside m these I heartily believe in the omission of rules. The work before us does not omit consideration of the most of the above geometrical magnitudes, but the rules are not ressly stated. This work of Professors Beman and Smith is, however, one that we can y cheerfully recommend. B. F. F.

A Brief Introduction to the Infinitesimal Calculus. Designed Especially to l in Reading Mathematical Economics and Statics. By Irving Fisher, Ph. D., ıistant Professor of Political Science in Yale University, Co-author of Phillips l Fisher's Elements of Geometry. 12mo. Cloth. 84 pages. Price, 75 cents. v York and London: The Macmillan Co.

This little work on the Calculus will be received with joy by a great army of students, teachers, and professors, who have lacked the time and courage to attack some of the more exhaustive works on the subject yet felt the need of a knowledge of the Calculus in order to enable them to read with intelligence the highest authorities on economic as well as other subjects. Dr. Fisher has prepared this little work with a special view of the needs of this class of students. Any one with a clear mind can very easily read and understand every sentence in this book. There is no metaphysical speculation nor obscure statements made in establishing its first principles.

In considering the formula $s=\frac{1}{2}gt^2$, where s=space a body falls under the influence of gravity in the time t, he says, pages 2 and 3, "Since the above formula holds true of *all* points, it holds true now, when the time is $t+\Delta t$ and the distance $s+\Delta s$. That is $s+\Delta s=16(t+\Delta t)^2$. This gives $s+\Delta s=16t^2+32t.\Delta t+16\Delta t^2$. But $s=16t^2$. Subtracting, we have

$$\Delta s=32t.\Delta t+16(\Delta t)^2,$$

$$\text{whence } \frac{\Delta s}{\Delta t}=32t+16\Delta t \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

This is the *average* velocity during the small interval Δt.

Thus, if $\Delta t=\frac{1}{2}$ second and t be five seconds, the average speed of the body during that half second (viz., the one beginning 5 seconds from the rest) is $32\times 5+19\times\frac{1}{2}$, or 168 per second. If we take $\frac{1}{100}$ of a second instead of $\frac{1}{2}$, we have $32\times 5+16\times\frac{1}{100}$, or 168.1 feet per second.

The speed at the very instant of completing the 5th second is obtained by putting $\Delta t=0$, which gives 160 as the instantaneous speed.

Now when $\Delta t=0$, we call it dt, because 0 would not remind us of the kind of quantity which vanished, whereas dt does suggest t, the magnitude which vanished. When Δt becomes 0, or dt, Δs evidently becomes zero too, for a body can not go any distance in no time. This zero we call ds. Equation (1) therefore becomes at the limit

$$\frac{ds}{dt}=32t+[16]dt,$$

$$\text{or } \frac{0}{0}=32t+0, \text{ which may be written}$$

$$\frac{ds}{dt}=32t\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

for we can neglect the zeros on the right, but not those on the left (the ratio of two zeros does not reduce to zero).

* * * * * * * * * * * * *

It may be objected to the reasoning in the last article that $\frac{ds}{dt}$ or $\frac{0}{0}$ is indeterminate. This is true. $\frac{0}{0}$ is equal to 2, or 19, or 1, or any number we we please. *But the limit* $\frac{\Delta s}{\Delta t}$ *is not indeterminate.* We thus use $\frac{ds}{dt}$ in two distinct senses, viz: $\lim\frac{\Delta s}{\Delta t}$ and $\frac{\lim\Delta s}{\lim\Delta t}$.

The first is determinate, the second is indeterminate, though for that very reason it may always be put equal to the first. Only the first, or $\lim\frac{\Delta s}{\Delta t}$ is important. This is the ultimate ratio of two vanishing quantities."

Excepting the statement in the next to the last sentence, viz., that $\lim \frac{\Delta s}{\Delta t}$ is determinate and $\frac{\lim \Delta s}{\lim \Delta t}$ indeterminate, we claim that Dr. Fisher has established the fundamental principles of the Differential Calculus in a simple, rigorous, and logical manner. By the principle of Limits, the $\lim \frac{\Delta s}{\Delta t} = \frac{\lim \Delta s}{\lim \Delta t}$, that is to say, the limit of the quotient of two variables equals the quotient of their limits. Then if one is determinate the other is determinate, or if one is indeterminate the other is indeterminate. They are, however, both determinate. The statement that dt is used instead of 0 to preserve the trace of the quantity that vanished will be considered by many mathematicians as the rankest sort of mathematical heresy, the reanimating of Berkeley's "ghost of departed quantities." But here too Dr. Fisher's position is absolutely impregnable, for, since $\frac{0}{0}$ is, *per se*, indeterminate, but determinate by the equation by which, in every case, it is defined, it may be replaced by the ratio of any two quantities which preserves the ratio that defines $\frac{0}{0}$. So in the case above, $\frac{0}{0}$ can be replaced by $\frac{ds}{dt}$ or $\frac{y}{x}$ or the quotient of any other two quantities which preserve the ratio $32t$. But $\frac{0}{0}$ is replaced by $\frac{ds}{dt}$ to preserve the trace of the quantities which vanished, and ds and dt can represent large or small parts of s and t. There are in general three possible ways by which the ratio $32t$ can be preserved by $\frac{ds}{dt}$. First, ds being considered a constant, and dt a variable; second, ds being a variable, and dt a constant; third, $\frac{ds}{dt}$ both being variables. Each of these three ways of viewing $\frac{ds}{dt}$ is used in the Calculus. This method of exposition is used in my classes with the result that students are enabled to use the Calculus, not as a machine by which to grind out problems, but as an instrument of research.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, \$2.50 per year in advance. Single Number, 25 cents. The American Monthly Review of Reviews Co., 13 Astor Place. New York City.

The American Monthly Review of Reviews for October has several articles of unusual interest to women readers. Miss Frances Willard tells the story of the World's W. C. T. U. movement; Mrs. Ellen M. Henrotin, president of the General Federation of Women's Clubs, outlines the benefits of those organizations; Mrs. Sheldon Amos, of England, writes of a London Women's Club, and Miss Mary Taylor Blauvelt contributes an enlightening article on the opportunities for women at the English universities. B. F. F.

The Arena. An Illustrated Monthly Magazine. Edited by John Clark Ridpath, LL. D. Price, \$2.50 per year in advance. Single Number, 25 cents. Boston: The Arena Co.

The Arena for October continues the battle for reform. The number is especially interesting and in some parts brilliant; it is in all parts aggressive and courageous. Hon. Charles A. Towne's article, "The New Ostracism," is in the author's best vein of critical analysis. In the course of the discussion he attacks with great vigor the plutocratic interference with professors in colleges and universities. Herman E. Taubeneck continues his cogent statistical attack on consecrated wealth. Judge Walter Clark sends out a powerful plea for the establishment of public rights over semi-public interests and institutions. The Editor of *The Arena* continues with unabated vigor his onslaught on the organized forces of plutocracy. His article, "Prosperity: the Sham and the Reality," is one of his strongest and best. Dr. Ridpath's exposition of the bottom purposes and methods of the money power is as caustic as it is true. Mary Platt Parmelee's article on the Political Philosophy of the Father of American Democracy is an original and forceful argument for popular liberties. B. O. Flower is again at his best pace in "The Latest Social Vision," in which he discusses the merits of Bellamy's "Equality." Perhaps the most radical and defiant article in the number is "The Dead Hand in the Church," by Rev. Clarence Lathbury, in which he attacks with destructive criticism the domination of the dead past over the living present in the church. "Hypnotism in its Scientific and Forensic Aspects" is the subject of an interesting and useful article by Marion L. Dawson. "Suicide: Is It Worth While?" is the caption of Charles B. Newcomb's startling study of one of the most interesting and painful themes of the age. The "Plaza of the Poets" is rich with the contributions of Ironquill, Junius Hempstead, Clinton Scollard, Reubie Carpenter, and Helena M. Richardson; while "The Editor's Evening" sparkles with its usual gems of social and poetical philosophy. Under "Book Reviews" the charming poems of Madison Cawein are set forth with merited commendation. B. F. F.

The Open Court. A Monthly Magazine Devoted to the Science of Religion, and the Religion of Science, and the Extension of the Religious Parliament Idea. Edited by Dr. Paul Carus. T. J. McCormack, Assistant Editor, and E. C. Hegeler and Mary Carus, Associate Editors. Price, $1.00 per year in advance. Single Copies, 10 cents. Chicago and London: The Open Court Publishing Co.

The following is the table of contents of the November number: "An Introduction to the Study of Ethnological Jurisprudence," by the Late Justice Albert Hermann Post, Bremen, Germany; "History of the People of Israel from the Beginning to the Destruction of Jerusalem by the Chaldeans," by C. H. Cornill, Professor of Theology in the University of Koenigsberg; "The Religion of Science; the Worship of Beneficence," by James Odgers Knutsford, England; "Death in Religious Art," by the Editor; "Vivisection from an Ethical Point of View: A Controversy," by Prof. Henry C. Mercer, and others; "Leonhard Euler," a biographical sketch by T. J. McCormack; "The Sacred Books of the Buddhists," by Albert J. Edmunds; "Brief Notes on some Recent French Philosophical Works;" Book Reviews, and Notes. Among the book reviews is a just estimate or criticism of "Finkel's Mathematical Solution Book;" the review contains about a page and a half, and is written by Assistant Editor T. J. McCormack. B. F. F.

ALEXANDER VASILIEVITCH VASILIEV.

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BIOGRAPHY.

VASILIEV.

BY GEORGE BRUCE HALSTED.

ALEXANDER VASILIEVITCH VASILIEV was born August 24 (old style), 1853, at Kazan. His father, orientalist already academician, was then Professor of Chinese Literature at the University of Kazan. His mother was a daughter of Simonov, Professor of Astronomy in Lobachévski's time and his predecessor as Rector. In 1855 on the transference of the Oriental Faculty to St. Petersburg, Vasiliev's father removed thither. In 1870 Vasiliev finished the course of the fifth St. Petersburg gymnasium as gold-medalist.

The love for mathematics, awakened in the gymnasium, where in Class VI. he studied Sturm's Differential Calculus, carried him to the mathematical department of the University of St. Petersburg, which then boasted Somov and the great Chebishev (Tchébychev).

As result of his earnest studies for 1870-73 appears the work "On the separation of roots," crowned with a gold medal. In 1874 on his taking his first degree he was invited by the University of Kazan to begin there his teaching as Privat-docent. Though he had planned to continue his studies at Berlin, he accepts this invitation to his birthplace and begins in November, 1874.

His Dissertatio pro venia legendi was entitled "On the separation of the roots of simultaneous equations." In January 1875 he begins to lecture on Functiontheory, all his scholars being older than the professor.

His thesis for the Master's examination, taken in 1878, was "On singular solutions in connection with the new views on the problem of integration of differential equations of first order."

His Master's Dissertation, accepted in May 1880, he prepared abroad, spending the year 1879 in Berlin with Weierstrass and Kronecker, and in Paris with Hermite. His subject was "On the rational functions analogous to the double-periodic." Soon after he was made Docent in the University of Kasan. He spent the next summer in Germany, and wrote "The teaching of mathematics in Berlin and Leipzig Universities."

A question which had so long interested him was treated again in his Doctor's dissertation in 1884, "Theory of the separation of the roots of systems of simultaneous equations." Now chosen Professor Extraordinarius, he was made Professor Ordinarius in 1887.

In 1884 Vasiliev was made president of the physico-mathematical section of the Scientific Society of Kazan University. In 1891 this section changed itself into the independent "Physico-mathematic Society." The eight volumes of Proceedings of this section from 1880 to 1890 contain a series of important articles and criticisms by Vasiliev. Since 1883 he has been the authority on all Russian works in Analysis for the "Fortschritte der Mathematik." In the years 1880-89 Vasiliev was particularly active as member of the local assembly, the Zemstvo, in the government of Kazan. By his influence, the number of folk-schools increased in 1883-89 from 43 to 90, of scholars from 1692 to 3100. Thus his district, Svijaschsk, attained a first rank in all Russia by passing from one scholar for 920 inhabitants to one scholar for 28 inhabitants.

Since 1891 Vasiliev has edited the "Bulletin de la Société Physico-Mathématique de Kasan," which now exchanges with 110 learned publications. In the brilliantly successful celebration of the hundredth birthday of Lobachevski by this society, and the foundation of the Lobachevski Prizes, more than a thousand persons from all over the world took part as subscribers.

The position now held by Vasiliev in the Russian mathematical world may be judged from his being chosen by the Academy of Sciences to report on a great work offered in competition for the Buniakovski Prize. The book received the half prize, while Vasiliev's report is to be honored by insertion in the Transactions of the Academy and the award of the Buniakovski Medal.

The great International Congress of Mathematicians just born into permanent life at its wonderfully successful first meeting, in Zurich, and next to meet at Paris, owes its inception to Vasiliev, who pushed the idea into prominence in every country. It was on his initiative that I brought the matter up in the American Mathematical Society and obtained the signatures of all the members present at the Brooklyn meeting to an endorsement of the idea giving specific credit to Vasiliev as originator. At the actual congress he was most active. From him, Laisant, and G. Cantor emanated the three important resolutions constituting the three commissions of the Congress.

The many works of Vasiliev, being inaccessible because in Russian, will be enumerated, but the depth of his thinking and charm of his style may be ged from his great Address on Lobachévski, which it was my good fortune to e to the world in a *literal* translation, not a paraphrase. This translation was eted by a tremendous outburst of enthusiasm in the mathematical world.

It must here suffice to give a few detached sentences from a mass of let- sent me. "I am astonished to find these researches of such deep philosoph- import," writes Professor Daniels of the University of Vermont. "I have d it with intense interest," says Cajori. "This life and work of Lobachevski l be a grand inspiration to mathematicians," says Zerr. "I rejoice that you, the midst of the virgin forests of Texas,' are able to do this work," says Pro- sor Carman. "It will arouse a deeper enthusiasm for scientific achievement l widen the horizon of every reader. Surely no mathematician should miss gem from farthest Russia," says Dr. L. E. Dickson. "By translating this st interesting Address, you have earned for yourself a title to the thanks of mathematical world," says Dr. Paul Staeckel, since so well known in this y line. I sent this translation in 1894 to Professor Friedrich Engel of Leip- , to whom I afterward offered for translation into German my translation of bachevski's largest work, "New principles of Geometry with complete theory parallels." He issued the Address in 1895, saying in his *Nachwort:* "Ich e die Wassiljefsche Rede nach dem Original uebersetzt, obwohl bereits eine glische Uebersetzung von *G. B. Halsted* (Austin, Texas, 1894) vorlag ; es ien mir aber fuer einen Deutschen nicht passend, eine russische Schrift nach er englishen Uebersetzung zu uebertragen. Selbstverständlich habe ich r die Halstedsche Uebersetzung ueberall verglichen und bekenne gern, dass mir an manchen Stellen gute Dienste geleistet hat."

A French translation and an (incomplete) Spanish translation have since eared.

This transcendently beautiful production, linking forever the name of siliev with that of Lobachevski, wins both for author and object, the love of ry reader.

A personal picture with scene at Kazan the ancient capital of the Tartars, st be reserved for a subsequent chapter : "A Visit to Vasiliev."

W AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from October Number.]

LVIII. Fig. 29.

$ALMI$ is equivalent to $2IAC-2BAE$ is equivalent to $ACDE$.

$BKML$ is equivalent to $BKNC$ is equivalent to $BCFH$.

$\therefore ABKI$ is equivalent to $ACDE+BCFH$.

LIX. Fig. 29.
QIK=*RAB*. *BOP*=*AFQ*. *OHKP*=*DEAR*.
∴ *ABKI* is equivalent to *ACDE*+*BCFH*.

LX. Fig. 29.
BHK is equivalent to *AFQ*+*DEAR*.
Then *BAIK* is equivalent to *BRQK*.
∴ *ABKI* is equivalent to *ACDE*+*ACFH*.

LXI. Fig. 29.
ABTS is equivalent to 2*ABH* is equivalent to *BCFH*.
STKI=*ALMI* is equivalent to *ACDE*.
∴ *ABKI* is equivalent to *ACDE*+*BCFH*.

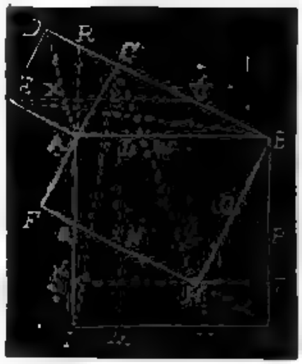


Fig. 29.

LXII. Fig. 29.
Same as in LXI, except that
STKI is equivalent to *ABUE* is equivalent to *ACDE*.

LXIII. Fig. 29.
WBKV, the half of *ABKI*, is equivalent to *BWH*+*BHK*+*HVK*.
But *BHK*=*BCA* is equivalent to *BWC*+*DXE*; and *HVK*=*AXE*.
∴ ½*ABKI* is equivalent to ½*ACDE*+½*BCFH*.
∴ *ABKI* is equivalent to *ACDE*+*BCFH*.

LXIV. Fig. 30.
MAF=*NFA*. Then, *KLI*=*FCD*.
ILN=*DEM*. *BHK*=*BCA*.
∴ *ABKI* is equivalent to *ACDE*+*BCFH*.

LXV. Fig. 30.
KHI=*DEF* is equivalent to ½*ACDE*.
HIL *ALF*.
ILA=*DEF* is equivalent to ½*ACDE*.
BHK=*BCA*.
∴ *ABKI* is equivalent to *ACDE*+*BCFH*.

Fig. 30.

LXVI. Fig. 30.
LNOC is equivalent to *LFDC*−*NFDO* is equivalent to *ACDE*.
For *LFDC* is equivalent to *ACDE*+2*FAE*, and 2*FAE* is equivalent to 2*FAD* is equivalent to *NFDO*.
Also, *KLCB* is equivalent to *BCFH*.
∴ *KNOB* is equivalent to *ACDE*+*BCFH*.
But, *ABKI* is equivalent to *KNOB*.
∴ *ABKI* is equivalent to *ACDE*+*BCFH*.

LXVII. Fig. 30.

ISPK is equivalent to $2IFK=2ADB$ is equivalent to $2ACB+ACDE$ is valent to $ACB+FHQ+ACDE$.

SABP is equivalent to *FABQ*.

∴ *ABKI* is equivalent to $ACDE+BCFH$.

LXVIII. Fig. 30.

$ILR=ACD$, and $ILF=AED$.

Then $IRK=IFA$. $BHK=BCA$.

∴ *ABKI* is equivalent to $ACDE+BCFH$.

LXIX. Fig. 30.

LNOC is equivalent to $2LAC=2FED$ is equivalent to *ACDE*.

KLCB is equivalent to *BCFH*.

∴ *ABKI* is equivalent to *KNOB* is equivalent to $ACDE+BCFH$.

[To be Continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

ORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from October Number.]

PROPOSITION XXIX. *Resuming Fig. 33 of the preceding proposition: I very straight AC, which cuts angle BAX, finally at a finite or terminated dis- (even in hypothesis of acute angle) will meet BX certain point P, if only AC be produced ever more rd the parts of the points X.*

Proof. And firstly indeed (lest straight *AC* ide space with *AX*) it must meet at finite dis- the straights *LK*, *HK*, *DK* in certain points *C*, *f*; must meet, I say, unless before (and that finite distance, just as we maintain) it meets in some point between the point *B* and one of the to *K*.

Then (from Corollary I. after XXIII.) the an- *ACK*, *ANK*, *AMK* will be obtuse.

Fig. 33.

Moreover those angles, always obtuse, approach n the preceding proposition) without any certain limit, to equality with a angle, when indeed that *AC* is supposed to meet *BX* only at an infinite dis-. Therefore such an ordinate *KMD* can be reached that at it the angle *AMK* exceeds a right angle by less than the angle DAC. But then angle DAC, or DAM, together with angle AMD will be greater than a right angle. Wherefore, the obtuse angle ADM being added, the thrée angles together of the triangle ADM will be greater than two right angles, which is against the hypothesis of acute angle.

Therefore every straight AC, which cuts that angle BAX, finally at a finite or terminated distance (hypothesis of acute angle) must meet BX in a certain point P. Quod etc.

COROLLARY I. Hence no straight AZ, which toward the parts of the points X makes an acute angle greater than BAX can ever meet BX, either at a finite, or at an infinite distance. For as far as so should happen, now AX dividing angle BAZ, ought (against the premised supposition) to meet BX at a finite distance, as this is demonstrated of the straight AC dividing angle BAX.

COROLLARY II. Moreover it follows that no determinate acute angle will be the maximum of all under which a straight line produced from point A meets BX at finite distance. For if toward the parts of the point X you assume any point higher than the point P, it follows that the straight joining point A with this higher point will make with AB a greater angle than angle BAP. And so ever without any intrinsic end. Wherefore angle BAX (since indeed AX both always approaches to BX, and meets it only at an infinite distance) will be the outside limit of all acute angles under which straights produced from that point A meet the aforesaid BX at a finite distance.

[To be Continued.]

SOPHUS LIE'S TRANSFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

By EDGAR ODELL LOVETT, Princeton University.

II.

THE GROUP OF ONE PARAMETER. THE INFINITESIMAL TRANSFORMATION. EXISTENCE OF AN INFINITESIMAL TRANSFORMATION IN A GROUP OF ONE PARAMETER.

5. Consider the plane as a point manifoldness, *i. e*, a space whose space element is the point. The plane will then be two-dimensional,* *i. e* contain ∞^2

*This idea and its bearing in the paragraph are emphasized here not to introduce any unnecessary ultra refinement but because of their use in geometrical illustrations to appear in succeeding articles. For example, the plane is one, two, three, or four dimensional according as, a circle with fixed center, the straight line, a circle of general position, or parabola, be taken as space element, since there are ∞^1, ∞^2, ∞^3, ∞^4, of these elements respectively in the plane. Similarly if the straight line is element it has no dimension, the point has one dimension, the plane two dimensions, and ordinary space four dimensions.

elements, or in other words the position of a point in the plane will be determined by two parameters, the coördinates of the point.

A point-transformation of the plane into itself is an operation by which every point in the plane is conveyed into the position of some point in the same plane. In order to represent this operation analytically, let us take as the coördinate system of reference an ordinary rectangular Cartesian system, x, y; then the point transformation is expressed by two equations of the form

$$x_1=\varphi(x, y), \quad y_1=\psi(x, y), \tag{1}$$

where (x, y) is the original point and (x_1, y_1) the transformed point. It is further assumed that the transformation is of such a nature that every point (x_1, y_1) of the plane may be regarded as having originated from some point in the plane by effecting the transformation. This geometrical assumptions finds its analytical condition in the demand that the two functions $\varphi(x, y)$ and $\psi(x, y)$ be independent functions and thus the preceding equations are soluble theoretically with regard to x and y. For, suppose that φ and ψ were not independent, and for example let

$$\varphi=n\alpha(x, y), \quad \psi=m\alpha(x, y);$$

then eliminating $\alpha(x, y)$ from the equations of the transformation

$$x_1=n\alpha(x, y), \quad y_1=m\alpha(x, y)$$

we find that the points (x, y) of the plane are transformed into the points (x_1, y_1) of the straight line

$$y_1=\frac{m}{n}\cdot x_1,$$

and hence point of general position no longer is conveyed into point of general position by the transformation.

If the equations (1) be solved with regard to the variables x, y, there result two equations of the form

$$x=\overline{\varphi}(x_1, y_1), \quad y=\overline{\psi}(x_1, y) \tag{2}$$

which represent a transformation that carries the point (x_1, y_1) back into the position (x, y); this transformation (2) is called the *inverse* of the transformation (1). If the transformation (1) be followed by the transformation (2), that is, if the two transformations be carried out successively, we have the two equations

$$x_1=x, \quad y_1=y.$$

These equations are very particular cases of equations (1) and hence should represent a transformation. The transformation which they represent obviously transforms a point into itself, or in other words, it leaves all points at rest, for this reason it is called the *identical* transformation.

6. If the equations of a transformation

$$x_1=\varphi(x, y, a), \quad y_1=\psi(x, y, a) \tag{3}$$

contain an arbitrary constant a, these equations no longer represent a single transformation but a family of ∞^1 transformations, since the arbitrary parameter a may assume all values from $-\infty$ to $+\infty$. Let us make the hypothesis that the equations (3) represent such a family of transformations that *the successive application of any two transformations of the family is equivalent to a transformation belonging to the same family*; in this case the family (3) is called a *group*; since the group contains *one* parameter a and hence ∞^1 transformations, the group is called a *one parameter group*, or a *group* of one parameter, or symbolically a G_1. Further, since the parameter varies continuously the group is said to be a *continuous* group. As the group contains a finite number of parameters, in this case but one, namely a, it is a *finite continuous group*. The sentence above in italics expresses the group property of the family. A footnote in the preceding article calls attention to the fact that the group property is peculiar to certain classes of families and not common to all of them.

The analytical criterion for a one-parameter group as just defined reveals itself in the following manner. The transformation

$$T_1 \qquad x_1=\varphi(x, y, a), \quad y_1=\psi(x, y, a), \tag{4}$$

changes the point (x, y) into the point (x_1, y_1); let T_1 be followed by the transformation T_2 which corresponds to the value a_1 of the parameter and changes the point (x_1, y_1) into the point (x_2, y_2), given by the equations

$$T_2 \qquad x_2=\varphi(x_1, y_1, a_1), \quad y_2=\psi(x_1, y_1, a_1). \tag{5}$$

The transformation, T_3, say, which will carry the point (x, y) directly into the position (x_2, y_2) is found by eliminating (x_1, y_1) from the equations (4) and (5). The elimination yields

$$T_3\equiv T_1T_2 \begin{cases} x_2=\varphi\{\varphi(x, y, a), \psi(x, y, a), a_1\}, \\ y_2=\psi\{\varphi(x, y, a), \psi(x, y, a), a_1\}. \end{cases} \tag{6}$$

If this transformation is to belong to the original family it must be capable of expression in the form

$$x_2=\varphi(x, y, \lambda), \quad y_2=\psi(x, y, \lambda),$$

where λ is a certain function of a and a_1 alone.

Hence the criterion sought is that the two equations

$$\varphi\{\varphi(x, y, a), \psi(x, y, a), a_1\}\equiv\varphi\{x, y, \lambda(a, a_1)\},$$

$$\psi\{\varphi(x, y, a), \psi(x, y, a), a_1\} \equiv \psi\{x, y, \lambda(a, a_1)\},$$

must exist identically for all values of x, y, a, and a_1.

7. In the sequel we shall study only those continuous groups* which contain the *inverse* transformation of every transformation of the group. *i. e.* to a tranformation corresponding to the parameter a,

$$T_1 \qquad x_1 = \varphi(x, y, a), \quad y_1 = \psi(x, y, a),$$

there corresponds a transformation of the family whose parameter is $\bar{a}$ say,

$$T_2 \qquad x_2 = \varphi(x_1, y_1, \bar{a}), \quad y_2 = \psi(x_1, y_1, \bar{a}),$$

such that T_2 cancels T_1 and gives

$$x_2 = x, \quad y_2 = y,$$

the identical transformation.

Accordingly, if the transformations of a group are inverse in pairs the group contains the *identical* transformation. Let a_0 be the value of the parameter which gives the identical transformation, then

$$\varphi(x, y, a_0) \equiv x, \quad \psi(x, y, a_0) \equiv y.$$

A transformation of the family whose parameter is $a_0 + \delta a$, where δa is an indefinitely small quantity will move the point (x, y) through only an infinitesimal distance, such a transformation is called an *infinitesimal transformation*, where by an *infinitesimal* transformation of the group is meant a transformation whose parameter differs by an infinitesimal from that value of the parameter which gives the *identical transformation.*

8. There is a most intimate connection between the notions *infinitesimal transformation* and *one parameter group.* It is proposed to derive now three fundamental theorems of Lie which establish this relationship. The first proves that *every one parameter group contains an infinitesimal transformation*, the second that *every infinitesimal transformation generates a one parameter group*, and the third that *a one parameter group contains but one infinitesimal transformation.*

The three theorems show that an infinitesimal transformation may be taken as the representative of a one parameter group.

That a group of one parameter contains an infinitesimal transformation may be seen geometrically in the following manner:

Let a transformation of the group which corresponds to the parameter a and which is designated for convenience by (a) carry the point p (x, y) to the position p_1, (x_1, y_1). By assumption the inverse of (a) is contained in the group. Let the parameter of this inverse transformation be $\bar{a}$; $\bar{a}$ is a certain

*The reader must be reminded that this limitation is really not a restriction. Lie has proved in volume III of the Theory of Transformation Groups, Theorem 26, that the defining equations of any continuous group can be derived from those of a group whose transformations are inverse in pairs.

function of α. The transformation $(\bar\alpha)$ changes all points p_1 into the points p again respectively. A transformation whose parameter differs infinitesimally from $\bar\alpha$, say $\bar\alpha+\delta\alpha$, will carry the point p_1 not back to p, but to a position at an infinitesimal distance from p, say p'. The successive performance of (α) and $(\bar\alpha+\delta\alpha)$ will carry p to p_1 and then to p'; but (α) and $(\bar\alpha+\delta\alpha)$ belong to the group, hence the third transformation to which they are equivalent belongs to the group; that is, the transformation which carries p to p', a point infinitesimally near, belongs to the group, or in other words the group contains an infinitesimal transformation.

This geometric process may now be clothed in analytic garb. The first transformation (α) is given by the equations

$$x_1=\varphi(x,\,y,\,\alpha),\quad y_1=\psi(x,\,y,\,\alpha)\,; \tag{7}$$

the second transformation $(\bar\alpha+\delta\alpha)$ by

$$x'=\varphi(x_1,\,y_1,\,\bar\alpha+\delta\alpha),\quad y'=\psi(x_1,\,y_1,\,\bar\alpha+\delta\alpha). \tag{8}$$

The elimination of x_1, y_1 from these equations gives the transformation which carries p to p', namely

$$x'=\varphi\{\varphi(x,\,y,\,\alpha),\,\psi(x,\,y,\,\alpha),\,\bar\alpha+\delta\alpha\},\ y'=\psi\{\varphi(x,\,y,\,\alpha),\,\psi(x,\,y,\,\alpha),\,\bar\alpha+\delta\alpha\}. \tag{9}$$

Developing* these values in powers of $\delta\alpha$ we have

$$\begin{aligned} x'&=\varphi\{\varphi(x,\,y,\,\alpha),\,\psi(x,\,y,\,\alpha),\,\bar\alpha\}+\frac{\partial\varphi\{\varphi(x,\,y,\,\alpha),\,\psi(x,\,y,\,\alpha),\,\bar\alpha\}}{\partial\bar\alpha}\,\frac{\delta\alpha}{1}+\ldots\,,\\ y'&=\psi\{\varphi(x,y,\,\alpha),\,\psi(x,\,y,\,\alpha),\,\bar\alpha\}+\frac{\partial\psi\{\varphi(x,\,y,\,\alpha),\,\psi(x,\,y,\,\alpha),\,\bar\alpha\}}{\partial\bar\alpha}\,\frac{\delta\alpha}{1}+\ldots \end{aligned} \tag{10}$$

Now since the transformetions (α) and $(\bar\alpha)$ are inverse

$$\varphi\{\varphi(x,\,y,\,\alpha),\,\psi(x,\,y,\,\alpha),\,\bar\alpha\}\equiv x,\quad \psi\{\varphi(x,\,y,\,\alpha),\,\psi(x,\,y,\,\alpha),\,\bar\alpha\}\equiv y\,;$$

hence the equations of the transformation which changes p into p' are

$$\begin{aligned} x'&=x+\frac{\partial\varphi\{\varphi(x,\,y,\,\alpha),\,\psi(x,\,y,\,\alpha)\,\bar\alpha\}}{\partial\bar\alpha}\,\frac{\delta\alpha}{1}+\ldots\,,\\ y'&=y+\frac{\partial\psi\{\varphi(x,\,y,\,\alpha),\,\psi(x,\,y,\,\alpha),\,\bar\alpha\}}{\partial\bar\alpha}\,\frac{\delta\alpha}{1}+\ldots\,, \end{aligned} \tag{11}$$

and in this form they represent an infinitesimal transformation since the values

*It is to be remarked once for all that all functions here considered are regular analytic functions and hence expansible by Taylor's Theorem.

of x' and y' differ from x and y respectively by infinitely small quantities. It is easy to see that the coefficients of $\delta\alpha$ do not vanish, for if we put for $\varphi(x, y, \alpha)$ and $\psi(x, y, \alpha)$ their equals x_1 and y_1 respectively, these coefficients equated to zero are

$$\frac{\partial\varphi(x_1, y_1, \bar{\alpha})}{\partial\bar{\alpha}}\equiv 0, \quad \frac{\partial\psi(x_1, y_1, \bar{\alpha})}{\partial\bar{\alpha}}\equiv 0.$$

But these last identities assert that φ and ψ are free from $\bar{\alpha}$, that is, in general the equations of the group contain no parameter which is contrary to hypothesis.

The quantity $\bar{\alpha}$ is a function of α, since to a transformation ($\bar{\alpha}$) there corresponds, by hypothesis, a completely determinate inverse transformation ($\bar{\alpha}$). The equations (1) of the infinitesimal transformation may be written in the form

$$x'=x+\xi(x, y, \alpha)\delta\alpha+\ldots, \quad y'=y+\eta(x, y, \alpha)\delta\alpha+\ldots *$$

LIE thus arrives at the following theorem :

I. *Every one parameter group whose transformations are inverse in pairs contains at least one infinitesimal transformation.*

Princeton University, 22 October, 1897.

[To be Continued.]

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

83. Proposed by the late REV. G. W. BATES, A. M., Pastor of M. E. Church, Dresden City, Ohio.

A has three notes; the first and second, \$1000 each, and the third \$457; all dated April 1, 1884. The first is due April 1, 1888, second, April 1, 1889, and the third, April 1, 1890, and each bearing interest at 6%. What must B pay for the three notes September 21, 1886, that the investment will bring him 8% compound interest ?

Solution by G. B. M. ZERR, A. M., Ph. D., President of Russell College, Lebanon, Mo.

(I). Regarding the notes as bearing simple interest. We get

\$1000 × 1.24=\$1240, amount of first note.

\$1000 × 1.30=\$1300, amount of second note.

\$457 × 1.36=\$621.52, amount of third note.

*These equations contain a constant α which can be arbitrarily chosen, hence we can find an infinitesimal transformation of the group in many different ways. But the sequel will show that all these, excepting a constant factor, are identical in their terms of the first order of infinitesimals.

From September 21, 1886, to April 1, 1888, is $1\frac{19}{36}$ years.
From September 21, 1886, to April 1, 1889, is $2\frac{19}{36}$ years.
From September 21, 1886, to April 1, 1890, is $3\frac{19}{36}$ years.
Let x=amount paid for first note ; y, for second ; z, for third.
$\therefore x(1.08)^{1\frac{19}{36}}=1240$, or $\log x=\log 1240-1\frac{19}{36}\log 1.08$.
$\therefore x=\$1102.448$.
$y(1.08)^{2\frac{19}{36}}=1300$, or $\log y=\log 1300-2\frac{19}{36}\log 1.08$.
$\therefore y=\$1070.176$.
$\log z=\log 621.52-3\frac{19}{36}\log 1.08$.
$\therefore z=\$473.743$.
$x+y+z=\$2646.367$=whole amount to be paid for the notes.
(II). If the notes bear compound interest we get,
$\$1000\times(1.06)^4=\$1262.477$, amount of first note.
$\$1000\times(1.06)^5=\$1338.226$, amount of second note.
$\$457\times(1.06)^6=\$648.263$, amount of third note.
$\therefore \log x=\log 1262.477-1\frac{19}{36}\log 1.08$.
$\therefore x=\$1122.43$.
$\log y=\log 1338.226-2\frac{19}{36}\log 1.08$.
$\therefore y=1101.646$.
$\log z=\log 648.263-3\frac{19}{36}\log 1.08$.
$\therefore z=\$494.127$.
$x+y+z=\$2718.20$=whole amount paid for the three notes.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

78. Proposed by J. A. MOORE, Ph. D., Professor of Mathematics, Millsaps College, Jackson, Miss.

Required the number of normals that can be drawn from any point (a, b) to the parabola $y^2=2px$.

I. Solution by the PROPOSER.

The equation of the normal to the parabola in terms of its slope, (s), is

$$y=sx-\tfrac{1}{2}(sp)(2+s^2)\ldots\ldots\ldots\ldots(1).$$

Substituting a, b for x, y in (1), and putting the equation in a new form, we have,

$$s^3+\tfrac{1}{2}p(p-a)s+(2b/p)=0\ldots\ldots\ldots\ldots(2).$$

Denoting Sturm's functions by F, F_1, F_2, etc., we have the following:

$F = z^3 + (2/p)(p-a)z + (2b/p)$.

$F_1 = 3z^2 + (2/p)(p-a)$.

$F_2 = -2(p-a)z - 3b$.

$F_3 = -b^2 - (8/27p)(p-a)^3$.

Consider five cases.

(1). Suppose $p-a<0$, and $(8/27p)(p-a)^3$ numerically greater than b^2. Sturm's Theorem gives

	F	F_1	F_2	F_3
For $z=+\infty$,	+	+	+	+
$z=-\infty$,	−	+	−	+

Hence the roots are real and unequal.

(2). Suppose $p-a<0$, and $(8/27p)(p-a)^3$ numerically less than b^2. Then

	F	F_1	F_2	F_3
$z=+\infty$,	+	+	+	−
$z=-\infty$,	−	+	−	−

Hence, there is one real root.

(3). Suppose $p-a>0$. Then

	F	F_1	F_2	F_3
$z=+\infty$,	+	+	−	−
$z=-\infty$,	−	+	+	−

Hence, one real root.

(4). Suppose $-b^2-(8/27p)(p-a)^3=0$.

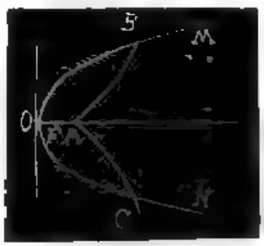
Then there are equal roots, as in this case F and F_1 have a common divi-
and all the roots are real.

(5). Suppose $-b^2-(8/27p)(p-a)^3=0$, and $p=a$.

Then $b=0$, and all the roots are equal, each being 0.

Hence if MON is the given parabola and BAC its evolute, that is, the
cubical parabola whose equation is

$$b^2=\frac{8}{27p}(a-p)^3.$$



Then, (1), if the point (a, b) is within (to the
t) of the evolute, three normals can be drawn to
parabola; (2), if the point (a, b) is on the evolute,
not at A, two normals can be drawn; (3), if the
t (a, b) is A, or is without (to the left) of the evo-
one normal can be drawn to the parabola.

II. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, O[...] sity, Athens, Ohio, and OTTO CLAYTON, Fowler, Ind.

If m be the tangent of the angle which the normal makes with t[...] of x, the normal is given by

$$y=mx-pm-(\tfrac{1}{2}p)m^3 \ldots\ldots\ldots\ldots\ldots$$

This passing through (a, b) gives

$$b=am-pm-(\tfrac{1}{2}p)m^3 \ldots\ldots\ldots\ldots\ldots$$

a cubic in m, showing that the required number is three.

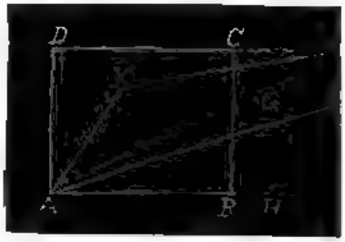
Also solved by *G. B. M. ZERR* and *J. F. LAWRENCE*.

79. Proposed by JOHN MACNIE, Professor of Mathematics, University of North Dakota, Univer[...]

To construct a quadrilateral of given area, the diagonals, one of which is gi[...] ting each other in given ratios and at a given angle.

I. Solution by JAS. F. LAWRENCE, Freshman Class, Classical Course, Drury College, Springfiel[...] the PROPOSER.

Let AC be a rectangle equivalent to the given area and having a [...] equal to one-half of the given diagonal. Produce AB to E, so that $BE=$[...] A construct $\angle EAF$ equal to the angle to be made by the diagonals, and meet DC produced in F. Divide AF in G in the ratio of division of one diagonal, and AE in H, in the ratio of the given diagonal. On an indefinite line drawn through G parallel to AB lay off GK, CL, equal to AH, HE, respectively, and FK, FL, AK, AL; $AKFL$ is the required quadrilateral.



Join EF. $\triangle FAE$ ⇔ AC. It is also equivalent to $AKFL$; for each is equivalent to one-half the paralle[...] formed by drawing parallels through the extremities of the diagonals A[...] Hence $AKFL$ ⇔ AC; it has also a diagonal $KL=AE=2AB$; its diagon[...] are divided in the given ratios, and make an angle $FGL=\angle EAF=$th[...] angle. Hence, etc.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Lebanon, Va.; OTTO CLAYTON, Fowler, Ind.; and F[...] EY, Ph. D., New Haven, Conn

Let AB be the given diagonal, COB the given angle, $\triangle$ the give[...] $m:n$ the given ratio for the known diagonal, $p:q$ the given ratio for t[...] known diagonal.

*From the well known theorem: Any quadrangle is equivalent to one-half the parallelogr[...] ed by drawing lines through its vertices parallel to its diagonals; follow the corollaries—

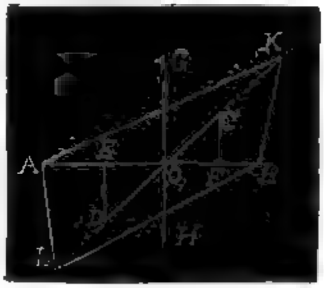
a. Two quadrangles are equivalent if their diagonals are respectively equal and interse[...] same angle. (Triangle a special case.)

b. Any quadrangle is equivalent to the rectangle of its diagonals multiplied by half the sin[...] angle.

Let x=unknown diagonal, $\beta=\angle COB$, h=altitude of triangle above AB, =the altitude of the triangle below AB, a=given diagonal.

$$x=2\Delta/a\sin\beta \ldots\ldots\ldots\ldots(1).$$

Divide AB at O in ratio $m:n$ and draw the indefinite line KL making an gle β with AB.



Let $CO=p$, $DO=q$, $OK=y$, $OL=z$.

$\therefore y=xp/(p+q)=2p\Delta/\{a(p+q)\sin\beta\}$,

$z=xq/(p+q)=2q\Delta/\{a(p+q)\sin\beta\}$.

$\therefore h=2p\Delta/\{a(p+q)\}$

$h_1=2q\Delta/\{a(p+q)\}$.

Draw OG, OH perpendicular to AB and , h_1; draw GK, LH parallel to AB, cutting in K and L. $\therefore AKBL$ is the quadrilateral.

III. Solution by A. H. BELL, Hillsboro, Ill., and F. R. HONEY, Ph. B., New Haven, Conn.

Let a, and b, equal the segments of the given diagonals.

Let $x+y$, and $x-y$, equal the segments of the other diagonals.

Let r=the given ratio of the later diagonal, and θ the angle. Then

$$\frac{x+y}{x-y}=r \ldots\ldots\ldots\ldots(1).$$

$$[\ldots]\sin\theta+bx\sin\theta=c \text{ the given area} \ldots\ldots\ldots\ldots(2).$$

$$(2) \text{ and } (1)\ x=\frac{c}{(a+b)\sin\theta}-\frac{r+1}{r-1}y \ldots\ldots\ldots\ldots(3).$$

$$=\frac{c(r-1)}{(a+b)(r+1)\sin\theta},\quad x+y=\frac{2cr}{(a+b)(r+1)\sin\theta},\quad x-y=\frac{2c}{(a+b)(r+1)\sin\theta}.$$

Now having all the segments of the two diagonals with the given angle be-[tw]een them, the construction of the required quadrilateral is very simple.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

52. Proposed by S. ELMER SLOCUM, Union College, Schenectady, New York.

A chain 16 feet long is hung over a smooth pin with one end 2 feet higher than the [oth]er end and then let go. Show that the chain will run off the pin in about 7·5 second. [W]right's *Mechanics*, page 92.]

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Let 16 feet$=2a$, 9 feet$=b$, and $x=$the length of the longer part of chain at any time t from the beginning of motion. Then, if m be the mass of a unit of length of the chain, $g=32$, the equation of motion is

$$\frac{d}{dt}\left(2ma\frac{dx}{dt}\right)=2mg(x-a) \quad \ldots\ldots(1).$$

Multiplying both members by $\frac{d(x-a)}{dt}$ and integrating,

$$a\left(\frac{d(x-a)}{dt}\right)^2=g(x-a)^2+C \quad \ldots\ldots(2).$$

When $x=b$, $\frac{d(x-a)}{dt}=0$, and $C=-g(b-a)^2$;

$$\therefore (2) \text{ is } a\frac{d(x-a)}{dt}=g\sqrt{(x-a)^2-(b-a)^2} \quad \ldots\ldots(3),$$

$$\text{or } dt=\sqrt{\frac{a}{g}}\frac{d(x-a)}{\sqrt{(x-a)^2-(b-a)^2}} \quad \ldots\ldots(4).$$

Integrating, $t=\sqrt{\frac{a}{g}}\log\left(x-a+\sqrt{(x-a)^2-(b-a)^2}\right)+C'$ ......(5).

When $x=b$, $t=0$; $\therefore$ $C'=\sqrt{\frac{a}{g}}\log(b-a)$, and (5) then becomes

$$t=\sqrt{\frac{a}{g}}\log\left\{\frac{x-a+\sqrt{(x-a)^2-(b-a)^2}}{b-a}\right\} \quad \ldots\ldots(6).$$

Introducing numbers, $t=1.38$ seconds.

II. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss., and HENRY HEATON, M. Sc., Atlantic, Iowa.

Let s denote the distance through which the lower end of the string descends in t seconds.

Then, since the acceleration equals the moving force divided by the mass moved,

$$\frac{d^2s}{dt^2}=\frac{2+2s}{16}g.$$

Integrating, $\left(\frac{ds}{dt}\right)^2=\frac{1}{8}g(s^2+2s)$, no constant being added since when

$=0$, $\frac{ds}{dt}=0$. From the last equation $\frac{ds}{\sqrt{s^2+2s}}=\frac{\sqrt{2g}}{4}dt$ which, by integration, gives $\log_e(s+1+\sqrt{2s+s^2})=\frac{\sqrt{2g}}{4}t$, the constant again being zero, since when $t=0$, $s=0$, and $\log 1=0$.

Taking this between the limits 7 and 0, $t=\frac{7}{5}$, approximately.

Also solved by *G. B. M. ZERR*, *C. W. M. BLACK*, *J. SCHEFFER*, and the *PROPOSER*.

53. Proposed by J. C. NAGLE, M. A., C. E., Professor of Civil Engineering, Agricultural and Mechanical College of Texas.

Find the locus of the center of gravity of an arc of constant length for a parabola.

Solution by G. B. M. ZERR, A. M., Ph. D., President of Russell College, Lebanon, Va.

Let u, v be the coördinates of the center of gravity, $y^2=4ax$, be the equation to the parabola for any point on the curve.

$$\therefore\ su=\int_0^x x\,ds=\tfrac{1}{4}(a+2x)\sqrt{ax+x^2}-\frac{a^2}{8}\log\left(\frac{a+2x+2\sqrt{ax+x^2}}{a}\right)$$

$$=\tfrac{1}{4}(a+2x)\sqrt{ax+x^2}-\frac{a^2}{4}\log\left(\frac{\sqrt{x}+\sqrt{a+x}}{\sqrt{a}}\right)$$

$$\frac{a^2}{4}\log\left(\frac{\sqrt{x}+\sqrt{a+x}}{\sqrt{a}}\right)=\frac{a}{4}(s-\sqrt{ax+x^2}).$$

$$\therefore\ su=\tfrac{1}{2}\sqrt{x}(a+x)^{\frac{3}{2}}-\tfrac{1}{4}as \ \ldots\ldots\ldots\ldots (1).$$

$$sv=\int_0^x y\,ds=\tfrac{4}{3}\sqrt{a}(a+x)^{\frac{3}{2}}-\tfrac{4}{3}a^2 \ \ldots\ldots\ldots\ldots (2).$$

a and x are both variable in (1) and (2). It does not appear easy to eliminate a and x and thus obtain an equation in u, v.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

53. Proposed by A. H. BELL, Hillsboro, Illinois.

Given $x^2-114\tfrac{1}{4}y^2=\mp3$ to find the least values of x and y in integers.

I. **Solution by the PROPOSER.**

It can be demonstrated that D, in $x^2-Ay=\pm D$, can be any denominator of the complete quotients from the $\sqrt{A}$, and that x and y are the numerator and denominator of the convergent preceding the term in which D is taken. Now the complete quotients for the $\sqrt{114\frac{1}{4}}$ are

$$\frac{0+\sqrt{114\frac{1}{4}}}{1};\quad \frac{10\frac{1}{2}+\sqrt{114\frac{1}{4}}}{4};\quad \frac{9\frac{1}{2}+\sqrt{114\frac{1}{4}}}{6};\quad \frac{8\frac{1}{2}+\sqrt{114\frac{1}{4}}}{7};\ \text{etc.}$$

No. term		1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	etc., reversing.
$\therefore \sqrt{114\frac{1}{4}}=\frac{1}{0}$, $\frac{10\frac{1}{2}}{1}$	:	5,	3,	2,	1,	2,	1,	6,	2,	1,	1,	10,	10,	etc., reversing.
Complete denom'rs=1	:	4,	6,	7,	12,	6,	14,	3,	8,	9,	12,	2,	2,	etc., reversing.

Hence x and y are found in the 6th convergent and also the 15th convergent, and x and y=2095 and 196, and also from the 15th term x=42, 807, 834, and y=3, 958, 154. [Also see problem 38.]

II. **Solution by JOSIAH H. DRUMMOND, LL. D., Portland Maine.**

I have not solved this problem as stated, but as I have solved it in this form $x^2-114\frac{1}{4}y^2\pm3=\square$ ......(1), and as that is a pretty question, I send my solution.

Multiplying by 4, it becomes $4x^2-457y^2\pm12=\square$ =say $(2x-m)^2=4x^2-4mx+m^2$), from which we find

$$x=\frac{457y^2+m^2\pm12}{4m}.$$

$(m\pm12)/4m$ evidently becomes integral when $m=6$; and we have

$$x=\frac{457y^2}{24}+2,\ \text{or}\ \frac{457y^2}{24}+1.$$

$457y^2/24$ becomes integral when $y=12n$, and $x=2742n+2$, or $=2743n+1$, according as the + or − sign before 3 is taken.

If $n=1$, $y=12$, and $x=2744$ or 2743 ; in the former, 3 is negative, and in the latter, positive, in order to make the expression a square.

64. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

In the expression $2x^2-2ax+b^2$, find two series of values for x in integral terms of a and b.

I. **Solution by the PROPOSER.**

$2x^2-2ax+b^2$ is evidently a square when $x=a$. Take $x=y+a$, and substituting, we have $2y^2+2ay+b^2=\square=(\text{say})(my-b)^2$.

Reducing, $y=2(a+bm)/(m^2-2)$. Taking m=2/1, 10/7, 58/41, etc., we have *one* integral series of the value of y, viz : $a+2b$, $49a+10b$, etc. Taking m=3/2, 17/12, 99/70, etc., we have *another* integral series of the value of y, viz:

$8a+12b$, $288a+408b$, etc. By adding a to each term of each series we have two series of the value of x. These series hold good when either a or b is zero; but if both are zero, $x=0$.

It will be noticed that this solution applies the terms of the question to the expression $2x^2+2ax+b=\square$, the value of x in the latter being a less than in the former.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

$2x^2-2ax+b^2=0$. $\therefore$ $x=\frac{1}{2}(a\pm\sqrt{a^2-2b^2})$.

Let $a=p^2+2q^2$, $b=2pq$. Then $x=p^2$ or $2q^2$.

$\therefore$ p	q	a	b	x,
2	1	6	4	4 or 2,
3	2	17	12	9 or 8,
4	3	34	24	16 or 18,
1	2	9	4	1 or 8.
etc.	etc.	etc.	etc.	etc.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

54. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

A man is at the center of a circle whose diameter is equal to three of his steps. If each step is taken in a perfectly random direction, what is the probability, (1), that he will step outside the circle at the second step, and, (2), that he will step outside at the third step?

I. Solution by the PROPOSER.

Let O be the center of the circle, A, the end of the first step, and B, the end of the second, and C, the end of the third.

Let $\angle OAB=\theta$, $\angle OBC=\phi$, $OB=x$; and $OC=y$.

Then if the length of the step be taken as the unit of measure, $x=2\sin\frac{1}{2}\theta$, and $y=(x^2+1-2x\cos\phi)^{\frac{1}{2}}=(4\sin^2\frac{1}{2}\theta+1-4\sin\frac{1}{2}\theta\cos\phi)^{\frac{1}{2}}$.

If $x=\frac{3}{2}$, B falls upon the circumference of the circle, and $\theta=2\sin^{-1}\frac{3}{4}$. If θ be $>2\sin^{-1}\frac{3}{4}$, and $<\pi$, the second step falls outside the circle. The probability of this is $P_1=(\pi-2\sin^{-1}\frac{3}{4})/\pi$.

If θ be $<2\sin^{-1}\frac{3}{4}$, and $y=\frac{3}{2}$, C falls upon the circumference of the circle, and $4\sin^2\frac{1}{2}\theta+1-4\sin\frac{1}{2}\theta\cos\phi=\frac{9}{4}$ or $\phi_1=\cos^{-1}(\sin\frac{1}{2}\theta-5/16\sin\frac{1}{2}\theta)$. Hence if ϕ be $>\phi_1$ the third step falls outside the circle. The chance that ϕ will be $>\phi_1$ and $<\pi$ is $(\pi-\phi_1)\pi$. The chance that θ has any particular value is $d\theta/\pi$. Hence the probability that the third step falls outside the circle is

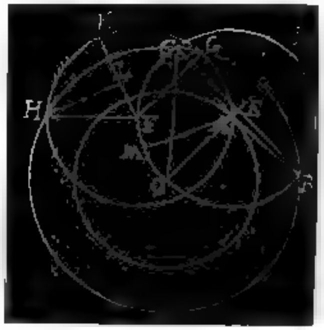
$$P_2=\frac{1}{\pi^2}\int_0^{2\sin^{-1}\frac{3}{4}}(\pi-\phi_1)d\theta.$$

This is not integrable in general terms but its value may be readily approximated by methods of mechanical quadrature.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va., and J. A. MOORE, Ph. D., Professor of Mathematics, Millsaps College, Jackson, Miss.

Let $AO=a$, then $CO=\frac{3}{2}a$.

(1). Let his first step place him at the point A, then in order that he may step outside on the second step he must step somewhere on the arc CDB.



Let $AE=t$, $EC=u$, $\angle DAC=\beta$, P=chance in (1), p=chance in (2).

Now $u^2=a^2-t^2=\frac{9}{4}a^2-(a+t)^2$, $\therefore t=\frac{1}{8}a$.

$\therefore \cos\beta=\frac{1}{8}$.

$\therefore P=\beta/\pi=\cos^{-1}\frac{1}{8}/\pi=.460106$.

(2). Let chord $OM=\frac{1}{2}a$, then in order that he may step out the third step he must step somewhere on the arc CM or its equal on the opposite side

$$\angle CAM=\delta=\pi-(\beta+OAM)$$

$$=\cos^{-1}\left(\frac{3\sqrt{105}-7}{64}\right).$$

P_1=chance he steps on this arc$=\delta/\pi=.379034$.

If his second step places him on arc CM then his third step must place him on the arc GKH. The $\angle KFH$ may vary from 0 to $\cos^{-1}(-\frac{1}{3})$.

$\therefore p_1$=chance that he steps on arc $GKH=\dfrac{\cos^{-1}(-\frac{1}{3})}{2\pi}$.

$\therefore p_1=.304086$.

Now $p=P_1\times p_1=\delta/\pi\times\dfrac{\cos^{-1}(-\frac{1}{3})}{2\pi}=.115259$.

Solved with a different result by *CHAS. C. CROSS.*

56. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

It has been clear for 15 consecutive days, what is the chance of the 16th day being cloudy?

Solution by the PROPOSER.

Let p=chance, p_1=chance that 16th day is clear.

$$\therefore p_1=\frac{\int_0^1 x^{16}dx}{\int_0^1 x^{15}dx}=\frac{16}{17}. \quad \therefore p=1-p_1=\frac{1}{17}.$$

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

49. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Give a general proof that the centre of gravity, or centroid, determines that point om which the sum of the distances to all other points of a given area is the minimum.

This problem is almost the same as No. 30, Miscellaneous, solutions of hich were published on pages 334-5 of Vol. II, and on pages 86-88 of Vol. III. o further solutions have been received. If any of our contributors will attempt ther solutions, they will be given in a future number. EDITOR.

50. Proposed by JOHN KEELEY ELLWOOD, A. M., Principal of the Colfax School, Pittsburg, Pa.

Describe and compute the actual path traversed by the moon in July and August, 96, taking into account the motion of the earth around the sun.

No solution of this problem has been received. Dr. S. Hart Wright re- arks that "a solution is not possible, as the *actual* path of the moon in space is quired, while the moon and the earth describe, in their orbits, neither circles or ellipses, but curved lines that are *undulatory*, being affected by perturbations ue to other planets. If the orbits of the earth and moon were circles or ellips- , the moon's path would be an epicycloidal curve, always concave towards the n." With the aid of a Nautical Almanac or data of the moon's path during e time asked, it would seem that a practically correct solution of the problem uld be effected. We shall be pleased to publish anything further from con- ibutors on this problem. EDITOR.

51. Proposed by F. M. SHIELDS, Coopwood, Miss.

A stock dealer traveled from his home H, due north across a lake L 40 miles wide to city, and bought 156 horses and 177 mules for \$23631; he then traveled farther due north A, and bought at same price 468 horses and 235 mules for \$52245; he then traveled from due west 130 miles to B, and bought 120 cows; he then traveled due north to C, nd bought 250 sheep; he then traveled from C due east 330 miles to D, and bought 300 oats,—paying 1-4 as much for cows as horses, and 1-9 as much for sheep as mules, and 1-2 much for goats as sheep; at D he turned and traveled in a straight line to the city, a istance equal to the sum of the entire distance he traveled due north from his home H; e sold all his stock at a profit of 20%. How far did he travel from his home H the entire ip around and back to the city? What was the cost of each head of stock, and what was e entire gain?

I. Solution by P. S. BERG, A. M., Principal of Schools, Larimore, N. D.; CHARLES C. CROSS, Laytonsville, d.; H. C. WILKES, Skull Run, W. Va.; J. SCHEFFER, A. M., Hagerstown, Md.; and G. B. M. ZERR, A. M., Ph. ., The Russell College, Lebanon, Va.

Let x=price of each horse, y=price of each mule.

Then $156x+177y=23631$; and $468x+235y=52245$.

$\therefore x=\$80,\ y=\$63.$

$\frac{1}{4}$ of $80=$20, price of each cow; $\frac{1}{9}$ of $63=$7, price of each sheep; $\frac{1}{2}$ of $7=$3.50, price of each goat. $120\times20=2400$; $250\times7=1750$; $300\times3.50=1050$.

$2400+$1750+$1050+$23631+$52245=$81076, entire cost. 20% of $81076=$16215.20, entire gain.

Let $AH=u$, $BC=v$.

$\therefore (40+u+v)^2=(u+v)^2+(200)^2.$

$\therefore u+v=480$ miles.

$\therefore 480+40+480+40+330+130=1500$ miles.

II. **Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.**

Draw a diagram of the traveling, and produce line HA to E in line CD. Represent the city by O. Then OED be a right triangle in which $ED=330-130=200$ miles.

Put $a=$distance from home to city. Let $x=OE$: then $OD=x+a$.

Whence $(x+a)^2=x^2+200^2$.

$$\therefore x=\frac{200^2-a^2}{2a}=\frac{200^2}{2a}-\tfrac{1}{2}a.$$

Now, in order that x may be positive, $\frac{1}{2}a<200^2/2a$; whence $a<200$.

But as the lake is 40 miles wide, a can not be less than 40. Therefore for positive values of x, a may have any value from 40 to 200.

The distance due north $=\dfrac{200^2-a^2}{2a}+a=\dfrac{200^2+a^2}{2a}$; and the entire distance traveled $=\dfrac{200^2+a^2}{a}+460$.

When $a=40$, or if H and O are situated on the lake, the entire distance traveled $=1500$ miles.

When $a=200$, $x=0$, and the city is the farthest north traveled. A would then coincide with O, and C with B.

When $a>200$, x is *negative*. Instead of traveling north from the city, he would then go west from the city to B, and thence *south*, the value of x, to C. For *any* positive value of x, A may be at any point in a due north line between O and E.

Let h, m, c, s, and g be the cost per head, respectively, of horses, mules, cows, sheep, and goats. Then $156h+177m=\$23631$, (1); $468h+235m=\$52245$, (2); $c=\frac{1}{4}h$, (3); $s=\frac{1}{9}m$, (4); and $g=\frac{1}{2}s$, (5). From (1) and (2), $h=\$80$, and $m=\$63$. Whence $c=\$20$, $s=\$7$, and $g=\$3\frac{1}{2}$.

$\therefore$ The stock cost $23631+$52245+$2400+$1750+$1050=$81076.

By selling his stock at a gain of 20%, he gained $\frac{1}{5}$ of $81076=$16215.20.

Also solved by *E. W. MORRELL*, and *JOSIAH H. DRUMMOND, LL. D.*

10. Proposed by I. J. WIRKBACK, M. D., St. Petersburg, Penn.

What is the volume of a segment of a right cone, whose diameter is 6 inches and **ndicular 9 inches? The section being parallel with the perpendicular of the cone** **ncludes ¼ of its circumference at the base.**

I. Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Mass.

Let OBE be section of cone perpendicular to section cutting off segment D. By considering projection of hyperbolic section on parallel plane through the axis, it is seen that asymptotes are intersections of latter plane with con- surface. Accordingly, if $OF=a$, $FA=b$, equation to erbola is

$$a^2y^2-b^2x^2=-a^2b^2.$$

Now $FA=CD=\frac{1}{2}$ side of square inscribed in circle $=\frac{3}{2}\sqrt{2}=b$. $OF:FA=OC:CB$, or $OF:\frac{3}{2}\sqrt{2}=9:3$; $=\frac{9}{2}\sqrt{2}=a$. Substituting in formula for area of hyperbola,

$$AD=(b/a)x\sqrt{x^2-a^2}-ab\log_e\left(\frac{x+\sqrt{x^2-a^2}}{a}\right),$$

$$=\tfrac{1}{3}\times 9\sqrt{81-\tfrac{81}{2}}-27\log_e\left(\frac{9+\sqrt{81-\frac{81}{2}}}{\frac{9}{2}\sqrt{2}}\right),$$

$$=\tfrac{27}{2}\sqrt{2}-\tfrac{27}{2}\log_e(\sqrt{2}+1).$$

ume of conical segment $OAD=\frac{1}{3}CD\times$ area AD, $=\frac{27}{2}-\frac{27}{4}\sqrt{2}\log_e(\sqrt{2}+1)$.
of circular segment $BD=9\pi/4-\frac{9}{2}$, $=\frac{9}{4}(\pi-2)$.
ume of conical segment $OBD=\frac{1}{3}OC\times$ area DB, $=\frac{27}{4}(\pi-2)$.
ume $ABD=$ volume $OBD-$ volume $OAD=\frac{27}{4}(\pi-2)-\frac{27}{2}+\frac{27}{4}\sqrt{2}\log_e(\sqrt{2}+1)$
$=\frac{27}{4}\pi-27+\frac{27}{4}\sqrt{2}\log_e(\sqrt{2}+1)$, $=2.619+$ cubic inches.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, **on, Va.**

Let $AFB-C$ be the cone, $GLDMF$ the section made by the plane cutting the given segment. Let $AB=6=2R$, $OC=9=h$, $=c$. Since $\angle GOF=\pi/2$, $GF=R\sqrt{2}$.

$$\therefore OE=\sqrt{OG^2-GE^2}=\sqrt{R^2-\tfrac{1}{2}R^2}=\tfrac{1}{2}R\sqrt{2}$$

$$=\tfrac{1}{2}(3\sqrt{2})=c.$$

Let $CN=x$, then $CO:OB=CN:NK$.

or $h:R=x:NK$. $\therefore NK=Rx/h=NL$.

Similarly $CP=CO-DF=h-\dfrac{h(R-c)}{R}=\dfrac{hc}{R}$.

Area of segment $LMK=\dfrac{R^2x^2}{h^2}\cos^{-1}\left(\dfrac{ch}{Rx}\right)-\dfrac{c}{h}\sqrt{R^2x^2-h^2c^2}$.

$$\therefore V=\int_{ch/R}^{h}\left\{\frac{R^2x^2}{h^2}\cos^{-1}\left(\frac{ch}{Rx}\right)-\frac{c}{h}\sqrt{R^2x^2-h^2c^2}\right\}dx,$$

$$=\tfrac{1}{3}h\left\{R^2\cos^{-1}\left(\frac{c}{R}\right)-2c\sqrt{R^2-c^2}+\frac{c^3}{R}\log\left(\frac{R+\sqrt{R^2-c^2}}{c}\right)\right\},$$

$$=3\{9\cos^{-1}\left(\frac{1}{\sqrt2}\right)-9+\frac{9\sqrt2}{4}\log(\sqrt2+1)\},\ =2.619 \text{ cubic inches.}$$

III. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

I Solution. Designating the radius of the base by r, the altitude by and choosing the center of the base for the origin of orthogonal coördinates, for the axis of z, the radius OB for the axis of x and a radius parallel to the s tion FDG for that of y, we find the equation of the cone to be

$$z=(h/r)(r-\sqrt{x^2+y^2},$$

and the volume V of $COFGD$

$$=\int_{-\sqrt{r^2-x^2}}^{-\sqrt{r^2+x^2}}zdy=\frac{h}{r}\int_0^x dx\int_{-\sqrt{r^2-x^2}}^{+\sqrt{r^2-x^2}}(r-\sqrt{x^2+y^2})dy,$$

$$=\frac{h}{r}\int_0^x\left(r\sqrt{r^2-x^2}-x^2\log\frac{r+\sqrt{r^2-x^2}}{x}\right)dx,$$

$$=\tfrac{2}{3}hx\sqrt{r^2-x^2}+\tfrac{1}{3}r^2h\sin^{-1}(x/r)-\tfrac{1}{3}\frac{hx^3}{r}\log\frac{r+\sqrt{r^2-x^2}}{x}.$$

Substituting $x=(r/2)\sqrt2$, we have for the volume $COFGD$ the expressi $\frac{r^2h}{12}[4+\pi-\sqrt2.\log(\sqrt2+1)]$, and for that of $B-FGD$ $\frac{r^2h}{12}[\pi-4+\sqrt2.\log(\sqrt2+$

II Solution. Let HK be a circle parallel to AB cutting the hyperb FDK in the points L and M, and let the diameter HK cut the axis DE at Put $OE=b$, $OF=r$, $CO=h$, $DQ=x$, $LQ=y$. We find from the geometry of t figure $y^2=(2br/x)x+(r^2/h^2)x^2$ as the equation of the hyperbola FDK.

$$\therefore \text{Area of } FDK=2\int dx\sqrt{\frac{2br}{h}x+\frac{r^2}{h^2}x^2} \text{ between the limits } O,$$

and $DE=\dfrac{(r-b)h}{r}$. Integrating we find for this area the expression,

$$\frac{h}{b}\left(r\sqrt{r^2-b^2}-b^2\log\frac{r+\sqrt{r^2-b^2}}{b}\right).$$

$$\therefore \text{Volume of } COFGD=\frac{h}{b}\int_0^b\left(r\sqrt{r^2-b^2}=b^2\log\frac{r+\sqrt{r^2-b^2}}{b}\right)db$$

$$=\tfrac{2}{3}hb\sqrt{r^2-b^2}+\tfrac{1}{3}r^2h\sin^{-1}\frac{b}{r}-\tfrac{1}{3}\frac{hb^3}{r}\log\frac{r+\sqrt{r^2-b^2}}{b}\Big),$$

d volume of $BFGD=\frac{1}{3}r^2h\cos^{-1}\frac{b}{r}-\frac{2}{3}hb\sqrt{r^2-b^2}+\frac{1}{3}\frac{hb^3}{r}\log\frac{r+\sqrt{r^2-b^2}}{b}$.

III Solution. Let HK be a circle parallel to AB, and N its centre. hrough N draw a diameter parallel to the hyperbolic section FGD. Put $N=x$, $OE=b$, $BO=r$, $CO=h$, then the area of the circular segment lying be- reen the diameter through N and the parallel chord LM

$$=\frac{r^2x^2}{h^2}\sin^{-1}\frac{bh}{rx}+\frac{br}{h}\sqrt{x^2-\frac{b^2h^2}{r^2}}.$$

$\therefore$ Volume of conical section $COFGD$

$$=\frac{r}{h}\left\{\frac{r}{h}\int x^2\sin^{-1}\frac{bh}{rx}+b\int dx\sqrt{x^2-\frac{b^2h^2}{r^2}}\right\},$$

he integrals to be taken between $h-DE=bh/r$ and h. Thus we find for this olume the expression

$$\tfrac{2}{3}hb\sqrt{r^2-b^2}+\tfrac{1}{3}r^2h\sin^{-1}\frac{b}{r}-\tfrac{1}{3}\frac{hb^3}{r}\log\frac{r+\sqrt{r^2-b^2}}{r};$$

nd for the volume of the conical section $DBFG$,

$$\tfrac{1}{3}r^2h\cos^{-1}\frac{b}{r}-\tfrac{2}{3}hb\sqrt{r^2-b^2}+\tfrac{1}{3}\frac{hb^3}{r}\log\frac{r+\sqrt{r^2-b^2}}{r}.$$

HISTORICAL NOTE. The famous astronomer KEPLER tried hard to find the volume of such conical ctions as the above, but all his efforts proved futile.

Also solved by *GEORGE LILLEY, Ph. D., LL. D.*, and *CHARLES C. CROSS*. Dr. Lilley obtained a merical result of 6.771 cubic inches, and Professor Cross obtained 2.[illegible]79 cubic inches.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

87. Proposed by E. W. MORRELL, A. M., Professor of Mathematics, Montpelier Seminary, Montpelier, Vt.

A and *B* set out from the same place, and in the same direction. *A* travels uniform- 18 miles per day, and after 9 days turns and goes back as far as *B* has traveled during ose 9 days; he then turns again, and, pursuing his journey, overtakes *B* 22½ days after e time they first set out. It is required to find the rate at which *B* uniformly traveled. rom *Greenleaf's Arithmetic.*]

88. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

Find the principal of a note given March 19, 1891, bearing interest at 6%. Pay- ents: September 1, 1892, \$243.50; January 19, 1893, \$6.90: April 13, 1894, \$19.10; Septem- r 19, 1894, \$110.90. Amount due February 22, 1897, \$229.10.

ALGEBRA.

81. Show that $\dfrac{a_1^r}{(a_1-a_2)(a_1-a_3)(a_1-a_4)\dots(a_1-a_n)}$

$$+\frac{a_2^r}{(a_2-a_1)(a_2-a_3)\dots(a_2-a_n)}+\dots\frac{a_n^r}{(a_n-a_1)(a_n-a_2)\dots(a_n-a_{n-1})}$$

is zero if r is less than $n-1$; to 1 if $r=n-1$, and to $a_1+a_2+_3+\dots a_n$ if $r=n$.

[*C. Smith's Treatise on Algebra.*]

82.

$$\left.\begin{array}{l} y^2+yz+z^2=a^2 \\ z^2+zx+x^2=b^2 \\ x^2+xy+y^2=c^2 \end{array}\right\} \text{find } x, y, \text{ and } z.$$

[*Ibid.*]

GEOMETRY.

83. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

θ being variable, find the locus of a point whose coördinates are

$$a\tan(\theta+\alpha),\quad b\tan(\theta+\beta).$$

84. **Proposed by FREDERICK R. HONEY, Ph. B., New Haven, Conn.**

Find the locus of a point which will trisect all arcs having a common chord.

85. **Proposed by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.**

Prove by pure geometry. Give direct proof, if possible.

If the bisectors of two angles of a triangle are equal, the triangle is ososceles.

[From *Wentworth's Plane Geometry*, exercise 43, page 72.]

MECHANICS.

61. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

A body is suspended from a fixed point by an elastic string, which is stretched to double its natural length when the body is in equilibrium. Find how much the body must be depressed, so that when let go, it may just reach the point of suspension.

62. **Proposed by J. SCHEFFER, A. M., Hagerstown, Md.**

A particle of mass m moves in the circumference of an ellipse with constant rate v. It is constrained to move in that circumference by attractive forces in the two foci. To determine the magnitude of these forces.

DIOPHANTINE ANALYSIS.

58. **Proposed by E. S. LOOMIS, Ph. D., Professor of Mathematics in Cleveland West High School, Berea, O.**

"The base of a right-angled triangle is 105; find all the perpendiculars and hypotenuses to fit it, such that their values shall be integers."

59. **Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.**

Find the sum of the mth powers of all the numbers less than P and prime to it, and then by substitution find the sume when $m=1, 2, 3, 4, 5$.

NOTES.

THE IRVING HOPKINS FALLACY.

After my having so recently pointed out in THE AMERICAN MATHEMATICAL ONTHLY (Vol. III., pages 122-123) the fallacy of Professor G. C. Edwards of e University of California in his Elements of Geometry in treating parallells, id in *Science* (N. S. Vol. VI. page 491) the gross blunder made by Andrew '. Phillips and Irving Fisher, professors in Yale University, in their Elements Geometry, could it have been supposed that so respectable a person as Irving opkins would deliberately have published the extended fallacy which has just ›peared in THE AMERICAN MATHEMATICAL MONTHLY (Vol. IV., pages 251-255) ıder the ambitious title "Euclidean Geometry without Disputed Axioms"?

It is a simple *petitio principii*. The question is begged in his Proposition /, which explicitly uses Euclid, III. 31. If any one will turn to III. 31 in any uclid they will find it proved by Euclid, I. 32. But Euclid I. 32 is the famous ıgle-sum proposition, which since 1733 has been known to be equivalent to the ırallel-postulate, the most disputed of all axioms.

In THE AMERICAN MATHEMATICAL MONTHLY's serial *Non-Euclidean Geomry*, (Vol. I., page 346) is given the Proposition : In any right-angled triangle ıe two acute angles remaining are taken together equal to one right angle, in the ypothesis of right angle ; greater than one right angle, in the hypothesis of ob-use angle ; but less in the hypothesis of acute angle. In other words, if the an-le inscribed in a semicircle is right the geometry is Euclidean ; if obtuse, Rie-nannian ; if acute, Lobachevskian. GEORGE BRUCE HALSTED.

Austin, Texas.

There are several errors in Mr. Hopkin's paper on "Euclidean Geometry Without Disputed Axioms," but one is enough to which to call attention. n several places he uses Euclid III., 31, which depends upon I., 32, which de-ends upon I., 29, which depends upon Axiom 12 !

When will we cease trying to accomplish what the masters have found to e impossible ? BENJ. F. YANNEY.

Mt. Union College, Alliance, Ohio.

NOTE ON DR. LILLEY'S ARTICLE IN THE OCTOBER NUMBER.

There is one statement in Professor Lilley's article in the October number ›out which I wish to say a few words. Concerning the quotient 0. I define vision thus : Having given the product of two factors, and one of the factors, find the other factor.

Thus, the product of two factors=12,

One of the factors=0,

The other factor=0, (Lilley).

Hence $0\times0=12$. Do you believe it ?

Take this illustration—

$$
\begin{aligned}
12 \div 3 &= 4 \\
12 \div 2 &= 6 \\
12 \div 1 &= 12 \\
12 \div .1 &= 120 \\
12 \div .01 &= 1200 \\
12 \div .001 &= 12000 \\
12 \div .0001 &= 120000 \\
&\vdots \\
12 \div -.0001 &= -120000 \\
12 \div -.001 &= -12000 \\
12 \div -.01 &= -1200 \\
12 \div -.1 &= -120 \\
12 \div -1 &= -12 \\
12 \div -2 &= -6 \\
12 \div -3 &= -4
\end{aligned}
$$

Here the dividend is constant. The divisor varying continuously, suppose, changes sign in passing through zero (absolute) and at the same time the quotient changes sign in passing through infinity.

The following definition of division may assist in reaching a conclusion: Division is the process of finding how many times a number may be subtracted from another without changing the sign of the remainder.

Apply this definition thus: How many times may zero (absolute) be subtracted from 12 without changing the sign of the remainder.

The answer is, an infinity of infinities, rather than zero.

MILTON L. COMSTOCK.

Knox College, Galesburg, Ill.

Upon some of the points about which I shall disagree with Dr. Lilley he can quote in his favor some of the most brilliant mathematicians that the world has produced. Nevertheless I shall endeavor to show that they and he have failed to take a common sense view of the subject. Upon one point I think I am safe in saying that the Doctor's position is unique. The source of his errors lies, in my opinion, in his conception of infinity and zero.

Concerning the former he says: "If $12/\Phi$=infinity or *the largest possible number*," etc.

From this I can not but infer that he thinks infinity is a constant and that that constant is the largest possible number. He says "$12/\Phi=\Theta$, where Θ represents no number of times." Again we infer that he believes that while Θ represents *no number of times*, ∞ must represent *some number of times*.

He uses too many zeros. He has Φ=absolute zero, Φ=*no number of times*, and ◎=an infinitesimal. He refers to the latter zero as follows: It is a consequence of confounding the 0 arising from dividing a by infinity, with the absolute zero, that so much confusion has arisen."

He has the authority of Davies and Peck's Mathematical Dictionary for his statement, but this does not make it true. Nothing could be more confusing to the average man of common sense than the Doctor's three zeros.

I have no use for more than one. My mind is perfectly clear as to what that is but it is not so clear as to what ∞ is. It is much easier to tell what ∞ is not than what it is.

If we suppose $a/h=N$, where h is a very small positive quantity, then N is a very large one. As h grows smaller and smaller, N grows larger and larger, but N will *not* become infinite *so long as h has the smallest shadow of value.* So long as h has the slightest value we can form some conception of the value of N. It is only when h becomes equal to 0 that N suddenly swings clear out of our powers of conception. It is then, and then only, that it becomes infinite.

I must dissent from even so great a mathematician as Professor De Morgan when he said that he dated his first clear conception of mathematical infinity from the time when he rejected the relation $a/0=\infty$.

The very fact that he had a clear conception of what he called infinity proved that it was not the real infinity.

I have no criticism to make on Dr. Lilley's disposition of 0/0. I would have liked it better if he had added Art. 175 of his Higher Algebra, which reads: "The symbol 0/0 does not always mean *indetermination.* It is often the result of a particular condition which makes a factor, common to both terms of a fraction become zero. Thus," etc. Here follows the well known illustration by using $\dfrac{a^2-x^2}{a-x}=a+x$.

He does not find it necessary to introduce the infinitesimal to prove that the expression equals $2a$ when $x=a$, as do many writers on the differential calculus when discussing the expression $\dfrac{y_1-y}{x_1-x}$. In this he is right, for if $a-x$ were an infinitesimal the value of the fraction would differ from $2a$ by an infinitesimal, and *an equation that differs from the truth by an infinitesimal is not true at all.*

HENRY HEATON.

Atlantic, Iowa.

We see no place for confusion in the use of the symbols 0 and ∞, and, therefore, of course, no necessity of introducing new symbols to avoid confusion. If 0 is a symbol used to denote the absence of quantity, and ∞ to denote a quantity larger than any assignable quantity however large, then all operations with these symbols are meaningless. For example, $5\div\infty$, $0\div5$, $5\div0$, $0\div0$, $0\times\infty$, etc., are impossible operations. Standing apart from conditions imposed upon quantities from which these symbols arise by certain limitations, they have no meaning whatever. Hence, when these symbols do arise in mathematical investigations, they must be interpreted in conformity to fundamental principles and conceptions. When we say that $5\div\infty=0$, we mean that the limit of $5\div$ a quantity which increases indefinitely $=0$, concisely expressed thus $\lim_{h\doteq\infty}\left[\dfrac{5}{h}\right]=0$.

This is an absolutely accurate statement. 0 is the absolute zero and not an *infinitesimal*. In like manner $5\div0=\infty$ is an abbreviated and inaccurate expression for the following: 5 divided by a quantity which decreases indefinitely gives a quotient larger than any quantity however large, or briefly and *accurately* thus $\lim_{e=0}\left[\frac{5}{e}\right]=\infty$.

Discussion on a subject of this sort is trivial, but if it results in giving clearer notions of the use of 0 and ∞, a good work will have been done.

B. F. F.

BOOKS AND PERIODICALS.

Plane and Solid Analytical Geometry. By Frederick H. Bailey, A. M. (Harvard), and Frederick S. Woods, Ph. D. (Göttengen), Assistant Professors of Mathematics in the Massachusetts Institute of Technology. 8vo. Cloth, 371 pages. Boston and Chicago: Ginn & Co.

Besides the usual subjects treated in the ordinary text-books of Analytical Geometry, the following additional ones are treated with sufficient fullness to give a student a fair knowledge of them, viz: Radical Axis, and Properties of Pole and Polars. More attention should be given to these subjects in the future by the ordinary student. Besides deriving the equations of the conics in the usual way, the authors have also derived the equations by passing a plane through a right circular cone, thus emphasizing the relation of the geometrical to the analytical method of treatment. About seventy pages are given to the treatment of Solid Analytical Geometry. The treatment here is clear and concise, affording the student an excellent introduction to this important subject. B. F. F.

Famous Problems of Elementary Geometry.—The Duplication of the Cube: The Trisection of an Angle; and The Quadrature of the Circle. Authorized translation Vorträge Ueber Ausgewälte Fragen der Elementargeometrie Ausgearbeitet von F. Tägert. By Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Eugene Smith, Professor of Mathematics in the Michigan State Normal College. 8vo. Cloth, 80 pages. Price, 55 cents. Boston and Chicago: Ginn & Co.

This book deals with the possibility of elementary geometric constructions in general, the nature of transcendental numbers, and with the transcendence of e and π. While no knowledge of the calculus is needed to read this book, the calculus not being employed in any of the discussions, yet a fair knowledge of the theory of equations and series is absolutely necessary to make it easy reading. The translators deserve the thanks of students and teachers of mathematics, and for putting out books of such scientific value at a very reasonable price, the publishers should receive encouragement by a large sale of this book.

B. F. F.

Popular Scientific Lectures. By Ernst Mach, formerly Professor of Physics in the University of Prague, now Professor of the History and Theory of Inductive Science in the University of Vienna. Translated by Thomas J. McCormack. Second Edition, Revised and Enlarged. 8vo. Cloth, 382 pages. Price, \$1.00. Chicago: The Open Court Publishing Co.

These sixteen lectures on various scientific subjects are full of interest to all classes of readers. The lecture "On the Relative Educational Value of the Classics and the Mathematico-Physical Sciences" is especially interesting, and is a fair exposition of the argument *pro* and *con*.

In acquiring an education two things are requisite: first, the development of thought, and second, the power to express thought in a clear and forcible manner. The first is gained by the study of mathematics and the natural sciences, the second, by the classics. Hence, in securing the most symmetrical and stable development of the mind, it is essential that the student pursue his study in the classics, especially Latin, as well as mathematics and the natural sciences. Dr. Mach makes this very pertinent statement: "Here I may count upon assent when I say that mathematics and the natural sciences pursued alone as means of instruction yield a richer education, an education in matter and form, a more general education, an education better adapted to the needs and spirit of the times, than the philological branches pursued alone would yield." In bringing out the translation of these valuable lectures, the translator has the thanks of English readers.

B. F. F.

Field-Manual for Railroad Engineers. By J. C. Nagle, M. A., M. C. E., Professor of Civil Engineering in the Agricultural and Mechanical College of Texas. 4½x6¾ inches, Flexible Morocco, xv+394 pages. Price, $2.50. New York: John Wiley & Sons.

This book is in every way a model field-manual. It contains six chapters. Chapter I.—Reconnoisance: Chapter II.—Preliminary Surveys: Chapter III.—Location, Art. 7, Projecting Location; Art. 8, Simple Curves; Art. 9, Compound Curves; Art. 10, Track Problems: Chapter IV.—Transition Curves; Art. 11, Theory of the Transition Curve; Art. 12, Field Work; Art. 13, Transition Curve Problems: Chapter V.—Frogs and Switches; Art. 14, Turnouts; Art. 15, Crossovers; Art. 16, Crossing-Frogs and Crossing-Slips: Chapter VI.—Construction; Art. 17, Definitions, General Consideration, Vertical Curves, Elevation of Outer Rail; Art. 18, Earthworks; Art. 19, Grade and Ballast Stakes, Culverts, Bridges, and Tunnels; Art. 20, Monthly and Final Estimates.

The above abridged outline of the table of contents indicates very imperfectly the scope and character of this work. In it may be found the most essential things to be known in civil engineering discussed in a way not only that may be understood, but that can be easily understood, by any one familiar with algebra, geometry, and trigonometry.

B. F. F.

A Chapter in the History of Mathematics. An Address by Vice President W. W. Beman, Chairman of Section A, before the Section of Mathematics and Astronomy, American Association for the Advancement of Science, Detroit Meeting, August, 1897. Pamphlet, 20 pages.

In this very able address by Professor Beman is gathered together some valuable history concerning the introduction in mathematics of the square root of negative numbers. The address bears evidence of careful research, and is of great interest to all who are concerned about the progress and development of that great body of doctrine known as mathematics.

B. F. F.

Darwin and After Darwin: Part II. Post-Darwinian Questions. Heredity and Utility 8vo. Cloth, xii and 344 pages. Price, $1.50. With portrait of Romanes. Chicago: The Open Court Publishing Co.

This, as all of Dr. Romanes' works, bears the evident marks of a profound thinker and scholar. The volume before us is chiefly devoted to a consideration of those Post-Darwinian theories which involve fundamental questions of Heredity and Utility, and contains the most valuable results of a deep study of the evolutionary problem. B. F. F.

The Probability of Hit when the Probable Error in Aim is Known with a Comparison of the Probabilities of Hit by the Method of Independent and Parallel Fires from Mortar Batteries. By Mansfield Merriman, Professor of Civil Engineering in Lehigh University. Pamphlet, 12 pages. Reprinted from the Journal of the U. S. Artillery, Vol. VIII, No. 2.

The problem considered in this paper is, To find the probability of hit on the target or deck of a ship whose area is $4aA$, where $2a$ is the width of the target in azimuth and $2A$ its length in range, a shot being fired with the intention of hitting the center. B. F. F.

Contributions to the Geometry of the Triangle. By Robert J. Aley, A. M., Professor of Mathematics in the University of Indiana. Pamphlet, 32 pages.

This thesis was accepted by the Department of Mathematics of the University of Pennsylvania in partial fulfillment of the requirements for the degree of Doctor of Philosophy, which is a sufficient testimonial of its importance and value. B. F. F.

Periodico di Mathematica Per L'Insegnamento Secondario. Dott. G. Lazzeri. November-December number.

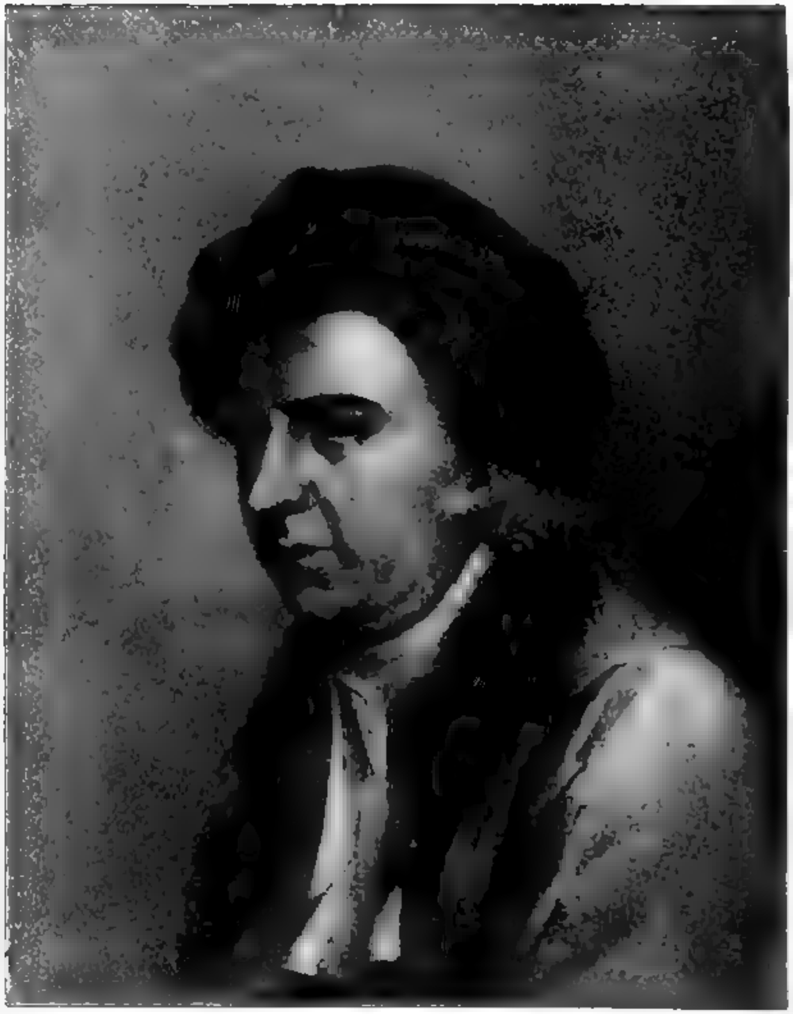
The Mathematical Gazette. Edited by F. S. Macauley, M. A., D. Sc. October number.

Bollettino della Associazione "Mathesis" Fra Gl'Insignanti di Mathematica delle Scuole Medie.

Revue Semestrielle des Publications Mathomatiques Rédigée sous les auspices de la Sociéte Mathematique d'Amsterdam. Par P. H. Schoute, D. J. Korteweg, J. C. Kluyver, W. Kapteyn, P. Zeeman.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.00 per year in advance. Single numbers, 25 cents. The American Monthly Review of Reviews Co., 13 Astor Place, New York.

The December number of the *American Monthly Review of Reviews* has several interesting features. Mr. Ernest Knaufft, editor of the *Art Student*, contributes an elaborate study of "John Gilbert and Illustration in the Victorian Era"; Dr. Clifton H. Levy tells "How the Bible Came Down to Us," with a number of reproductions from ancient Biblical manuscripts and printed texts; Lady Henry Somerset pays a tribute to the late Duchess of Teck; an English officer in the Indian service writes about the Ameer of Afghanistan; Mr. E. V. Smalley discusses Canadian reciprocity, and Mr. Alex. D. Anderson summarizes the progress of the American Republics. There is also a 23-page illustrated department devoted to the season's new books, with an introductory chapter, by Albert Shaw, on "Some American Novels and Novelists." Altogether, the *Review* is not lacking in novelty or variety. B. F. F.



LEONHARD EULER

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BIOGRAPHY.

LEONHARD EULER.

BY B. F. FINKEL.

LEONHARD EULER (oi'ler), one of the greatest and most prolific mathematicians that the world has produced, was born at Basel, Switzerland, on the 15th day of April, 1707, and died at St. Petersburg, Russia, November the 18th (N. S.), 1783. Euler received his preliminary instruction in mathematics from his father who had considerable attainments as a mathematician, and who was a Calvinistic* pastor of the village of Riechen, which is not far from Basel. He was then sent to the University of Basel where he studied mathematics under the direction of John Bernoulli, with whose two sons, Daniel and Nicholas, he formed a life-long friendship. Geometry soon became his favorite study. His genius for analytical science soon gained for him a high place in the esteem of his instructor, John Bernoulli, who was at the time one of the first mathematicians of Europe. Having taken his degree as Master of Arts

*_The Encyclopedia Brittanica_ says Euler's father was a Calvinistic minister, while W. W. R. Ball, in his _History of Mathematics_, says he was a Lutheran minister. Euler himself was a Calvinist in doctrine, as the following, which is his apology for prayer, indicates: "I remark, first, that when God established the course of the universe, and arranged all the events which must come to pass in it, he paid attention to all the circumstances which should accompany each event; and particularly to the dispositions, to the desires, and prayers of every intelligent being; and that the arrangement of all events was disposed in perfect harmony with all these circumstances. When, therefore, a man addresses God a prayer worthy of being heard it must not be imagined that such a prayer came not to the knowledge of God till the moment it was formed. That prayer was already heard from all all eternity; and if the Father of Mercies deemed it worthy of being answered, he arranged the world expressly in favor of that prayer, so that the accomplishment should be a consequence of the natural course of events. It is thus that God answers the prayers of men without working a miracle."

in 1723, Euler afterwards applied himself, at his father's desire, to the study of theology and the Oriental languages, with the view of entering the ministry, but, with his father's consent, he returned to his favorite pursuit, the study of mathematics. At the same time, by the advice of the younger Bernouillis, who had removed to St. Petersburg in 1725, he applied himself to the study of physiology, to which he made useful applications of his mathematical knowledge; he also attended the lectures of the most eminent professors of Basel. While he was eagerly engaged in physiological researches, he composed a dissertation on the nature and propagation of sound. In his nineteenth year he also composed a dissertation in answer to a prize-question concerning the masting of ships, for which he received the second prize from the French Academy of Sciences.

When his two close friends, Daniel and Nicholas Bernoulli, went to Russia, they induced Catherine I, in 1727, to invite Euler to St. Petersburg, where Daniel, in 1733, was assigned to the chair of mathematics. Euler took up his residence in St. Petersburg, and was made an associate of the Academy of Sciences. In 1730 he became professor of physics, and in 1733 he succeeded his friend Daniel Bernoulli, who resigned on a plea of ill health.

At the commencement of his astonishing career, he enriched the Academical collection with many memoirs, which excited a noble emulation between him and the Bernouillis, though this did not in any way affect their friendship. It was at this time that he carried the integral calculus to a higher degree of perfection, invented the calculation of sines, reduced analytical operations to greater simplicity, and threw new light on nearly all parts of pure or abstract mathematics. In 1735, an astronomical problem proposed by the Academy, for the solution of which several eminent mathematicians had demanded several months' time, was solved by Euler in three days with the aid of improved methods of his own, but the effort threw him into a fever which endangered his life and deprived him of his right eye, his eyesight having been impaired by the severity of the climate. With still superior methods, this same problem was solved later by the illustrious German mathematician, Gauss.

In 1741, at the request, or rather command, of Frederick the Great, he moved to Berlin, where he was made a member of the Academy of Sciences, and Professor of Mathematics. He enriched the last volume of the *Mélanges* or Miscellanies of Berlin, with five memoirs, and these were followed, with astonishing rapidity, by a great number of important researches, which were scattered throughout the annual memoirs of the Prussian Academy. At the same time, he continued his philosophical contributions to the Academy of St. Petersburg, which granted him a pension in 1742.

The respect in which he was held by the Russians was strikingly shown in 1760, when a farm he occupied near Charlottenburg happened to be pillaged by the invading Russian army. On its being ascertained that the farm belonged to Euler, the general immediately ordered compensation to be paid, and the Empress Elizabeth sent an additional sum of four thousand crowns. The despotism of Anne I. caused Euler, who was a very timid man, to shrink from public affairs, and to devote all his time to science. After his call to Berlin, the Queen of Prussia who received him kindly, wondered how so distinguished a scholar should be so timid and reticent. Euler replied, "Madam, it is because I come from a country where, when one speaks, one is hanged."

In 1766, Euler, with difficulty, obtained permission from the King of Prussia to return to St. Petersburg, to which he had been originally called by Catherine II. Soon after returning to St. Petersburg a cataract formed in his left eye, which ultimately deprived him of sight, but this did not stop his wonderful literary productiveness, which continued for seventeen years—until the day of his death. It was under these circumstances that he dictated to his amanuensis, a tailor's apprentice who was absolutely devoid of mathematical knowledge, his *Anleitung zur Algebra*, or *Elements of Algebra*, 1770, a work which, though purely elementary, displays the mathematical genius of its author, and is still considered one of the best works of its class. Euler was one of the very few great mathematicians who did not deem it beneath the dignity of genius to give some attention to the recasting of elementary processes and the perfecting of elementary text-books, and it is not improbable that modern mathematics is as greatly indebted to him for his work along this line as for his original creative work.

Another task to which he set himself soon after returning to St. Petersburg was the preparation of his *Lettres à une Princesse d'Allemagne sur quelques sujects de Physique*, (3 vols. 1768-72). These letters were written at the request of the princess of Anhalt-Dessau, and contain an admirably clear exposition of the principal facts of mechanics, optics, acoustics, and physical astronomy. Theory, however, is frequently unsoundly applied in it, and it is to be observed generally that Euler's strength lay rather in pure than in applied mathematics. In 1755, Euler had been elected a foreign member of the Academy of Sciences at Paris, and sometime afterwards the academical prize was adjudged to three of his memoirs *Concerning the Inequalities in the Motions of the Planets*. The two prize-problems proposed by the same Academy in 1770 and 1772 were designed to obtain a more perfect theory of the moon's motion. Euler, assisted by his eldest son, Johann Albert. was a competitor for these prizes and obtained both. In his second memoir, he reserved for further consideration the several inequalities of the moon's motion, which he could not determine in his first theory on account of the complicated calculations in which the method he then employed had engaged him. He afterward reviewed his whole theory with the assistance of his son and Krafft and Lexell, and pursued his researches until he had constructed the new tables, which appeared with the great work in 1772. Instead of confining himself, as before, to the fruitless integration of three differential equations of the second degree, which are furnished by mathematical principles, he reduced them to three ordinates which determine the place of the moon ; and he divides into classes all the inequalities of that planet, as far as they depend either on the elongation of the sun and moon, or upon the eccentricity, or the parallax, or the inclination of the lunar orbit. The inherent difficulties of this task were immensely enhanced by the fact that Euler was virtually blind, and had to carry all the elaborate computations involved in his memory. A further difficulty arose from the burning of his house and the destruction of a greater part of his property in 1771. His manuscripts were fortunately preserved. His own life only was saved by the courage of a native of Basel, Peter Grimmon, who carried him out of the burning house.

Some time after this, the celebrated Wenzell, by couching the cataract, restored his sight; but a too harsh use of the recovered faculty, together with some carelessness on the part the surgeons, brought about a relapse. With the assistance of his sons, and of Krafft and Lexell, however, he continued his labors, neither the loss of his sight nor the infirmities of an advanced age being sufficient to check his activity. Having engaged to furnish the Academy of St. Petersburg with as many memoirs as would be sufficient to complete its acts for twenty years after his death, he in seven years transmitted to the Academy above seventy memoirs, and left above two hundred more, which were revised and completed by another hand.

Euler's knowledge was more general than might have been expected in one who had pursued with such unremitting ardor, mathematics and astronomy, as his favorite studies. He had made considerable progress in medicine, botany, and chemistry, and he was an excellent classical scholar and extensively read in general literature. He could repeat the *Ænied* of Virgil from the beginning to the end without hesitation, and indicate the first and last line of every page of the edition which he used. But such lines from Virgil as, "The anchor drops, the rushing keel is staid," always suggested to him a problem and he could not help enquiring what would be the ship's motion in such a case.

Euler's constitution was uncommonly vigorous and his general health was always good. He was enabled to continue his labors to the very close of his life so that it was said of him, that he ceased to calculate and to breath at nearly the same moment. His last subject of investigation was the motions of balloons, and the last subject on which he conversed was the newly-discovered planet Herschel.

On the 18th of September, 1783, while he was amusing himself at tea with one of his grandchildren, he was struck with apoplexy, which terminated the illustrious career of this wonderful genius, at the age of seventy-six. His works, if printed in their completeness, would occupy from 60 to 80 quarto volumes. However, no complete edition of Euler's writings has been published, though the work has been begun twice.

He was simple, upright, affectionate, and had a strong religious faith. His single and unselfish devotion to the truth and his joy at the discoveries of science whether made by himself or others, were striking attributes of his character. He was twice married, his second wife being a half-sister of his first, and he had a numerous family, several of whom attained to distinction. His *éloge* was written for the French Academy by Condorcet, and an account of his life, with a list of his works, was written by Von Fuss, the secretary of the Imperial Academy of St. Petersburg.

As has been said, Euler wrote an immense number of works, chief of which are the following: *Introductio in Analysin infinitorum*, 1748, which was intended to serve as an introduction to pure analytical mathematics. This work produced a revolution in analytical mathematics, as the subject of which it treated had hitherto never been presented in so general and systematic a manner. The first part of the Analysis Infinitorum contains the bulk of the matter which is to be found in modern text-books on algebra, theory of equations, and trigonometry. In the algebra, he paid particular attention to the expansion of various functions in series, and to the summation of given series; and pointed out explicitly that an infinite series can not be safely employed in mathematical investigations unless it is convergent. In trigonometry, he introduced (simultaneously with Thomas Simpson in England) the now current abbreviations for trigonometric functions, and simplified formulæ by the simple expedient of designating the angles of a triangle by A, B, C, and the opposite sides by a, b, c. He also showed that the trigonometrical and exponential functions are connected by the the relation $\cos\theta + i\sin\theta = e^{i\theta}$. Here too we meet the symbol e used to denote the base of the Naperian logarithms, namely the incommensurable number 2.7182818 . . . and the symbol π used to denote the incommensurable number 3.14159265 . . . The use of a single symbol to denote the number 2.7182818 . . . seemes to be due to Cotes, who denoted it by M. Newton was probably the first to employ the literal exponential notation, and Euler using the form a^z, had taken a as the base of any system of logarithms. It is probable that the choice of e for a particular base was determined by its being the vowel consecutive to a, or, still more probable because e is the initial of the word *exponent*.

The use of a single symbol to denote 3.14159265 . . . appears to have been introduced by John Bournilli, who represented it by c. Euler in 1734 denoted it by p, and in a letter of 1736 in which he enunciated the theorem that the sum of the square of the reciprocals of the natural numbers is $\frac{1}{6}\pi^2$. he uses the letter c. Chr. Goldbach in 1742 used π, and after the publication of Euler's Analysis, the sympol π was generally employed, the choice of π being determined by the initial of the word, περιφέρεια=*periphereia*.

The second part of the *Analysis Infinitorum* is on analytical geometry. Euler begins this part by dividing curves into algebraic and transcendental, and establishes a number of propositions which are true for all algebraic curves. He then applied these to the general equation of the second degree in two dimensions, showed that it represents the various conic sections, and deduces most of their properties from the general equation. He also considered the classification of cubic, quartic, and other algebraic curves. He next discussed the question as to what surfaces are represented by the general equation of the second degree in three dimensions, and how they may be discriminated one from the other. Some of these surfaces had not been previously investigated. In this work he also laid down the rules for the transformation of coördinates in space. Here also we find the first attempt to bring the curvature of surfaces within the domain of mathematics, and the first complete discussion of tortuous curves.

In 1755 appeared *Institutiones Calculi Differentialis*, to which the *Analysis Infinitorum* was intended as an introduction. This is the first text-book on the differential calculus which has any claim to be regarded as complete, and it may be said that most modern treatises on the subject are based upon it.

At the same time, the exposition of the principles of the subject is often prolix and obscure, and sometimes not quite accurate.

This series of works was completed by the publication in three volumes in 1768 to 1770 of the *Institutiones Calculi Integralis*, in which the results of several of Euler's earlier memoirs on the same subjects and on differential equations are included. In this treatise as in the one on the differential calculus was summed up all that was at that time known on the subject. The beta and gamma functions were invented by Euler, and are discussed here, but only as methods of reduction and integration. His treatment of elliptic integrals is superficial. The classic problems on isoperimetrical curves, the brachistochrone in a resisting medium, and theory of geodesics had engaged Euler's attention at an early date, and the solving of which led him to the calculus of variations. The general idea of this was laid down in his *Curvarum Maximi Minimive Proprietate Gaudentium Inventio Nova ac Facilis*, published in 1744, but the complete development of the new calculus was first effected by Lagrange in 1759. The method used by Lagrange is described in Euler's integral calculus, and is the same as that given in most modern text-books on the subject.

In 1770, Euler published the *Anleitung zur Algebra* in two volumes. The first volume treats of determinate algebra. This work includes the proof of the binomial theorem for any index, which is still known by Euler's name. The proof, which is not accurate according to the modern views of infinite series, depends upon the principle of the permanence of equivalent forms, and may be seen in C. Smith's *Treatise on Algebra*, pages 336-7. Euler's proof with important additions due to Cauchy, may be seen in G. Chrystal's Algebra, Part II.

It is a fact worthy of note that Euler made no attempt to investigate the convergency of the series, though he clearly recognized the necessity of considering the convergency of infinite series. While Euler recognized the convergency of series, his conclusions in reference to infinite series are not always sound. In his time no clear notion as to what constitutes a convergent series existed, and the rigid treatment to which infinite series are now subjected was undreamed of. Euler concluded that the sum of the oscillating series $1-1+1-1+1-1+\dots=\frac{1}{2}$, for the reason, that by stopping with an even number of terms the sum is 0, and by stopping with an odd number of terms the sum is 1. Hence, the sum of the series is $\frac{1}{2}(0+1)=\frac{1}{2}$. Guido Grandi went so far as to conclude that $\frac{1}{2}=0+0+0+0\dots$ The paper in which Euler cautions against divergent series contains the proof that $\dots\frac{1}{n^2}+\frac{1}{n}+1+n+n^2+n^3\dots=0$. His proof is as follows, $n+n^2+n^3+\dots=\frac{n}{1-n}$, $1+\frac{1}{n}+\frac{1}{n^2}+\dots=\frac{n}{n-1}$. $\frac{n}{n-1}+\frac{n}{1-n}=0$. Euler had no hesitation in writing $1-3+5-7+9-\dots=0$, and he confidently believed that $\sin\phi-2\sin 2\phi+3\sin 3\phi-\dots=0$.

A remarkable development, due to Euler, is what he named the hypergeometrical series, the summation of which he observed to be dependent upon the integration of linear differential equations of the second order, but it remained for Gauss to point out that for special values of the letters, this series represented nearly all the functions then known. By giving the factors 641×6700417 of the number $2^{2^n}+1=4294967297$ when $n=5$, he pointed out the fact that this expression did not always represent primes, as was supposed by Fermat.

The sources from which this biography has been obtained are Cajori's and *History of Mathematics*, and the *Encyclopedia Britannica*.

MOMENTS OF INERTIA.

By G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

It is the purpose of this paper to put on record formulæ for the Moments Inertia of the plane areas, $\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}=1$, and the solid bounded the surface, $\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}+\left(\frac{z}{c}\right)^{\frac{2}{2p+1}}=1$.

Let μ be the mass of a unit, (a) area, (b) volume.

(a) Areas, when n and m are positive integers.

For the x-axis,

$$I=4\mu\int\int y^2dxdy=\frac{\frac{4ab^3\mu}{4}}{(2m+1)(2n+1)}\cdot\frac{\Gamma(m+\frac{1}{2})\Gamma(3n+\frac{3}{2})}{\Gamma(m+3n+3)}$$

$$=\frac{\mu ab^3(2m+1)(2n+1)(6n+1)\Gamma(m+\frac{1}{2})\Gamma(3n+\frac{1}{2})}{2(m+3n+2)(m+3n)(m+3n+1)\Gamma(m+3n)}\text{.............}(1)$$

$$=\frac{1.3.5......(2m+1)\times 1.3.5.....(6n+1)}{2.4.6......2(m+3n+2)}\,.\,2\pi\mu ab^3(2n+1)\,.....\,(2).$$

For the y-axis,

$$=4\mu\int\int x^2dxdy=\frac{\mu a^3b(2m+1)(2n+1)(6m+1)\Gamma(3m+\frac{1}{2})\Gamma(n+\frac{1}{2})}{2(3m+n+2)(3m+n+1)(3m+n)\Gamma(3m+n)}\text{........}(3)$$

$$=\frac{1.3.5......(6m+1)\times 1.3.5......(2n+1)}{2.4.6.....2(3m+n+2)}\,.\,2\pi\mu a^3b(2m+1)....\,(4).$$

For an axis through its center perpendicular to its plane,

$$I_2=I+I_1\text{............................}(5).$$

The product of inertia of a quadrant about its axes is,

$$=\mu\int\int xydxdy=\frac{\frac{\mu a^2b^2}{4}}{(2m+1)(2n+1)}\cdot\frac{\Gamma(2m+2)\Gamma(2n+1)}{\Gamma(2m+2n+3)}$$

$$=\frac{\mu a^2b^2mn(2m+1)(2n+1)\Gamma(2m)\Gamma(2n)}{4(m+n+1)(m+n)(2m+2n+1)\Gamma(2m+2n)}\text{...............}(6)$$

$$=\frac{1.2.3.4\ldots\ldots(2m+1)\times 1.2.3.4\ldots\ldots(2n+1)}{1.2.3.4\ldots\ldots(2m+2n+2)}\cdot\frac{\mu a^2 b^2}{4}\ldots\ldots\ldots(7).$$

Let $m=n=0$. Then for the ellipse, $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$.

$$I=\tfrac{1}{4}\pi\mu ab^3,\quad I_1=\tfrac{1}{4}\pi\mu a^3 b,\quad I_2=\tfrac{1}{4}\pi\mu ab(a^2+b^2),\quad p=\tfrac{1}{8}\mu a^2 b^2.$$

Let $m=n=1$. Then for the hypocycloid, $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

$$I=\tfrac{21}{512}\pi\mu ab^3,\quad I_1=\tfrac{21}{512}\pi\mu a^3 b,\quad I_2=\tfrac{21}{512}\pi\mu ab(a^2+b^2),\quad p=\tfrac{1}{80}\mu a^2 b^2.$$

(b) Solids, when m, n, and p are positive integers.

With regard to the plane (yz),

$$\mathbf{I}=8\mu\int\int\int x^2\,dx\,dy\,dz$$

$$=\frac{8\mu a^3 bc}{\dfrac{8}{(2m+1)(2n+1)(2p+1)}}\cdot\frac{\Gamma(3m+\frac{3}{2})\Gamma(n+\frac{1}{2})\Gamma(p+\frac{1}{2})}{\Gamma(3m+n+p+\frac{5}{2})}$$

$$=\frac{4\mu a^3 bc(2m+1)(2n+1)(2p+1)(6m+1)\Gamma(3m+\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(p+\frac{1}{2})}{(6m+2n+2p+5)(6m+2n+2p+3)(6m+2n+2p+1)\Gamma(3m+n+p+\frac{1}{2})}\ldots(8)$$

$$=\frac{1.3.5\ldots\ldots(6m+1)\times 1.3.5\ldots\ldots(2n+1)\times 1.3.5\ldots\ldots(2p+1)}{1.3.5\ldots\ldots(6m+2n+2p+5)}$$

$$\times 4\mu\pi a^3 bc(2m+1)\ldots\ldots(9).$$

With regard to the plane (xz),

$$\mathbf{I}_1=8\mu\int\int\int y^2\,dx\,dy\,dz$$

$$=\frac{4\mu ab^3 c(2m+1)(2n+1)(2p+1)(6n+1)\Gamma(m+\frac{1}{2})\Gamma(3n+\frac{1}{2})\Gamma(p+\frac{1}{2})}{(2m+6n+2p+5)(2m+6n+2p+3)(2m+6n+2p+1)\Gamma(m+3n+p+\frac{1}{2})}\ldots(10)$$

$$=\frac{1.3.5\ldots\ldots(2m+1)\times 1.3.5\ldots\ldots(6n+1)\times 1.3.5\ldots\ldots(2p+1)}{1.3.5\ldots\ldots(2m+6n+2p+5)}$$

$$\times 4\mu\pi ab^3 c(2n+1)\ldots\ldots(11).$$

With regard to the plane (xy),

$$\mathbf{I}_2=8\mu\int\int\int z^2\,dx\,dy\,dz$$

$$=\frac{4\mu abc^3(2m+1)(2n+1)(2p+1)(6p+1)\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(3p+\frac{1}{2})}{(2m+2n+6p+5)(2m+2n+6p+3)(2m+2n+6p+1)\Gamma(m+n+3p+\frac{1}{2})}\ldots(12)$$

$$\frac{.3.5\ldots\ldots(2m+1)\times 1.3.5\ldots\ldots(2n+1)\times 1.3.5\ldots\ldots(6p+1)}{1.3.5\ldots\ldots(2m+2n+6p+5)}$$

$$\times 4\mu\pi abc^3(2p+1)\ldots\ldots(13).$$

$\mathbf{I}_3=\mathbf{I}+\mathbf{I}_1$, for z-axis, $\mathbf{I}_4=\mathbf{I}+\mathbf{I}_2$, for y-axis,

$\mathbf{I}_5=\mathbf{I}_1+\mathbf{I}_2$, for x-axis, $\mathbf{I}_6=\mathbf{I}+\mathbf{I}_1+\mathbf{I}_2$, for center.

Product of inertia of an octant of the solid with regard to the (y, z) axes,

$$\mu\int\int\int yz\,dx\,dy\,dz=\frac{\frac{\mu ab^2c^2}{8}}{(2m+1)(2n+1)(2p+1)}\cdot\frac{\Gamma(m+\frac{1}{2})\Gamma(2n+1)\Gamma(2p+1)}{\Gamma(m+2n+2p+\frac{7}{2})}.$$

$$\frac{4\mu ab^2c^2np(2m+1)(2n+1)(2p+1)\Gamma(m+\frac{1}{2})\Gamma(2n)\Gamma(2p)}{2m+4n+4p+5)(2m+4n+4p+3)(2m+4n+4p+1)\Gamma(m+2n+2p+\frac{1}{2})}\ldots(14)$$

$$\frac{2.3\ldots\ldots(2n+1)\times 1.2.3\ldots\ldots(2p+1)\times\frac{1}{2}.\frac{3}{2}.\frac{5}{2}\ldots\ldots\left(\frac{2m+1}{2}\right)}{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}.\frac{7}{2}\ldots\ldots\left(\frac{2m+4n+4p+5}{2}\right)}\cdot\frac{\mu ab^2c^2}{4}$$

$$=\frac{1.2.3.4\ldots\ldots(2n+1)\times 1.2.3.4\ldots\ldots(2p+1)}{(2m+3)(2m+5)\ldots\ldots(2m+4n+4p+5)}\cdot\mu ab^2c^2.2^{2(n+p)}\ldots.(15).$$

With regard to the axes (x, z),

$$P_1=\mu\int\int\int xz\,dx\,dy\,dz.$$

$$\frac{4\mu a^2bc^2mp(2m+1)(2n+1)(2p+1)\Gamma(2m)\Gamma(n+\frac{1}{2})\Gamma(2p)}{4m+2n+4p+5)(4m+2n+4p+3)(4m+2n+4p+1)\Gamma(2m+n+2p+\frac{1}{2})}\ldots(16)$$

$$=\frac{2^{2(m+p)}1.2.3.4\ldots\ldots(2m+1)\times 1.2.3.4\ldots\ldots(2p+1)}{(2n+3)(2n+5)\ldots\ldots(4m+2n+4p+5)}\mu a^2bc^2\ldots\ (17).$$

With regard to the axes (x, y),

$$P_2=\mu\int\int\int xy\,dx\,dy\,dz$$

$$\frac{4\mu a^2b^2cmn(2m+1)(2n+1)(2p+1)\Gamma(2m)\Gamma(2n)\Gamma(p+\frac{1}{2})}{4m+4n+2p+5)(4m+4n+2p+3)(4m+4n+2p+1)\Gamma(2m+2n+p+\frac{1}{2})}\ldots(18)$$

$$=\frac{2^{2(m+n)}1.2.3.4\ldots\ldots(2m+1)\times 1.2.3.4\ldots\ldots(2n+1)}{(2p+3)(2p+5)\ldots\ldots(4m+4n+2p+5)}\cdot\mu a^2b^2c\ldots(19).$$

Let $m=n=p=0$. Then for $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$,

$\mathbf{I}=\frac{4}{15}\mu\pi a^3bc$, $\mathbf{I}_1=\frac{4}{15}\mu\pi ab^3c$, $\mathbf{I}_2=\frac{4}{15}\mu\pi abc^3$,

$\mathbf{I}_3=\frac{4}{15}\mu\pi abc(a^2+b^2)$, $\mathbf{I}_4=\frac{4}{15}\mu\pi abc(a^2+c^2)$,

$\mathbf{I}_5=\frac{4}{15}\mu\pi abc(b^2+c^2)$, $\mathbf{I}_6=\frac{4}{15}\mu\pi abc(a^2+b^2+c^2)$,

$P=\frac{1}{15}\mu ab^2c^2$, $P_1=\frac{1}{15}\mu a^2bc^2$, $P_2=\frac{1}{15}\mu a^2b^2c$.

Let $m=n=p=1$. Then for $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1$,

$\mathbf{I}=\frac{4}{715}\mu\pi a^3bc$, $\mathbf{I}_1=\frac{4}{715}\mu\pi ab^3c$, $\mathbf{I}_2=\frac{4}{715}\mu\pi abc^3$,

$\mathbf{I}_3=\frac{4}{715}\mu\pi abc(a^2+b^2)$, $\mathbf{I}_4=\frac{4}{715}\mu\pi abc(a^2+c^2)$,

$\mathbf{I}_5=\frac{4}{715}\mu\pi abc(b^2+c^2)$, $\mathbf{I}_6=\frac{4}{715}\mu\pi abc(a^2+b^2+c^2)$,

$P=\frac{64\mu ab^2c^2}{15.13.11.7.5}$, $P_1=\frac{64\mu a^2bc^2}{15.13.11.7.5}$, $P_2=\frac{64\mu a^2b^2c}{15.13.11.7.5}$.

Thus we could multiply examples without number.

Formulæ (1), (3), (6), (8), (10), (12), (14), (16), (18), will hold for m, n, p fractional as well as integral.

For the radius of gyration we have

$$k_n^2=\frac{I_n}{\mu A},\quad K_n^2=\frac{\mathbf{I}_n}{\mu V},$$

where A and V are known, (see AMERICAN MATHEMATICAL MONTHLY, page 380, Vol. I., No. 11.)

A SIMPLE DEDUCTION OF THE DIFFERENTIAL OF LOGx.

By J. W. NICHOLSON, A. M., LL. D., Professor of Mathematics in Louisiana State University.

Let $f(x)=\log x$ ....(1), then $f(xy)=f(x)+f(y)$ ....(2).

Differentiate, $f'(xy)(ydx+xdy)=f'(x)dx+f'(y)dy$ ....(3).

Since (3) is true when x and y are independent,

$f'(xy)ydx=f'(x)dx$ ....(4), and $f'(xy)xdy=f'(y)dy$. ....(5).

$$(4)\div(5),\quad \frac{f'(x)}{f'(y)}=\frac{y}{x}=\frac{1/x}{1/y}\ \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

$\therefore f'(x)=\frac{m}{x}$, $f'(y)=\frac{m}{y}$ ....(7). $\therefore$ $d\log x=\frac{m}{x}dx$ ......(8).

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

ORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from November Number.]

PROPOSITION XXX. *To any terminated straight AB stands at right angles . 36.) a certain unbounded straight BX. I say firstly, that the straight AY, ed perpendicularly toward the same parts upon will be one intrinsic limit of all those straights, h drawn from the point A out toward the same s have (in hypothesis of acute angle) a common endicular in two distinct points with the other unded straight BX. I say secondly that no e angle will be the minimum of all, produced r which a straight from the aforesaid point A he aforesaid hypothesis) has in two distinct ts a common perpendicular with BX.*

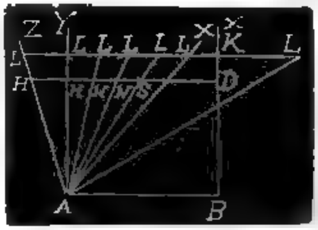

Fig. 36.

Proof of the first part.

For since AY has in common at two distinct points A and B the perpen-lar AB with BX; if any straight AZ is drawn toward the same parts under btuse angle, it follows there can be toward these parts in two distinct points ommon perpendicular to AZ, BX. Otherwise from the resulting quadrilat-containing four angles greater than four right angles, we hit (from Proposi-XVI.) upon the already rejected hypothesis of obtuse angle, against the hy-esis of acute angle in this place assumed.

Therefore that perpendicular AY will be from that side an intrinsic limit of he straights which drawn from the point A toward the same parts have (in hypothesis of acute angle) at two distinct points a common perpendicular the other unbounded straight BX. Quod erat primum.

Proof of the second part.

For if it were possible, let a certain acute angle be the least of all, drawn er which AN has with BX in two distinct points the common perpendicular Then in BX a higher point K being assumed, from this erect to BX the endicular KL, upon which from the point A let fall (by Euclid I. 12) the endicular AL.

But now, if this AL meets ND in any point S, it certainly follows that e BAL will be less than BAN, which therefore will not be the least of all rn under which AN has with BX in two distinct points a common perpen-lar ND.

But furthermore that the aforesaid perpendicular ND is cut by this per-licular AL in some intermediate point of it S is thus demonstrated.

And first indeed, that BK cannot be cut by AL in any point M follows absolutely from Euclid I. 17, since otherwise in the same triangle MKL we would have two right angles at the points K and L, apart from the fact that in this case we would have our assertion about that angle BAN, that it is not in such circumstances the least of all.

But again AL cannot be the continuation of AN; because otherwise in the quadrilateral $NDKL$ we would have four right angles, against the hypothesis of acute angle.

But neither can it cut DN produced in any exterior point H; because angle AHN (from Euclid I. 16) would be acute, on account of the external angle AND supposed right; and therefore angle DHL would be obtuse, and so in the quadrilateral $DHLK$ we would have four angles, which taken together would be greater than four right angles, against the aforesaid hypothesis of acute angle.

Therefore it follows that the angle BAN must be cut by this AL, and therefore cannot be declared the least of all, drawn under which AN has with BX in two distinct points a common perpendicular ND.

Quod erat secundo loco demonstrandum. Itaque constat etc.

Corollary. But hence is permitted to observe, that under a lesser angle BAL is obtained (in hypothesis of acute angle) a common perpendicular LK, more remote indeed from the base AB, as follows from the construction, but moreover less than the other nearer common perpendicular ND, which is obtained under a greater angle BAN.

The reason of this latter is because in the quadrilateral $LKDS$ the angle at the point S is acute in the aforesaid hypothesis, since the three remaining angles are supposed right.

Wherefore (from Corollary I. to Proposition III.) the side LK will be less than the opposite side SD, and so much less than the side ND.

[To be Continued.]

SOPHUS LIE'S TRANSFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

By EDGAR ODELL LOVETT, Princeton University.

III.

Construction of a One Parameter Group from an Infinitesimal Transformation.

9. Let there be given the one parameter continuous group

$$x_1=\varphi(x,\ y,\ a),\quad y_1=\psi(x,\ y,\ a)\ ; \qquad (1)$$

assume further that it contains the inverse transformation of every transformation in it, *i. e.* that the solutions of the equations (1) with regard to x and y have the form

$$x=\varphi(x_1, y_1, b), \quad y=\psi(x_1, y_1, b),$$

in which b is a canstant depending only on a. In the preceding paragraphs the theorem of LIE that every one parameter group whose transformations are inverse in pairs contains an infinitesimal transformation was arrived at both geometrically and analytically. Either process may be formulated symbolically as follows. If T_a represent the transformation of the group corresponding to the parameter a, its inverse T_a^{-1} is also contained in (1) by hypothesis. Further $T_{a+\delta a}$ will represent the transformation corresponding to the parameter $a+\delta a$, and therefore the transformation of the group (1) that differs from T_a by an infinitesimal. The successive application or the product of $T_{a+\delta a}$ and T_a^{-1}, namely $T_{a+\delta a}T_a^{-1}$ (which belongs to the group by virtue of our first supposition that the product of any two transformations of the group is itself a transformation of the group), differs infinitesimally from $T_aT_a^{-1}$, the identical transformation, and hence is itself an infinitesimal transformation belonging to the group (1).

10. On the other hand there is always a completely determinate continuous group of transformations which contains a given infinitesimal transformation. The truth of this assertion may be made to appear symbolically in the following manner.

Let S be any arbitrary transformation in the xy-plane. Construct the transformations which are equivalent to the repetition of S once, twice, and so on to n-times ; also the inverse of S, S^{-1}, and those equivalent to the repetition of this inverse once, twice, and so on to n-times ; we then have an infinite family of transformations,

$$\ldots, S^{-n}, \ldots, S^{-2}, S^{-1}, S^0, S^1, S^2, \ldots, S^n, \ldots,$$

where S^0 is the identical transformation, while n represents every possible positive whole number. This infinite family is a group, since if p and q are two positive or negative numbers, the product of S^p and S^q is equivalent to S^{p+q}, but the group is a discontinuous one.

In this manner, beginning with an arbitrary transformation S an infinite number of discontinuous groups in x and y may be constructed. Passing now to the limiting case, if, in particular, S is an infinitesimal transformation, then S^n and S^{n+1} differ from each other by an infinitesimal, and we have accordingly a continuous group constructed from, and containing the infiniteslmal transformation, S.

11. LIE has invented an ingenious kinematical illustration of this limiting case, which serves as a concrete introduction to the rigorous demonstration of the theorem.

The infinitesimal transformation is defined by two equations of the form

$$x'=x+\xi(x, y)\delta t+\ldots, \quad y'=y+\eta(x, y)\delta t\ldots, \qquad (2)$$

where ξ and η are any two given functions of x and y, the quantity δt an infinitesimal, and the terms omitted convergent power series in δt beginning with δt^2.

The coördinates of the transformed point (x', y') differ from those of the original point (x, y) by the infinitesimal increments

$$\delta x = \xi(x, y)\delta t, \quad \delta y = \eta(x, y)\delta t,$$

when terms of the second order of infinitesimals are neglected. The infinitesimal transformation makes correspond to every point (x, y) an infinitesimal arrow (say) whose length is

$$\sqrt{\delta x^2 + \delta y^2} = \sqrt{\xi^2 + \eta^2}\,\delta t,$$

and direction

$$\frac{\delta y}{\delta x} = \frac{\eta}{\xi};$$

and in general to different points arrows of different lengths and different directions. The infinitesimal transformation thus puts all the points (x, y) of the plane in motion, and if the variable t be taken as the time, these points describe in the element of time δt, the infinitesimal paths $\sqrt{\xi^2 + \eta^2}\,\delta t$, whose projections on the axes are $\xi\delta t$ and $\eta\delta t$. In the first element of time δt the point (x, y) goes over into (x', y') describing the path $\sqrt{\xi(x, y)^2 + \eta(x, y)^2}\,\delta t$, in the next element δt it runs over the infinitesimal path $\sqrt{\xi(x', y')^2 + \eta(x', y')^2}\,\delta t$, and so on. The original point (x, y) assumes, by the continued application of the infinitesimal transformation, a continuous series of positions which may be represented by a curve. This motion of the points of the plane is characterized by the fact that the components of the velocity of every point (x, y) have the values

$$\frac{dx_1}{dt} = \xi(x_1, y_1), \quad \frac{dy_1}{dt} = \eta(x_1, y_1),$$

which depend only on the position and not on the time. Since the change of position is to repeat itself from moment to moment, the motion is a so-called stationary motion and can be compared to the flow of the particles of a compressible fluid. That the phenomena of a stationary motion exhibit the group property is readily seen, for if the stationary motion carries the points (x, y) to the position (x_1, y_1) in the time t_1 and then these new points (x_1, y_1) to the positions (x_2, y_2) in the time t_2, it is clear that the motion carries the original points (x, y) to the positions (x_2, y_2) in the time $t_1 + t_2$; *i. e.* the successive performance of two transformations (t_1) and (t_2) of the family is equivalent to a single transformation $(t_1 + t_2)$ of the family.

12. This kinematical illustration may now be replaced by the following rigorous analytical reasoning.

The two differential equations

$$\frac{dx_1}{dt} = \xi(x_1, y_1), \quad \frac{dy_1}{dt} = \eta(x_1, y_1), \tag{3}$$

determine x_1 and y_1 as functions of t, and the initial values corresponding to $t=0$

which we take as $x_1=x$, $y_1=y$. In order to determine these functions x_1 and y_1, it is necessary to integrate the simultaneous system

$$\frac{dx_1}{\xi(x_1,y_1)}=\frac{dy_1}{\eta(x_1,y_1)}=dt, \tag{4}$$

with the initial conditions that $x_1=x$ and $y_1=y$ for $t=0$.

This integration is effected as follows: The differential equation in x_1, y_1

$$\frac{dx_1}{\xi(x_1,\, y_1)}=\frac{dy_1}{\eta(x_1,\, y_1)}$$

has an integral, $\Omega(x_1, y_1)$, which, since it is free from t; is also an integral of the whole simultaneous system (4). In order to find the second integral of the system which contains t, we eliminate say y_1 between the two equations

$$\Omega(x_1,\, y_1)=\text{constant}=c, \quad \text{and} \quad \frac{dx_1}{\xi(x_1,\, y_1)}=dt,$$

and obtain a differential equation,

$$\frac{dx_1}{\theta(x_1,\, c)}=dt.$$

Since the left hand member of this equation does not contain t it can be integrated by a quadrature* and its integral has the form $f(x_1, c)-t$. But this is not an integral of the system (4) until c has been eliminated by means of the equation $\Omega(x_1, y_1)=c$. Eliminating c, the second integral of the system (4) appears in the form $W(x_1, y_1)-t$.†

Finally, determining the constants of integration by the initial conditions that $x_1=x$, $y_1=y$ for $t=0$, we have as the result of the integration

$$\begin{aligned}\Omega(x_1,\, y_1)&=\Omega(x,\, y),\\ W(x_1,\, y_1)-t&=W(x,\, y).\end{aligned} \tag{5}$$

Without solving these equations for x_1, y_1 it is easy to see that they define a one parameter group, for the transformation of the family (5) which corresponds to the parameter value t carries the points (x, y) into the points (x_1, y_1), whose coördinates can be found by solving the equations (5) for x_1, y_1. A sec-

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*By the term quadrature is meant an integral of the form $\int F(x)dx$. It is assumed that a quadrature can always be performed.

†The reader will observe that this same integral would have been found had we begun by eliminating x_1 from $\frac{dy_1}{\eta(x_1,\, y_1)}=dt$ by means of $\Omega(x_1, y_1)=c$. This elimination would have given the differential equation $\frac{dy_1}{\lambda(y_1,\, c)}=dt$; the integral of the latter, $\mu(y_1, c)-t$, is found by a quadrature; eliminating c by means of $\Omega(x_1, y_1)=c$, we have finally the second integral of the system, $W(x_1, y_1)-t$.

ond transformation of the same family with the parameter value t_1 will change the points (x_1, y_1) into the points (x_2, y_2) whose coördinates are found from the equations,

$$\Omega(x_2, y_2)=\Omega(x_1, y_1),$$
$$W(x_2, y_2)-t=W(x_1, y_1). \tag{6}$$

In order to find the transformation which carries the original points (x_1, y_1) directly into the final positions (x_2, y_2), it is only necessary to eliminate x_1, y_1 from the equations (5) and (6). The elimination gives at once

$$\Omega(x_2, y_2)=\Omega(x, y),$$
$$W(x_2, y_2)-(t+t_1)=W(x, y).$$

But these equations represent the transformation of the family (5) corresponding to the parameter value $t+t_1$; hence the family (5) possesses the group property. The group contains also the inverse transformation of every transformation in it and the identical transformation.

The equations (5) can be solved with regard to x_1, y_1 in the form

$$x_1=\Phi(x, y, t), \quad y_1=\Psi(x, y, t). \tag{7}$$

These two functions can be expanded in powers of t by Maclaurin's theorem. In order to effect the expansion we must have the values

$$\left(\frac{dx_1}{dt}\right)_{t=0}, \quad \left(\frac{d^2x_1}{dt^2}\right)_{t=0}, \ldots .$$

From equations (4) we have $\frac{dx_1}{dt}=\xi(x_1, y_1)$, with $x_1=x$, $y_1=y$, for $t=0$;

hence,
$$\left(\frac{dx_1}{dt}\right)_{t=0}=\xi(x, y).$$

The equations (4) give also

$$\frac{d^2x_1}{dt^2}=\frac{\partial\xi(x_1, y_1)}{\partial x_1}\frac{dx_1}{dt}+\frac{\partial\xi(x_1, y_1)}{\partial y_1}\frac{dy_1}{dt}$$
$$=\frac{\partial\xi(x_1, y_1)}{\partial x_1}\xi(x_1, y_1)+\frac{\partial\xi(x_1, y_1)}{\partial y_1}\eta(x_1, y_1);$$

hence
$$\left(\frac{d^2x_1}{dt^2}\right)_{t=0}=\frac{\partial\xi(x, y)}{\partial x}\xi(x, y)+\frac{\partial\xi(x, y)}{\partial y}\eta(x, y).$$

Similarly, $\left(\frac{dy_1}{dt}\right)_{t=0}=\eta(x, y)$, $\left(\frac{d^2y_1}{dt^2}\right)_{t=0}=\frac{\partial\eta(x, y)}{dx}\xi(x, y)+\frac{\partial\eta(x, y)}{dy}\eta(x, y)$.

Accordingly equations (7) become by Maclaurin's theorem,

$$x_1 = x + \xi(x, y)t + \left(\xi \frac{\partial \xi}{\partial x} + \eta \frac{\partial \xi}{\partial y} \right) \frac{t^2}{1.2} + \dots,$$

$$y_1 = y + \eta(x, y)t + \left(\xi \frac{\partial \eta}{\partial x} + \eta \frac{\partial \eta}{\partial y} \right) \frac{t^2}{1.2} + \dots \tag{8}$$

The reader will observe that $t=0$ in the equations (8) gives the identical ınsformation, and $t=\delta t$ gives an infinitesimal transformation which to terms of e second order agrees with the original infinitesimal transformation (2).

All these facts may now be summed up in the following theorem of LIE:

Every infinitesimal transformation

$$x_1 = x + \xi(x, y)\delta t + \dots, \quad y_1 = y + \eta(x, y)\delta t + \dots,$$

longs to at least one one parameter group with inverse transformations, when infinsimals of the second and higher orders are neglected. The finite equations of this rup are found by integrating the simultaneous system

$$\frac{dx_1}{\xi(x_1, y_1)} = \frac{dy_1}{\eta(x_1, y_1)} = dt,$$

th the initial conditions

$$x_1 = x, \quad y_1 = y, \quad \textit{for } t=0,$$

the form

$$\Omega(x_1, y_1) = \Omega(x, y),$$

$$W(x_1, y_1) - t = W(x, y);$$

solved with regard to x, y, and developed in powers of t, in the form

$$x_1 = x + \xi(x, y)\frac{t}{1!} + \left(\xi \frac{\partial \xi}{\partial x} + \eta \frac{\partial \xi}{\partial y} \right) \frac{t^2}{1!} + \dots,$$

$$y_1 = y + \eta(x, y)\frac{t}{1!} + \left(\xi \frac{\partial \eta}{\partial x} + \eta \frac{\partial \eta}{\partial y} \right) \frac{t^2}{2!} + \dots,$$

The one parameter group thus generated accordingly possesses an infinitesiıl transformation which in its terms of the first order is identical with the original finitesimal transformation.

We have now proved that every G_1 contains an infinitesimal transforman and conversely that every infinitesimal transformation generates a G_1. We all prove in the next article that a G_1 contains but one infinitesimal transforıtion, with the converse that an infinitesimal transformation belongs to but one . The theorems will be illustrated by concrete examples. These theorems ablish the equivalence of the notions one parameter group and infinitesimal nsformation; that these notions may be used interchangeably is the fundantal principle of LIE's Theory of the Group of One Parameter.

Princeton University, 14 December, 1897.

[To be Continued.]

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

74. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, N. J.

Solve according to the conditions given :

$$\sqrt{x+1}+\sqrt{x}=\frac{3}{\sqrt{1+x}}$$

First, square without transposing and then solve; second, transpose $\sqrt{x+1}$ and then solve. Obtain the same roots as in the first way of solving.

I. Solution by J. M. BOORMAN, Counselor, Inventor, etc., etc., Hewlett, L. I., N. Y.

Solve ("conditions given") $\sqrt{x+1}+\sqrt{x}=\frac{3}{\sqrt{1+x}}$ .... (A).

The equation is of first degree. $\therefore$ *can have but one root, e. g.*

FIRST. The *conditioned* operation gives, $2(x+1)+2\sqrt{(x+1)}\sqrt{x}=1+\frac{9}{1+x}$. Thence $\sqrt{x+1}.\sqrt{x}=-(x+1)+\frac{1}{2}+\frac{9}{2(1+x)}$.

Square, etc., and reduce : $\therefore 8\frac{3}{4}(1+x)^2-4\frac{1}{2}(1+x)=20\frac{1}{4}$.

Divide and supply : $\therefore (1+x)^2-\frac{18}{35}(1+x)+\frac{81}{1225}=\frac{2916}{1225}$.

$\therefore x=\frac{4}{5}$; $x_1=-\frac{16}{7}$.

BUT, apply x_1 to the *given equation*. $\therefore \left(\frac{3+4}{\sqrt{7}}\right)\sqrt{-1}=\frac{3\sqrt{7}}{3\sqrt{-1}}=\frac{\sqrt{7}}{\sqrt{-1}}$.

Now $\frac{3+4}{\sqrt{7}}=\sqrt{7}$. $\therefore \sqrt{7}\sqrt{-1}=\sqrt{7}\left(\frac{1}{\sqrt{-1}}\right)$.

$\therefore -1\sqrt{7}=\sqrt{7}$; or $2\sqrt{7}=0$! ! So $x_1=-\frac{16}{7}$ *is not a root* of equation (A), but of its factor $\sqrt{x+1}+\sqrt{x}=\frac{-3}{\sqrt{1+x}}$, that *inevitably* results by the conditioned involution. Hence $x=\frac{4}{5}$ *only*.

SECOND (direct) "way." Transpose and square.

$\therefore x=x+1-6+\frac{9}{1+x}$. Thence $x=\frac{4}{5}$, [the "*same root as in the first way.*"]

PROOF. Apply *this* $x=\frac{4}{5}$, in equation (A).

$\therefore \sqrt{\frac{9}{5}}+\sqrt{\frac{4}{5}}=\frac{3\sqrt{5}}{\sqrt{9}}$. $\therefore$ as $\frac{3+2}{\sqrt{5}}=\frac{5}{\sqrt{5}}=\sqrt{5}$, so $\sqrt{5}=\frac{3\sqrt{5}}{\sqrt{9}}=\frac{3}{3}\sqrt{5}=1\sqrt{5}$,

i. e. $\sqrt{5}=\sqrt{5}$, *satisfies* equation (A). $\therefore x=\frac{4}{5}$ is the *one* root of (A). Q. E. D.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Squaring the equation as it stands, we get $2x+1+2\sqrt{x(x+1)}=\frac{9}{x+1}$.

Clearing of fractions and leaving the radical by itself in the first member, e get $2(x+1)\sqrt{x(x+1)}=8-3x-2x^2$. Squaring, arranging, and cancelling, we t the quadratic $35x^2+52x=64$, the two roots of which are $x=\frac{4}{5}$ and $-\frac{16}{7}$, the e former of which satisfies the equation $\sqrt{x+1}+\sqrt{x}=\frac{3}{\sqrt{x+1}}$, and the latter e equation $\sqrt{x+1}-\sqrt{x}=\frac{3}{\sqrt{x+1}}$.

Clearing the original equation of its denominator $\sqrt{x+1}$, we have $+1+\sqrt{x(x+1)}=3$, or $\sqrt{x(x+1)}=2-x$. Squaring, we have $5x=4$. $\therefore x=\frac{4}{5}$.

III. Solution by F. M. McGAW, A. M., Professor of Mathematics in Bordentown Military Institute, Bordenwn, N. J.; CHAS. C. CROSS, Laytonsville, Md.; G. B. M. ZERR, A. M., Ph. D., The Russell College, Lebanon, L.; and J. P. BURDETTE, Class of '97, Dickinson College, Carlisle, Pa.

(1), $\sqrt{x+1}+\sqrt{x}=\frac{3}{\sqrt{1+x}}$. $2x+1+2\sqrt{x+x^2}=\frac{9}{1+x}$.

$\therefore 8-2x^2-3x=2(x+1)\sqrt{x+x^2}$. $\therefore 35x^2+52x=64$. $\therefore x=\frac{4}{5}$, or $-2\frac{2}{7}$.

(2). Regarding $\sqrt{x+1}$ as affected by the $\pm$ sign

$\sqrt{x}=\frac{2-x}{\sqrt{1+x}}$ or $\frac{4+x}{\sqrt{1+x}}$.

$\therefore x=(4-4x+x^2)/(1+x)$, or $(16+8x+x^2)/(1+x)$.

$\therefore x=\frac{4}{5}$, or $x=-2\frac{2}{7}$.

Also solved by *A. H. BELL.*

75. Proposed by the late B. F. BURLESON, Oneida Castle, N. Y.

Mr. B's farm is in shape a quadrilateral, both inscriptible and circumriptible, and contains an area of $k=10752$ square rods. The square described n the radius of its inscribed circle contains $r^2=2304$ square rods; while the uare described on the radius of its circumscribed circle contains an area $R^2=7345$ square rods. Required the lengths of the sides of his farm.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, banon, Va.

Let a, b, c, d be the sides required. By the conditions of the problem,

$+c=b+d$; $abcd=k^2=115605504$ .... (1).

$\frac{1}{2}r(a+b+c+d)=k$, or $a+b+c+d=2k/r=448$.

$\therefore a+c=b+d=k/r=224$ .... (2).

$$R=\frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(bc+ad)}{abcd}}.$$

$\therefore (ab+cd)(ac+bd)(bc+ad)=16R^2k^2=13585958830080$.............(3).

Substituting (2) in (1) and (3), we get

$$(224a-a^2)(224b-b^2)=115605504 \quad\text{..................(4).}$$

$$\{ab+(224-a)(224-b)\}\{a(224-a)+b(224-b)\}\{b(224-a)+a(224-b)\}$$
$$=13585958830080 \quad\text{........................(5).}$$

Eliminating b from (4) and (5), we get, after reducing and factoring,

$$(a-168)(a-128)(a-96)(a-56)=0.$$

$\therefore$ The sides are 168, 128, 96, 56 rods, respectively.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.; and COOPER D. SCHMITT, A. M., Professor of Mathematics in the University of Tennessee, Knoxville, Tenn.

Denoting the four consecutive sides of the quadrilateral by a, b, c, d, we have, from well-known geometrical formulæ and principles:

$abcd=k^2$ ... .(1) ; $r^2=abcd/[a+c]^2=k^2/[a+c]^2$..............(2);

$a+c=b+d=k/r$.......(3) ; $R^2=\{[ac=bd][ad+bc][ab+cd]\}/16k^2$.......(4).

Putting $ac=x$, and $bd=y$, we have in (4),

$[x+y]\{ac[b^2+d^2]+y[a^2+c^2]\}=16k^2R^2$; or

$[x+y]\{x[(k^2/r^2)-2y]+y[(k^2/r^2)-2x]\}=16k^2R^2$; or

$[x+y]\{[x+y][k^2/r^2]-4xy\}=16k^2R^2$, and since $xy=abcd=k^2$,

$[x+y]\{[x+y]-4r^2\}=16R^2r^2$; or, reduced

$[x+y]^2-4r^2[x+y]=16R^2r^2$; whence $x+y=2r^2+2r\sqrt{r^2+4R^2}$,

and combining this with $xy=k^2$, we find x and y. Thus, we find for the given numerical values $x+y=21696$, $xy=115605504$, whence $x=12848$, $y=9408$. Now we have $ac=12288$, $a+c=224$, and $bd=9408$. $b+d=224$.

Whence $a=128$, $c=96$, $b=168$, $d=56$

III. Solution by the PROPOSER.

Let $ABCD$ represent the farm, and let $x=CD$, $y=DA$, $z=AB$, $w=BC$, in order. We have $x+z=y+w$..................(1). Also the following, where $s=x+y+z+w$, the perimeter of the quadrilateral:

$k^2=\frac{1}{16}\{[s-2x][s-2y][s-2z][s-2w]\}$. ....(2);

$rs=2k$..............(3) ; $k^2=xyzw$..............(4) ;

$R^2=\{[xy+zw][xz+yw][xw+yz]\}$
$\qquad+\{[s-2x][s-2y][s-2z][s-2w]\}$... (5).

Substitute in (2), (4), and (5) :

$m=xy+xz+xw+yz+yw+zw$;

$n=xyz+xyw+xzw+yzw$; and $p=xyzw$.

Then we shall have by involving terms and re-factoring,

$^2=16p-s^4+4s^2m-8sn$ ....(6) ; $k^2=p$. ....(7) ;

$R^2=16p-s^4+4s^2m-8sn=n^2-4pm+ps^2$ ....(8).

From (3), (6), (7), and (8), we obtain by elimination and resolution,

$2k/r=448$; $m=[k^2+r^2n]/r^2k=71872$; $n=2rk+2k\sqrt{[4R^2+r^2]}=4859904$;

$p=k^2=115605504$.

We now, by the "Theory of Equations," construct the biquadratic, the roots of which will be the values of x, y, z, and w.

$x^4-448x^3+71872x^2-4859904x=-115605504$ ....(9).

The four roots of equation (9), we find to be 56, 96, 128, and 168. Arging these values in conformity with equation (1), we have, $CD=x=56$ rods, $=y=96$ rods, $AB=z=168$ rods, and $BC=w=128$ rods.

IV. **Solution by A. H. BELL, Hillsboro, Illinois.**

Since circumscriptible quadrilaterals have the sums of their opposite sides al, take $x+y$, $x+z$, $x-y$, $x-z$, for the sides AB, BC, DC, and AD.

$\therefore 2rx=k$, $x=k/2r$ ....(1).

$\overline{BD}^2=[x+y]^2+[x-z]^2-2[x+y][x-z]\cos A$ ....(3).

$\overline{BD}^2=[x+z]^2+[x-y]^2+2[x-y][x+z]\cos A$,

$\{\cos C=\cos[180-A]=-\cos A\}$ ....(4).

$\therefore \cos A=\dfrac{x[y-z]}{x^2-y^2}$. $\sin^2 A=1-\cos^2 A=\dfrac{[x^2-y^2][x^2-z^2]}{[x^2-y^2]^2}$ ....(5).

$\overline{BD}^2=\dfrac{\{[x^2-y^2]+[x^2-z^2]\}[x^2+y^2]}{[x^2-y^2]}$; also $R^2=\dfrac{\overline{BD}^2}{4\sin^2 A}$ ....(6).

$2k=[x+y][x-z]\sin A+[x-y][x+z]\sin A$, or $[x^2-y^2]\sin A=k$ ....(7).

Substituting the value of $\sin A$, (5) in (7), and

$[x^2-y^2][x^2-z^2]=k^2$ ....(8),

Then (6) becomes, $[x^2-y^2+x^2-z^2][x^4-y^2z^2]=4Rk^2$ ....(9).

Let the product of the opposite sides $v=x^2-y^2$. $\therefore y^2=x^2-v$....(10) ;

$w=x^2-z^2$. $\therefore z^2=x^2-w$ ....(11).

Then (8), and (1), $vw=k^2=4r^2x^2$ ....(12) ;

(9) reduces to $[v+w]^2-4r^2[v+w]=16R^2r^2$ ....(13).

$\therefore\ v+w=2r^2\pm2r\sqrt{4R^2+r^2}$ ....(14).

$(14)^2-4(7)$, etc. $v-w=2r[4R^2+2r^2-4x^2\pm2r\sqrt{4R^2+r^2}]^{\frac{1}{2}}$ ....(15).

(14)±(15) after substituting the given values, $v=12288$, and $w=9408$.

(1), (5), and (6) $x=112$, $y=56$, and $z=16$, and the required sides AB, BC, CD, and AD are 168 rods, 128 rods, 56 rods, and 96 rods, respectively.

Also $BD=158.22$ rods.

Also solved by *CHARLES C. CROSS* and *H. C. WILKES*.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

61. Proposed by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, La.

If $r=a\sin n\theta$ is the polar equation of a curve, show (1) that the curve consists of n or $2n$ loops according as n is an odd or an even integer; (2) that its area is $\frac{1}{4}$ or $\frac{1}{2}$ of the circumscribing circle according as n is an odd or an even integer.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

$r=a\sin n\theta$. Let $r=0$, then $\sin n\theta=0$.

$\therefore\ \theta=0,\ 2\pi/n,\ 4\pi/n,\ 6\pi/n,\ 8\pi/n,\ \ldots\ldots$, are the angles at which the the curve cuts the polar axis at the pole.

$dr/d\theta=na\cos n\theta=0$. $\therefore\ \theta=\pi/2n,\ 3\pi/2n,\ 5\pi/2n,\ 7\pi/2n,\ \ldots\ldots$, gives the points where r has its greatest value, namely, $\pm a$.

When n is odd the values of $n\theta$ for the angles $0,\ 2\pi/n,\ 4\pi/n,\ 6\pi/n,\ 8\pi/n,\ \ldots\ldots$, are $0,\ 2\pi,\ 4\pi,\ 6\pi,\ 8\pi,\ \ldots\ldots$

When n is even the values of $n\theta$ for the angles $0,\ 2\pi/n,\ 4\pi/n,\ 6\pi/n,\ 8\pi/n,\ \ldots\ldots$, are $0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ \ldots\ldots$

$\therefore$ When n is even the polar axis is cut, at the pole, $2n$ times, but only n times when n is odd.

A=area of one loop.

$$A=\tfrac{1}{2}a^2\int_0^{\pi/n}\sin^2 n\theta d\theta,\ =\frac{\pi a^2}{4n}.$$

$$\therefore\ \frac{\pi a^2}{4n}\times n=\frac{\pi a^2}{4},\ \text{for } n \text{ odd};\ \frac{\pi a^2}{4n}\times 2n=\frac{\pi a^2}{2},\ \text{for } n \text{ even}.$$

II. Solution by E. L. SHERWOOD, A. M., Superintendent of City Schools, West Point, Miss.

Equation given $\rho=a\sin n\theta$. We may oberve that,

$$\rho=0,\ a,\ 0,\ -a, \text{ etc., when}$$

$$\sin n\theta=0,\ 1,\ 0,\ -1, \text{ etc., when}$$

$$n\theta=0,\ \tfrac{1}{2}\pi,\ \pi,\ \tfrac{3}{2}\pi, \text{ etc., when}$$

$\theta=\{[c/n].\frac{1}{2}\pi\}$, where c is 0, 1, 2, 3, 4, etc., up to $4n$ ($4n$ being determined y $\theta=2\pi$).

The series of values will be as follows:

$$\theta=0.\frac{\pi}{2n},\quad \frac{\pi}{2n},\quad 2.\frac{\pi}{2n},\quad 3\frac{\pi}{2n}\ldots\ldots n.\frac{\pi}{2n},\quad [n+1]\frac{\pi}{2n}\ \ldots\ldots 2n\frac{\pi}{2n},$$

$$[2n+1]\frac{\pi}{2n}\ \ldots\ldots\ ;$$

If n is *even*, $\rho=0,\ a,\ 0,\ -a\ldots\ldots0\pm a\ldots\ldots0,\ a.$

If n is *odd*, $\rho=0,\ a,\ 0,\ -a\ldots\ldots\pm a,\ 0\ldots\ldots0,\ -a\ldots\ldots$

$$\left\{\begin{array}{llll} 3n.\frac{\pi}{2n}, & [3n+1]\frac{\pi}{2n} & \ldots\ldots\ldots\ldots & 4n.\frac{\pi}{2n} \\ 0 & \pm a & \ldots\ldots\ldots\ldots & 0 \\ \pm a & 0 & \ldots\ldots\ldots\ldots & 0 \end{array}\right.$$

In each series are $4n$ terms (the first coincides with the last), and $\rho=a$ umerically in $2n$ of them. But when n is odd, the radius vector traces each loop wice for $\pi/2n$ and a is the same point as $\{[2n+1][\pi/2n]\}$ and $-a$.

$\therefore$ There are $2n$ loops when n is even, and n loops when n is odd.

$$\text{Area}\ =\tfrac{1}{2}\int\rho^2 d\theta, \text{ where } \rho^2=a^2\sin^2 n\theta,$$

$$=\tfrac{1}{2}a^2\int\sin^2 n d\theta,$$

$$=\tfrac{1}{2}a^2\left[\tfrac{1}{2}\theta-\frac{\sin 2n\theta}{4n}\right]_0^{\pi/2n}=\frac{\pi a^2}{8n} \text{ for } \tfrac{1}{2} \text{ loop},$$

or $\pi a^2/4n$ for an entire loop.

$\therefore$ For $2n$ loops, area$=\pi a^2/2$; and for n loops, area$=\pi a^2/4$.

Also solved by *J. SCHEFFER* and *C. W. M. BLACK*.

62. Proposed by A. H. HOLMES, Brunswick, Maine.

A bucket is in the form of a frustum of a cone having its smaller end as a base. It a inches in diameter at base and b inches in diameter at top, and its perpendicular height c inches. It contains water the perpendicular height of which is $\frac{1}{2}c$ inches. What is the reatest height, from the plane on which the vessel rests, to which the surface of the water ill rise when the bucket is overturned, no allowance being made for the thickness of the aterial of the bucket?

Partial Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Mass.

Let $AEFL$ be section of frustum. Complete the cone to the apex N. Let NM be axis, HK surface of water, PQ plane on which vessel rests, AB height of surface of water above PQ. Denote $\angle ONA$ by α, ON by l, HK by x, HM by y.

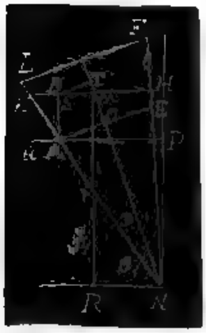

Fig. 1.

Then $EO=OA=\frac{1}{2}a$, $FL=b$. Denote angle which axis makes with PQ, $\angle ONR$ by θ, MR perpendicular to NR by h.

Now as vessel is tipped over, until H reaches E, volume of cone NHK is constant, and $=\frac{1}{3}\pi[\frac{1}{2}a+\frac{1}{2}b)^2[l+\frac{1}{2}c]$. Denote it by C.

Base HK is an ellipse, whose major axis is

$$x=h\cot[\theta-\alpha]-h\cot[\theta+\alpha]\ \ldots\ldots\ldots\ldots(1).$$

Also $HM=y$, $=h\cot\theta+h\cot[\theta+\alpha]$ ....(2).

Let z=semi-minor axis, =ordinate in circular section through S, middle point of HK. Let r=radius of section.

$$\text{Then } z=\sqrt{r^2-\overline{TS}^2},\ =\sqrt{r^2-[(x/2)-y]^2\sin^2\theta}\ \ldots\ldots\ldots\ldots(3),$$

since $\angle TMS=\angle ONR$, $=\theta$. Also,

$$r=NT\tan\alpha,\ =\left[\frac{h}{\sin\theta}+\left(\frac{x}{2}-y\right)\cos\theta\right]\tan\alpha\ \ldots\ldots\ldots(4).$$

Volume $NHK=[\frac{1}{3}\pi]h[x/2]$, $=C$ .... (5).

Let $\cot\theta=\beta$, $\cot\alpha=k$. (1) becomes, $x=\dfrac{2hk[\beta^2+1]}{k^2-\beta^2}$ ....(6);

(2) becomes, $y=\dfrac{h[\beta^2+1]}{k+\beta}$ ....(7), and $\dfrac{x}{2}-y=\dfrac{h\beta\,[\beta^2+1]}{k^2-\beta^2}$ ....(8).

From (4) by (8), $r=\dfrac{hk\sqrt{\beta^2+1}}{k^2-\beta^2}$ ....(9).

Substituting in (3),

$$z=\sqrt{\frac{h^2k^2[\beta^2+1]}{[k^2-\beta^2]^2}-\frac{h^2\beta^2[\beta^2+1]}{[k^2-\beta^2]^2}},\ =h\sqrt{\frac{\beta^2+1}{k^2-\beta^2}}\ \ldots\ldots\ldots(10).$$

$$\text{(5) becomes, } [\tfrac{1}{3}\pi]h\times\frac{hk[\beta^2+1]}{k^2-\beta^2}\times h\sqrt{\frac{\beta^2+1}{k^2-\beta^2}}=[\tfrac{1}{3}\pi]h^3k\left(\frac{\beta^2+1}{k^2-\beta^2}\right)^{\frac{3}{2}}=C;$$

$$\text{whence } h=\sqrt[3]{\frac{3C}{\pi k}}\sqrt{\frac{k^2-\beta^2}{\beta^2+1}}\ \ldots\ldots\ldots\ldots(11).$$

w $AB = h - l\sin\theta + \frac{1}{2}a\cos\theta, \; = \sqrt[3]{\frac{3C}{\pi k}}\sqrt{\frac{k^2-\beta^2}{\beta^2+1}} - \frac{l}{\sqrt{\beta^2+1}} + \frac{a\beta}{2\sqrt{\beta^2+1}},$

$$= \frac{\sqrt[3]{\frac{3C}{\pi k}}\sqrt{k^2-\beta^2} - l + \frac{a\beta}{2}}{\sqrt{\beta^2+1}} \qquad \ldots\ldots\ \ldots\ldots\ldots\ldots(12),$$

which β is the only variable. $dAB/d\beta=$

$$\frac{\overline{{}^2+1}\left(-\sqrt[3]{\frac{3C}{\pi k}}\frac{\beta}{\sqrt{k^2-\beta^2}} + \frac{1}{2}a\right) - \left(\sqrt[3]{\frac{3C}{\pi k}}\sqrt{k^2-\beta^2} - l + \frac{a\beta}{2}\right)\frac{\beta}{\sqrt{\beta^2+1}}}{\beta^2+1} = 0$$

Equating to zero, and clearing of fractions,

$${}^2+1)\left(-\sqrt[3]{\frac{3C}{\pi k}}\frac{\beta}{\sqrt{k^2-\beta^2}} + \frac{1}{2}a\right) - \left(\sqrt[3]{\frac{3C}{\pi k}}\sqrt{k^2-\beta^2} - l + \frac{a\beta}{2}\right)\beta = 0,$$

$$\text{or } \sqrt[3]{\frac{3C}{\pi k}}\left(\frac{\beta[\beta^2+1]}{\sqrt{k^2-\beta^2}} + \beta\sqrt{k^2-\beta^2}\right) = \frac{1}{2}a + l\beta.$$

aring and clearing, $[3C/\pi k]^{\frac{2}{3}}\,\beta^2[1+k^2]^2 = [\frac{1}{2}a + l\beta]^2[k^2-\beta^2]$. Whence,

$${}^4 + al\beta^3 + \{[3C/\pi k]^{\frac{2}{3}}[1+k^2]^2 + \tfrac{1}{4}a^2 - l^2k^2\}\beta^2 - alk^2\beta - \tfrac{1}{4}a^2k^2 = 0 \ldots\ldots\ \ldots(13).$$

Now $l = \frac{1}{2}a\cot\alpha, = \frac{1}{2}ak$; also, $k = \{c/\frac{1}{2}[b-a]\}, = \{2c/[b-a]\}$ ....(14).

$\therefore\ l = \frac{ac}{b-a}$. Also, $C = \pi\left(\frac{2a+b}{6}\right)^2[l + \frac{1}{2}c], = \pi\frac{[2a+b]^3c}{108[b-a]}$.

$\therefore\ 3C/\pi k = \{[2a+b]^3/216\}$ ....(16).

Substituting (14), (15), (16) in (13),

$$\frac{{}c^2}{-a]}\beta^4 + \frac{a^2c}{b-a}\beta^3 + \left(\frac{[2a+b]^2\{[b-a]^2+4c^2\}}{36[b-a]^2}\right.$$

$$\left. + \frac{1}{4}a^2 - \frac{4a^2c^4}{[b-a]^4}\right)\beta^2 - \frac{4a^2c^3}{[b-a]^3}\beta - \frac{a^2c^2}{[b-a]^2} = 0 \quad \ldots\ \ldots(17).$$

By solving this for β we get the maximum values of AB, provided (Fig. 1) loes not pass E. In Fig. 2, representing this condition $\angle EAB = \theta$,
$\therefore\ AB = a\cos\theta$.

Accordingly (17) will produce critical values of θ; provided $\cos\theta$ is not > /a, AB to be determined from (12).

It is evident that for any position of HK which cuts EA, the value of AB will be greater than that determined by the supposition made above. We must seek for maxima in this case by a different method.

Fig. 2.

DKA (Fig. 3) represents section of volume of water.

$$\text{Volume}=[\pi c/9]\{[\tfrac{1}{2}a]^2+[\tfrac{1}{3}a+\tfrac{1}{6}b]^2+\tfrac{1}{2}a[\tfrac{1}{3}a+\tfrac{1}{6}b]\},$$

$$=[\pi c/324][19a^2+7ab+b^2] \quad \ldots\ldots\ldots\ldots (18).$$

Equation (1)—(4) and (6)—(10) apply here as in Fig. 1.

Now volume ADK=cone NDK−cone NDA,$=\tfrac{1}{3}h\times$ [area elliptical segment DK]$-\tfrac{1}{3}l\times$[area circular segment AD] ……………… (19).

Let $AB=s$, $\angle DAC=\theta$, $\angle BKA=\theta-\alpha$.

$$DK=DB+BK,=s\tan\theta+s\cot[\theta-\alpha],=s\left(\frac{1}{\beta}+\frac{k\beta+1}{k-\beta}\right),=\left(\frac{sk[1+\beta^2]}{\beta[k-\beta]}\quad \ldots (20).$$

$$\text{Area segment } DK=\frac{\pi zx}{2}-\frac{zx}{2}\left(\cos^{-1}\frac{[DK-\frac{1}{2}x]}{\frac{1}{2}x}-\frac{DK-\frac{1}{2}x}{\frac{1}{2}x}\sqrt{1-\left(\frac{DK-\frac{1}{2}x}{\frac{1}{2}x}\right)^2}\right.$$

Substitute from (6), (10), and (20):

$$\text{Area } DK=\frac{h^2k[\beta^2+1]^{\frac{3}{2}}}{[k^2-\beta^2]^{\frac{3}{2}}}\left[\pi-\cos^{-1}\left(\frac{s[k+\beta]}{h\beta}-1\right)\right.$$

$$\left.+\left(\frac{s[k+\beta]}{h\beta}-1\right)\sqrt{\frac{s[k+\beta]}{h\beta}\left(2-\frac{s[k+\beta]}{h\beta}\right)}\right] \quad \ldots (21).$$

Fig. 3.

Now $s=h-l\sin\theta+\tfrac{1}{2}a\cos\theta$.

$$\therefore\ h=s+l\sin\theta-\tfrac{1}{2}a\cos\theta,=s+\frac{2l-a\beta}{2\sqrt{\beta^2+1}} \quad \ldots (22).$$

$$AD=s\sec\theta,=\frac{s\sqrt{\beta^2+1}}{\beta}.$$

$$\text{Area segment } AD=\tfrac{1}{8}a^2\left[\pi-\cos^{-1}\left(\frac{2s\sqrt{\beta^2+1}}{a\beta}-1\right)\right.$$

$$\left.+2\left(\frac{2s\sqrt{\beta^2+1}}{a\beta}-1\right)\sqrt{\frac{s\sqrt{\beta^2+1}}{a\beta}\left(1+\frac{s\sqrt{\beta^2+1}}{a\beta}\right)}\right] \quad \ldots\ldots (23).$$

Substitute from (18), (21), (22), and (23) in (19),

$$\frac{[s\sqrt{\beta^2+1}+l-\frac{1}{2}a\beta]^3}{3[k^2-\beta^2]^{\frac{3}{2}}}\left\{\pi-\cos^{-1}\left[\frac{s[k+\beta]}{\beta\{s+[(2l-a\beta)/(2\sqrt{1+\beta^2})]\}}-1\right]\right.$$

$$+\left[\frac{s[+\beta]}{\beta\{s+[(2l-a\beta)/(2\sqrt{\beta^2+1})]\}}-1\right]\times$$

$$\left.\sqrt{\frac{s[k+\beta]}{\beta\{s+[(2l-a\beta)/(2\sqrt{\beta^2+1})]\}}\left[2-\frac{s[k+\beta]}{\beta\{s+[(2l-a\beta)/(2\sqrt{\beta^2+1})]\}}\right]}\right\}$$

$$-\frac{a^3k}{24}\left\{\pi-\cos^{-1}\left[\frac{2s\sqrt{\beta^2+1}}{a\beta}-1\right]\right.$$

$$\left.-2\left[\frac{2s\sqrt{\beta^2+1}}{a\beta}-1\right]\sqrt{\frac{s\sqrt{\beta^2+1}}{a\beta}\left[1-\frac{s\sqrt{\beta^2+1}}{a\beta}\right]}\right\}=\tfrac{1}{324}\pi c[19a^2+7ab+b^2],$$

hich equation contains only s, β, and constants. However, the chance of solv-
ig it after differentiation seems extremely slight.

PROBLEMS FOR SOLUTION.

MISCELLANEOUS.

56. Proposed by S. HART WRIGHT, A. M., M. D., Ph. D., Penn Yan, N. Y.

In latitude 40° N.$=\lambda$, when the moon's declination is 5° 23′ N.$=\delta$, and ie sun's declination 9° 52′ S.$=-\delta'$, how long after sunset will the cusps of the ioon's crescent set synchronously, the moon having recently passed its conjunc- on with the sun?

57. Proposed by GEORGE LILLEY, Ph. D., LL. D., Professor of Mathematics in the Oregon State University, ugene, Oregon.

A particle is placed very near the center of a circle, round the circumfer- nce of which n equal repulsive forces are symmetrically arranged; each force aries inversely as the mth power of its distance from the particle. Show that he resultant force is approximately $\dfrac{m_1n(m-1)}{2r^{m+1}}\times CP$, and tends to the center of he circle, where m_1 is the mass of the particle, CP its distance from the center f the circle, and r the radius of the circle.

EDITORIALS.

The credit of preparing the index for this volume is due Editor Colow.

Dr. Artemas Martin, of the U. S. Coast and Geodetic Survey, has been promoted to Chief of the Library and Archives Division, at a salary of $1800 per annum, the promotion taking effect July 1, 1897. Dr. Martin has just been elected a member of the "Circolo Matematico di Palermo," Italy.

We regret to announce the death of Prof. B. F. Burleson, which occurred at his home, in Oneida Castle, New York, on December 2. Mr. Burleson was born in Stockbridge, July 7, 1835, but had resided in Oneida Castle for many years, where he was highly esteemed. For a number of years he occupied the position of Principal of the Union School, and in this position he proved a most successful and acceptable teacher. He was extremely fond of mathematics and was very expert in solving difficult problems. For many years he was a frequent contributor to most of the mathematical journals published in this country, and enjoyed a wide acquaintance with well-known mathematical teachers in various parts of the country. Five years ago he was stricken with paralysis and had since suffered from several strokes, which was the final cause of his death. Mr. Burleson was one of those promising but unfortunate men who possessed only the advantages of a common school education. His knowledge of mathematics was obtained by self application and in this way he became a very able analyzer of difficult mathematical problems as his solutions of many difficult problems will show. Had he possessed the advantages of a mathematical course in one of our leading universities, his influence would undoubtedly have been felt in a larger way. All honor is due him for what he made of the opportunities he possessed and the advantages afforded him. There survive him, his widow, a daughter, and one son, George Burleson, of Buffalo, and a sister residing at Oneida Castle.

BOOKS AND PERIODICALS.

Elements of the Differential and Integral Calculus. By William S. Hall, E. M., C. E., M. S., Professor of Technical Mathematics in Lafayette College. 250 pages. Price, $2.25. (1897). New York: D. Van Nostrand.

Great activity has been displayed within the last year in the production of texts on the subject of the Calculus. Among the more recent books on this subject, Professor Hall's treatise is entitled to very favorable consideration. The two branches of the Calculus are treated together to great advantage. The formulas for differentiation are established by the method of limits, but the method of infinitesimals is also explained, and the differential notation used when there is advantage gained by it. The numerical problems illustrating the text and showing applications in engineering practice is an excellent feature of the book. The table of integrals for convenience of reference is more extended than is usual in books of the same scope. Throughout the work there is a great compactness both in the methods and form of treatment, and we find more subjects presented than in most of the elementary texts. The chapter on Differential Equations is one of the best features of the book.

The Calculus for Engineers, with Applications to Technical Problems. By Professor Robert H. Smith. Pages 176. Price, $3.00. 1896. London: Charles Griffin and Company. Philadelphia: J. B. Lippincott Company.

The aim of this treatise is to introduce the student at once to the more important uses of the Integral Calculus, and incidentally to those of the Differential Calculus. The development of the *rattionale* of the subject is based on essentially *concrete* conceptions. Considerable use is made of the graphic method where admissable. The effort has been made to make the treatment less formal than usual, and the meaning and use of results is illustrated by many applications to mechanics, thermodynamics, electrodynamics, problems in engineering design, etc. One of the most distinctive and important features of the book is the very complete and extended Classified Reference Tables of Integrals and Methods of Integration, which occupy 42 pages. The chapter on the integration of Differential Equations will prove an important aid in pointing out methods of dealing with various classes of problems. The book has some practical features that will especially recommend it to engineers and physicists. J. M. C.

The Tutorial Trigonometry. By William Briggs, M. A., F. R. A. S., and G. H. Bryan, Sc. D., F. R. S. London: W. B. Clive. New York: Hinds & Noble. Pages 326. Price, $1.00. 1897.

This latest issue in the series of Tutorial texts is a very satisfactory book. The definitions of the trigonometric functions is wisely introduced early. Most of the articles are written with commendable clearness, and it is only in minor points that we have noticed any defects or inaccuracies in the book. The chapter on the ambiguous case in the solution of triangles is especially clearly and concisely stated. The large number of well-chosen examples attached to each chapter add much to the completeness of the book for class use. The relative importance of subject-matter is indicated by the use of different type, which somewhat mars the appearance of the printed page, but this is a slight objection as compared with the advantage gained in clearness and in effective presentation of the subject to students. While very much after the order of the long list of trigononometries now in use, this book seems to cover about the right ground and bears the marks of a well-constructed text-book. J. M. C.

Regular Points of Linear Differential Equations of the Second Order. By Maxime Bôcher, Ph. D., Assistant Professor of Mathematics in Harvard University. Pages 23. 1896. Cambridge: Harvard University Press.

This excellent little treatise is intended quite as much for students of mathematical physics who may not be able to carry the subject further than is here done as for those intending to make a more extended study of the modern theory of linear differential quotations. J. M. C.

Past and Present Tendencies in Engineering Education. By Mansfield Merriman, Professor of Civil Engineering, Lehigh University, South Bethlehem, Pennsylvania.

This pamphlet of 17 pages, reprinted from Volume IV of the Proceedings of the Society for Promotion of Engineering Education, contains the instructive presidential address of Professor Merriman before that society, at its meeting on August 20, last.

Macfarlane on Discharge of Condenser. This pamphlet contains the interesting discussion of Dr. Macfarlane's paper, which was presented at the meeting of the American Institute of Electrical Engineers in May last, in which Mr. Steinmetz, Dr. Kennelly, and Dr. Perrine took part, and also the communicated reply of Dr. Macfarlane.

Numerical Problems in Plane Geometry. By J. G. Estill, of the Hotchkiss School, Lakeville, Conn. 144 pages. 1897. New York: Longmans, Green & Co.

These problems are meant to be used with other geometries. The book contains a graded set of problems on the five books of geometry, as the division into Books is generally made. The use of the metric system is begun at the very first. The problems, and the entrance papers in the latter part of the book, seem to have been selected with great care and excellent judgment. The discussion of logarithms, and the explanation of their use, and the use of the table, have been clearly made. In as much as some knowledge of the metric system and the ability to solve numerical problems in plane geometry is now required for admission to most colleges, this little treatise should be especially acceptable to preparatory schools. J. M. C.

Euclid: Books I.—IV. By Rupert Deakin, M. A., Headmaster of King Edward's Grammar School, Stourbridge. Price, 70 cents. 1897. London: W. B. Clive. New York: Hinds & Noble.

This edition of Euclid was prepared for the well-known "Tutorial Series." The notes at the end of each book supply excellent comments upon and analysis of the propositions, especially aiding the student to group together propositions in which similar methods of proof are used. Special care has been taken to encourage the working of "riders," and a section is given in which methods of attack are suggested, while exercises on the various methods have been interspersed throughout the text. The book is attractively printed, and should furnish an important aid in teaching elementary Euclid. J. M. C.

School Geometry. By J. Fred Smith, A. M., Principal of Iowa College Academy. 320 pages. 1897. Chicago: Scott, Foresman & Co.

While there is no special novelty or marked improvement in this on other text-books of like purpose and scope, yet it is well written and has several good features. The subject is approached gradually, and as far as may be the abstract through the concrete; it is more elementary than many of the books in common use, and in the earlier part separate figures indicate the successive steps of a construction instead of one figure for all the steps combined. The equation is early introduced and frequently used. Emphasis is placed upon the importance of original work, and a large number of theorems and problems are given as additional exercises. The side references, usual in other books, showing the authority for each step in a demonstration are omitted, but we doubt if this feature will be much in its favor with most teachers. The book is well printed, but not very neatly nor substantially bound. J. M. C.

Elementary Arithmetic. By William W. Speer, Assistant Superintendent Schools, Chicago. 314 pages. 1897. Boston : Ginn & Company.

To the first book of this series we have previously directed special attention. The thor emphasizes the importance of early bringing into view the definite relations quantity. The idea of relative magnitude is made the basis of treatment in this new ies of books. Hence simple ratios are made the key to the solution of all problems. e treatise is sufficiently different from others of a similar purpose to give it field for al, and in the hands of *competent* teachers we predict it will give profitable results.

J. M. C.

The American Monthly Review of Reviews. An International Illustrated onthly Magazine. Edited by Dr. Albert Shaw. Price, $2.00 per year in adnce. Single numbers, 25 cents. The American Monthly Review of Reviews ., 13 Astor Place, New York.

The January number of the *American Monthly Review of Reviews* is one of the best is-es in the history of that magazine. From cover to cover it is thoroughly "live," alert, d forceful. The opening editorial department of "The Progress of the World" gives a ar and exhaustive New Year's summary of political conditions in both hemispheres at threshold of 1898. The elaborate article on "The Future of Austria-Hungary," by an strian, is by all odds the best account yet given in the English language of the warring ces which threaten to undermine the dual monarchy of central Europe; Mr. Charles A. nant's clean-cut analysis of the present demands for currency reform in the United States something that no practical man of affairs should fail to read; Dr. W. H. Tolman's sum-ng up of the municipal progress of New York City under Mayor Strong is just what is eded at this time as an encouragement of efforts for civic betterment everywhere; Lord assey's remarkable paper on "The Position of the British Navy," with Assistant Secre-y Roosevelt's comments, is full of food for thought when read in connection with the mpact digest of the United States annual naval report, which follows, and the review of ptain Mahan's new book; two noteworthy letters of Count Tolstoi on the doctrines of nry George, one addressed to a German disciple of George and the other to a Siberian asant, are also published in this number. Besides these important and spirited special tures, the magazine's regular departments of "Current History in Caricature," "Lead- Articles of the Month," "Periodicals Reviewed," and "New Books" cover such timely ics as Hawaiian annexation and the great strike in England.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edd by John Brisben Walker. Price, $1.00 per year in advance. Single numr, 10 cents. Irvington-on-the-Hudson.

Among the leading articles in the January number are the following: Stephen Girl and His College; The Real Klondike; Harold Frederick's "Gloria Mandi"; and A Brief story of our late War with Spain.

The Arena. An Illustrated Monthly Magazine. Edited by John Clark dpath, LL. D. Price, $2.50 per year in advance. Single number, 25 cents. ston : The Arena Co.

The Open Court. A Monthly Magazine devoted to the Science of Religion, the Religion of Science, and the Extension of the Religious Parliament Idea. lited by Dr. Paul Carus ; T. J. McCormack, Assistant Editor ; E. C. Hegeler, d Mary Carus, Associate Editors. Price, $1.00 per year in advance. The en Court Publishing Co., Chicago, Ill.

The following periodicals have been received: *Journal de Mathématiques Élémentaires*, (1^{er} Decembre 1897); *American Journal of Mathematics*, (October, 1897); *The Mathematical Gazette*, (October, 1897); *L'Intermédiare des Mathématiciens*, (Novembre 1897); *Miscellaneous Notes and Queries*, (October, 1897); *The Kansas University Quarterly*, (October, 1897); *The Monist*, (October, 1897); *The Educational Times*, (December, 1897); *Science*, (Nos. for the year to September 24, 1897); *Bulletin of the American Mathematical Society*, (November, 1897); *The Ohio Teacher*, (November, 1897).

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B. F. FINKEL, J. M. COLAW, Editors.

SOME ERRATA IN NO. 11.

Page 282, line 2, for "Ay" read Ay^2.
Page 282, line 17, for "$(2x-m^2$" read $(2x-m)^2$.
Page 283, line 5, for "$+b$" read $+b^2$.
Page 286, supply letter D in the figure.
Page 286, line 14, insert *will* before OED.
Page 286, line 5 from bottom, for "$\frac{1}{2}h$" read $\frac{1}{4}h$.
Page 287, line 17, for "27" read 27/2.
Page 288, line 2, for "R^3" read R^2.
Page 288, line 11, for "$\int_{-\sqrt{r^2-x^2}}^{-\sqrt{r^2+x^2}} zdy$" read $\int_0^r dx\int_{-\sqrt{r^2-x^2}}^{+\sqrt{r^2-x^2}} zdy$.
Page 288, line 12, for "$\int_0^s$" read $\int_0^r$.
Page 288, line 18, for "OF" read OB.
Page 288, line 19, for "$(2br/x)$" read $(2br/h)$.
Page 288, line 20, for "O" read 0.
Page 290, last line, for "sume" read sums.